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ASYMPTOTIC SOLUTION OF A HARMONIC CONTACT PROBLEM FOR A PERMEABLE STAMP ON A LIQUID SATURATED BASE

АСИМПТОТИЧНИЙ РОЗВ'ЯЗОК ГАРМОНІЧНОЇ КОНТАКТНОЇ ЗАДАЧІ ДЛЯ ПРОНИКНОГО ШТАМПА НА НАСИЧЕНІЙ РІДИНОЮ ОСНОВІ

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A plane harmonic problem of vertical vibrations of a rigid permeable stamp on a liquid saturated poroelastic base is considered. The equations of two-phase Biot media, taking into account inertial and viscous interactions of phases, are used. The asymptotic properties of the contact stress in dependence on small frequency of vibration are studied.

Розглянуто гармонічну задачу про вертикальні коливання жорсткого проникного штампа на пористо-пружній основі, насиченій рідиною. Використано рівняння для двофазних середовищ Віо з урахуванням інерційних та в'язких взаємодій між фазами. Досліджено низькочастотні асимптотичні властивості контактної напруги.

1. Introduction. Many specifics of the construction-base interaction can be explained when studying solutions of stamp vibration problems on a poroelastic liquid saturated (PELS) half-space, a foundation being simulated as a permeable or nonpermeable stamp. The sought for quantities are, in the first case, the contact stresses of the base skeleton, in the second case, the stresses in both phases. The most widely accepted theory for description of wave processes in PELS media is the theory developed by M. Biot [1], which takes into account macroscopic characteristics of the solid and the liquid phases, as well as their interaction.

This paper deals with low-frequency asymptotic behavior of contact stress in solving the plane harmonic problem of vertical vibrations of a rigid permeable stamp. It considers both the case of a given stamp displacement law and the case of stamp vibrations under a force load. Sections 2, 3 of the paper contain Biot's equations and integral representations of the Lamb problem for a PELS half-space in a 2-D formulation [2]. Note that although when wave problems are considered for PELS media, additional (in comparison with elastic media) difficulties occur, nevertheless the use of integral Fourier transformation allows to write explicit solutions for nonmixed boundary problems for Biot's equations on a half-space. Integral representation of the Lamb problem solution allows to reduce the considered mixed boundary problem to an integral equation on a segment. It is shown that the equation has a logarithmic kernel in the main part with a special entry of dimensionless vibration frequency ζ (Section 4). In Section 5 of the paper, abstract statements of an independent interest are given. They deal with unique solvability and asymptotic behavior of the corresponding class of integral equations for $\zeta \to 0$.

The principal results of the paper are contained in Section 6 where, on the base of the previous sections, a low-frequency asymptotic analysis of porous stamp vertical vibration problem on a PELS base was conducted. It is shown that under the given uniform stamp displacement law the contact stress at frequency ζ tends to zero, possesses root singularities near the stamp boundaries with an amplitude tending to 0 as $1/|\ln\zeta|$. In the case of stamp vibrations under a force loading it is found that the contact stress at ζ tends to the statistical solution of the plane problem about the rigid stamp pressure on an elastic half-space.

2. Biot's equations. Let us consider dynamic stationary Biot's equations of motion in the plane $(\overline{x},\, \overline{y})$ (further the harmonic time factor $e^{i\omega\overline{t}}$ is omitted). Let us introduce the dimensionless coordinates $x,\, y$, the time t and the vibration frequency ζ,

$$x = \frac{\overline{x}}{a}, \quad y = \frac{\overline{y}}{a}, \quad t = \frac{c_2\overline{t}}{a}, \quad \zeta = \frac{a\omega}{c_2},$$

where a is a parameter of length dimension and c_2 is a transverse wave velocity in a two-phase media not taking into account dissipation, $c_2^2 = N[\rho_{11} - \rho_{12}^2/\rho_{22}]^{-1}$. From this point on N is the elastic skeleton shear module, $\rho_{11} = (1-m)\rho_s - \rho_{12}$, $\rho_{22} = m\rho_f - \rho_{12}$ are the effective densities of, correspondingly, solid and liquid phases. Here, $m\ \rho_{12} \le 0$ is the porosity, $\rho_{12} \le 0$ is the dynamic connection coefficient of the phases, $\rho_s,\, \rho_f$ are the densities of the solid and the liquid phases. Let us denote the displacement of the solid phase by the index s, that of the liquid phase by f. The equations of motion of the poroelastic medium in terms of Biot's equations for displacement vectors of the solid $\vec{u} = \{u_s,\ v_s\}$ and the liquid $\vec{U} = \{u_f,\ v_f\}$ phases in dimensionless variables are:

$$N\Delta\vec{u} + (A+N)\ \mathrm{grad\ div}\ \vec{u} + Q\ \mathrm{grad\ div}\ \vec{U} =$$

$$= -\zeta^2 c_2^2(\rho_{11}\vec{u} + \rho_{12}\vec{U}) + i\zeta B c_2^2 \rho_{11}(\vec{u} - \vec{U}), \tag{1}$$

$$Q\ \mathrm{grad\ div}\ \vec{u} + R\ \mathrm{grad\ div}\ \vec{U} = -\zeta^2 c_2^2(\rho_{12}\vec{u} + \rho_{22}\vec{U}) - i\zeta B c_2^2 \rho_{11}(\vec{u} - \vec{U}),$$

where B is the dimensionless dissipation coefficient in the two-phase medium. At low vibration frequencies, when dissipation effects in the elastic skeleton prevail, the coefficient B has the form [3]

$$B = \frac{a}{c_2\rho_{11}}\frac{m^2\theta_0}{K_{pr}}. \tag{2}$$

The internal friction of the skeleton may be treated by introducing a complex shear module [4],

$$N = |N|e^{i\gamma}, \quad \gamma > 0. \tag{3}$$

The displacement vectors $\vec{u}$, $\vec{U}$ allow the representation [5]

$$\vec{u} = \nabla\phi_1 + \nabla\times\{\psi_1\vec{e}_z\}, \quad \vec{U} = \nabla\phi_2 + \nabla\times\{\psi_2\vec{e}_z\},$$
$$\phi_1 = \Phi_1 + \Phi_2, \quad \phi_2 = M_1\Phi_1 + M_2\Phi_2, \quad \psi_1 = \Psi, \quad \psi_2 = M_3\Psi, \tag{4}$$

where the scalar potentials Φ_j, Ψ are solutions of the Helmholtz equations

$$(\Delta + k_j^2)\Phi_j = 0, \quad j = 1,2, \quad (\Delta + k_3^2)\Psi = 0, \tag{5}$$

with the dimensionless wave numbers

$$k_j^2 = \frac{\zeta^2 c_2^2 z_j}{c^2}, \quad j = 1,2,$$
$$k_3^2 = \frac{\zeta^2\rho c_2^2\left[\Gamma_{11} + M_3\Gamma_{12} + (1-M_3)i\Gamma\right]}{N}. \tag{6}$$

The values E are determined as roots of the quadratic equation $z = z_j$, $j = 1,2$,

$$(q_{11}q_{22} - q_{12}^2)z^2 - (q_{11}\Gamma_{22} + q_{22}\Gamma_{11} - 2q_{12}\Gamma_{12} + i\Gamma)\,z + \left(\Gamma_{11}\Gamma_{22} - \Gamma_{12}^2 + i\Gamma\right) = 0,$$

whose coefficients are given by the expressions

$$\Gamma = -\frac{B\rho_{11}}{\zeta\rho}, \quad \Gamma_{ij} = \frac{\rho_{ij}}{\rho}, \quad \rho = (1-m)\rho_s + m\rho_f,$$

$$c^2 = \frac{H}{\rho}, \quad q_{11} = \frac{A+2N}{H}, \quad q_{12} = \frac{Q}{H}, \quad q_{22} = \frac{R}{H}, \quad H = A + 2N + R + 2Q. \tag{7}$$

The constants M_j in (4) and (6) have the form

$$M_{1,2} = \frac{\left\{\Gamma_{11}q_{22} - \Gamma_{12}q_{12} - (q_{11}q_{22} - q_{12}^2)z_{1,2} + (q_{22}+q_{12})i\Gamma\right\}}{\{\Gamma_{22}q_{12} - \Gamma_{12}q_{22} + (q_{22}+q_{12})i\Gamma\}},$$

$$M_3 = \frac{-\Gamma_{12} + i\Gamma}{\Gamma_{22} + i\Gamma}.$$

The equations (5) show [5, 6] that in an elastic porous media saturated with a viscous compressible liquid, longitudinal waves of three types can propagate. Their propagation constants depend on characteristics of the media as well as on the vibration frequency ζ. Let's define the values $\beta_j = k_j/\zeta$, $j = 1,2,3$, and introduce the notations

$$m_j = \left[1 + \frac{A + QM_j}{2N}\right]\beta_j^2,$$

$$n_j = \left[\frac{Q + RM_j}{2N}\right]\beta_j^2, \quad j = 1,2.$$

Based on the representation (4), we obtain, for the displacements, the following representations in terms of the potentials $\Phi_1, \Phi_2,\ \Psi$:

$$u_s = \frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial x} + \frac{\partial\Psi}{\partial y}, \quad v_s = \frac{\partial\Phi_1}{\partial y} + \frac{\partial\Phi_2}{\partial y} - \frac{\partial\Psi}{\partial x},$$
$$u_f = M_1\frac{\partial\Phi_1}{\partial x} + M_2\frac{\partial\Phi_2}{\partial x} + M_3\frac{\partial\Psi}{\partial y}, \quad v_f = M_1\frac{\partial\Phi_1}{\partial y} + M_2\frac{\partial\Phi_2}{\partial y} - M_3\frac{\partial\Psi}{\partial x}. \tag{8}$$

For the stresses, according to Hooke's law and taking into consideration the equations

$$\nabla\cdot\vec{u} = -k_1^2\Phi_1 - k_2^2\Phi_2, \quad \nabla\cdot\vec{U} = -M_1k_1^2\Phi_1 - M_2k_2^2\Phi_2,$$

we get the expressions

$$\frac{\sigma_x^s}{2N} = \frac{\partial}{\partial x}\left\{\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial x} + \frac{\partial\Psi}{\partial y}\right\} - [m_1 - \beta_1^2]\zeta^2\Phi_1 - [m_2 - \beta_2^2]\zeta^2\Phi_2,$$
$$\frac{\sigma_y^s}{2N} = \frac{\partial}{\partial y}\left\{\frac{\partial\Phi_1}{\partial y} + \frac{\partial\Phi_2}{\partial y} - \frac{\partial\Psi}{\partial x}\right\} - [m_1 - \beta_1^2]\zeta^2\Phi_1 - [m_2 - \beta_2^2]\zeta^2\Phi_2, \tag{9}$$
$$\frac{\tau_{xy}^s}{2N} = \frac{\partial^2}{\partial x\partial y}\Big(\Phi_1 + \Phi_2\Big) + \frac{1}{2}\frac{\partial^2\Psi}{\partial y^2} - \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}, \quad \frac{\sigma^f}{2N} = -n_1\zeta^2\Phi_1 - n_2\zeta^2\Phi_2.$$

3. Lamb problem. Let us examine a plane Lamb problem of dynamic displacements of a PELS half-space $y < 0$ with permeable surface under a given vertical loading,

$$\tau_{xy}^s(x,0) = 0, \quad \frac{\sigma_y^s(x,0)}{2N} = -p(x), \quad \sigma^f(x,0) = 0. \tag{10}$$

We consider the case where there is an internal friction $\gamma > 0$ in the elastic skeleton (see (3)) and, in this connection, it is assumed that the wave numbers k_j, in the considered frequency range, are not real. Then the boundary conditions (10) are complemented with the conditions of the displacements $\vec{u}, \vec{U}$ to decrease as $y \to -\infty$. From here on we shall use the notations

$$\xi_j = \xi_j(\xi) = \sqrt{\xi^2 - k_j^2}, \quad \xi \in (-\infty,\infty), \quad \mathrm{Re}\,\xi_j > 0, \quad j = 1,2,3.$$

The potentials Φ_j, Ψ, according to (6), are sought as the Fourier integrals

$$\Phi_j(x,y) = \int_{-\infty}^{\infty} A_j(\xi) e^{\xi_j y} e^{i\xi x} d\xi,$$
$$\Psi(x,y) = i \int_{-\infty}^{\infty} B(\xi) e^{\xi_3 y} e^{i\xi x} d\xi \tag{11}$$

with unknown densities $A_1(\xi),\ A_2(\xi),\ B(\xi)$. For the stresses in the elastic skeleton, σ_{ij}^s, and the force σ^f, acting on the liquid, we get the expressions

$$\frac{\tau_{xy}^s}{2N} = i \int_{-\infty}^{\infty} e^{i\xi x}\Big\{A_1(\xi)\xi\xi_1 e^{\xi_1 y} + A_2(\xi)\xi\xi_2 e^{\xi_2 y} + B(\xi)\Big(\xi^2 - \frac{k_3^2}{2}\Big)e^{\xi_3 y}\Big\} d\xi,$$

$$\frac{\sigma_y^s}{2N} = \int_{-\infty}^{\infty} e^{i\xi x}\Big\{A_1(\xi)[\xi^2 - m_1\zeta^2]e^{\xi_1 y} + A_2(\xi)[\xi^2 - m_2\zeta^2]e^{\xi_2 y} + B(\xi)\xi\xi_3 e^{\xi_3 y}\Big\} d\xi,$$

$$\frac{\sigma^f}{2N} = -\zeta^2 \int_{-\infty}^{\infty} e^{i\xi x}\Big\{A_1(\xi) n_1 e^{\xi_1 y} + A_2(\xi) n_2 e^{\xi_2 y}\Big\} d\xi.$$

The fulfillment of the boundary conditions (10) and the inversion of the Fourier transform result in a system of linear algebraic equations with respect to the functions $A_1(\xi),\ A_2(\xi),\ B(\xi)$, whose solution is [2]

$$\zeta^2 F(\xi) A_1(\xi) = n_2\zeta^2(2\xi^2 - k_3^2)\bar{P}(\xi), \quad \zeta^2 F(\xi) A_2(\xi) = -n_1\zeta^2(2\xi^2 - k_3^2)\bar{P}(\xi),$$
$$\zeta^2 F(\xi) B(\xi) = -2\xi\zeta^2[n_2\xi_1 - n_1\xi_2]\bar{P}(\xi), \tag{12}$$

where the determinant is

$$F(\xi) = 2(2\xi^2 - k_3^2)\{n_1[\xi^2 - m_2\zeta^2] - n_2[\xi^2 - m_1\zeta^2]\} + 4\xi^2\xi_3[n_2\xi_1 - n_1\xi_2] \tag{13}$$

and the inverse Fourier transform of a given loading is

$$\bar{P}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(x) e^{-i\xi x} dx.$$

For vertical displacement of the solid phase boundary, according to the formulas (8), (11), (12), we have the integral representation

$$v_s(x,0) = \beta_3^2 \int_{-\infty}^{\infty} \frac{\bar{P}(\xi) F_1(\xi)}{F(\xi)} e^{ix\xi} d\xi,$$
$$F_1(\xi) = \zeta^2 (n_1 \xi_2 - n_2 \xi_1). \tag{14}$$

This representation in terms of a given loading is used for building solution for the stamp vibration problem on a PELS half-space. Here, the asymptotic analysis of the corresponding integrand is of primary importance. On the other hand, for the low-frequency analysis it is necessary to know the asymptotic behavior of this integrand at $\zeta \to 0$. Both asymptotic forms can be obtained in the same way by using specifics of appearance of the argument ξ and the vibration frequency ζ in the functions.

The functions under study, $F(\xi)$ and $F_1(\xi)$, are even, therefore it is enough to analyze their behavior at $\xi \to +\infty$. Let us define the parameters $\overline{n}_j = n_j/\beta_j^2$, $\overline{m}_j = m_j/\beta_j^2$, $j = 1, 2$, and introduce the variables $x_j = k_j^2/\xi^2$, $j = 1, 2, 3$.

Then the determinant $F(\xi)$ can be written as $\zeta^2 F(\xi) = \xi^6 D(x_1, x_2, x_3)$, with the function

$$D(x_1, x_2, x_3) = 2(2 - x_3)[\overline{n}_1 x_1 (1 - \overline{m}_2 x_2) - \overline{n}_2 x_2 (1 - \overline{m}_1 x_1)] +$$
$$+ 4(1 - x_3)^{1/2} [\overline{n}_2 x_2 (1 - x_1)^{1/2} - \overline{n}_1 x_1 (1 - x_2)^{1/2}].$$

Using the asymptotic identity

$$(1 - x_j)^{1/2} = 1 - x_j/2 + \mathrm{O}(x_j^2), \quad x_j \to 0,$$

we get, for $|x_1| + |x_2| + |x_3| \to 0$,

$$D(x_1, x_2, x_3) = 2x_1 x_2 [2(\overline{m}_1 \overline{n}_2 - \overline{m}_2 \overline{n}_1) - (\overline{n}_2 - \overline{n}_1)] + O(\kappa),$$
$$\kappa = |x_3| (|x_1 x_2| + |x_3| \max\{|x_1|, |x_2|\}).$$

Similarly, we use the representation

$$F_1(\xi) = \xi^3 F_1^{(0)}(x_1, x_2), \quad F_1^{(0)}(x_1, x_2) = \overline{n}_1 x_1 (1 - x_2)^{1/2} - \overline{n}_2 x_2 (1 - x_1)^{1/2},$$

and, for $|x_1| + |x_2| + |x_3| \to 0$, the following relation is fulfilled:

$$F_1^{(0)}(x_1, x_2) = \overline{n}_1 x_1 - \overline{n}_2 x_2 + O(\kappa_1), \quad \kappa_1 = |x_1 x_2| \max\{|x_1|, |x_2|\}.$$

The considerations above show, in particular, that for the Raley's determinant the following asymptotics at $\xi \to +\infty$ is valid:

$$F(\xi) = d_0\xi^2 + O(1), \tag{15}$$

$$d_0 = \zeta^2 \left\{ \left(m_1 - \frac{\beta_1^2}{2} \right) n_2 - \left(m_2 - \frac{\beta_2^2}{2} \right) n_1 \right\}.$$

For the function $F_1(\xi)$ we get the following asymptotic expressions at $\xi \to +\infty$ (we omit the computations):

$$F_1(\xi) = C_1\xi + O(\xi^{-1}), \quad C_1 = \zeta^2(n_1 - n_2). \tag{16}$$

Thus, from (15) and (16) it follows that for the integrand in (14) the following estimate hold:

$$\frac{F_1(\xi)}{F(\xi)} = \frac{C_1}{d_0}\xi^{-1} + O(\xi^{-3}). \tag{17}$$

The use of the estimate (17) and the well-known Fourier transform properties show that when the condition $p(x) \in L_2(\mathrm{R})$ is fulfilled, the integrals (14) are absolutely convergent for any $x \in \mathrm{R}$ and are continuous functions tending to zero at $x \to \infty$.

When studying the integrands' asymptotics for $\zeta \to 0$, two cases should be considered depending on the presence or absence of dissipation due to the pore liquid friction (see expression (2). In the case $B = 0$, the roots z_j of the quadratic equation (6), as well as the numbers $\beta_j = k_j/\zeta$ and the coefficients M_j, depend on the frequency, so that

$$\frac{k_1^2}{k_2^2} = \frac{z_1}{z_2} := \Omega_1^2 \equiv \text{const},$$

$$\frac{k_3^2}{k_2^2} = \frac{\rho c^2}{N}\frac{[\Gamma_{11}\Gamma_{22} - \Gamma_{12}^2]}{z_2\Gamma_{22}} := \Omega^2 \equiv \text{const}.$$

In the presence of dissipation, i.e., for $B \neq 0$, the quadratic equation (6) for determining the roots z_j, from which the wave numbers k_1, k_2 are computed, contains frequency dependent coefficients. Set $b_0 = \zeta\Gamma \equiv -B\rho_{11}/\rho$, then for $\zeta \to 0$ we obtain

$$z_1 = \frac{ib_0\zeta^{-1} + b_2}{b_1} - 1 + O(\zeta), \quad z_2 = 1 + O(\zeta),$$

with the constants $b_1 = q_{11}q_{22} - q_{12}^2$, $\quad b_2 = q_{11}\Gamma_{22} + q_{22}\Gamma_{11} - 2q_{12}\Gamma_{12}$. So, for the squares of the wave numbers, we have the estimates

$$k_1^2 = \zeta\frac{c_2^2}{c^2}\left(\frac{ib_0}{b_1} + \left(\frac{b_2}{b_1} - 1\right)\zeta\right) + O(\zeta^3),$$

$$k_2^2 = \zeta^2\frac{c_2^2}{c^2} + O(\zeta^3), \quad \zeta \to 0.$$

Taking into account the equation $\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22} = 1$ and the relation

$$M_3 = 1 + i\frac{\Gamma_{12} + \Gamma_{22}}{\Gamma} + O(\zeta^2), \quad \zeta \to 0,$$

we get $k_3^2 = \zeta^2(c_2^2\rho)/N + O(\zeta^3)$, $\zeta \to 0$. So, the modulus of the wave number k_1, for $\zeta \to 0$, tends to zero as $\zeta^{1/2}$, while the modulus of the numbers k_2, k_3 tend to zero as ζ.

4. Vertical vibrations of a rigid permeable stamp. Consider the plane problem for the force $Pe^{i\zeta t}$ ($P =$ const) acting on a permeable rigid stamp of the width $2\bar{a}$, height $\bar{h}$ and density ρ_0, on a PELS half-space [7]. The normal reaction distribution along the stamp contact with the base and its vertical displacement w, in the absence of friction along the contact, is studied. As inital data, the Biot's equations for a two-phase medium and stamp motion equations in the case of harmonic vibrations (in dimensionless variables) are assumed [8],

$$M_0\left(\frac{c_2}{\bar{a}}\right)^2\zeta^2 w = R - P, \quad R = -\bar{a}\int_{-1}^{1}\sigma_y^s(x,0)dx, \quad |x| < 1, \tag{18}$$

where $M_0 = 2\bar{h}\bar{a}\rho_0$ is the mass of the stamp, R the resultant of the normal contact stresses (the time factor $e^{i\zeta t}$ is omitted). On the boundary of the stamp and the 2-phase base, the following conditions are fulfilled:

$$\tau_{xy}^s(x,0) = \sigma^f(x,0) = 0, \quad v_s(x,0) = w, \quad |x| < 1, \tag{19}$$

where $v_s(x,0)$ are vertical displacements of the solid phase boundary. Outside of the stamp, the stress and pore pressure are absent, that is,

$$\tau_{xy}^s(x,0) = \sigma_y^s(x,0) = \sigma^f(x,0) = 0, \quad |x| > 1. \tag{20}$$

Let us define a normalized unknown contact stress

$$p(x) = -\frac{\sigma_y^s(x,0)}{N}, \quad p(x) = p(-x), \quad |x| < 1.$$

Using the representation (14) for the shear $v_s(x,0)$, on the base of the conditions (19), (20), we get the integral equation for finding the even function $p(x) \in L_1[-1,1]$,

$$(\mathcal{K}(\zeta)p)(x) := \frac{1}{\pi}\int_{-1}^{1}K(s-x;\zeta)p(s)ds = f(x), \quad x \in (-1,1), \tag{21}$$

where the kernel and the right-hand side of the equation are

$$K(x;\zeta) = \int_0^\infty \frac{\beta_3^2 F_1(\xi)}{F(\xi)}\cos(x\xi)d\xi, \quad f(x) = \frac{w}{\bar{a}} = \text{const.} \tag{22}$$

Here, the constant w contains, according to (18), both the given load P and the integral of the unknown function $p(x)$.

If the constant w is considered as given in the equation (22), than the integral equation (21) will be used for finding the contact stress for a given uniform motion of the massless stamp.

5. Asymptotic solution of the integral equation with logarithmic kernel. According to (17), the function $F_1(\xi)/F(\xi)$ behaves at infinity like ξ^{-1} and this means that the kernel $K(s-x;\zeta)$, independent of the value of the parameter $\zeta > 0$, has logarithmic singularity as $|x-s| \to 0$. On the other hand, it will be shown later that for $\zeta \to 0$ the singularity of the kernel $K(s-x;\zeta)$ is characterized by the function $\ln \zeta |s-x| E$. In this section a statement is given on unique solvability and asymptotic behavior of equations of this kind as the parameter $\zeta \to 0$.

Let us recall the Noether's operator theory necessary for the further account [9, 10]. Let X, Y be Banach spaces, $\mathcal{R}(X,Y)$ a Banach space of linear continuous operators acting from X to Y. The norm in the space X will be denoted by $\|\cdot\|_X$. Let I be the identity operator in the corresponding space. By the kernel of the operator $A \in \mathcal{R}(X,Y)$ we call the subspace $\ker A = \{\phi \in X : A\phi = 0\} \subset X$ and its co-kernel is the kernel of the conjugate operator $A^* \in \mathcal{R}(Y^*, X^*)$, $\operatorname{coker} A = \{\psi \in Y^* : A^*\psi = 0\} \subset Y^*$. The operator $A \in \mathcal{R}(X,Y)$ is called a Noether operator, if its image AX is closed in the space Y and the kernel and the co-kernel are finite-dimensional subspaces,

$$n = \dim \ker A < \infty, \quad m = \dim \operatorname{coker} A < \infty.$$

The difference $\kappa(A) = n - m$ is called the Noether operator index, and the Noether operator with zero index is called Fredholm operator.

Further by H^α, $\alpha \in (0,1]$, we denote the Banach space of functions, given on $[-1,1]$ and satisfying the Hoelder condition (see, e.g. [10], § 1),

$$|f(x_1) - f(x_2)| \le c|x_1 - x_2|^\alpha, \quad x_1,\, x_2 \in [-1,1].$$

Set $r(x) = \sqrt{1-x^2}$, $|x| < 1$.

Let a_1, a_2 be certain constant values, and $L(x,s;\zeta)$ a family of continuous on the square $|x| \le 1$, $|s| \le 1$ functions, continuously differentiable with respect to x. Le us assume

$$L_{1,0}(\zeta) \equiv \max_{|x|\le 1,\ |s|\le 1} \Big\{ |L(x,s;\zeta)| + |L_x'(x,s;\zeta)| \Big\} < \infty, \quad \zeta > 0. \tag{23}$$

For $\zeta > 0$, introduce a family of integral operators $\mathcal{L}_0(\zeta)$, $\mathcal{L}_1(\zeta)$, and $\mathcal{L}(\zeta)$, acting on functions given on the segment $[-1,1]$ by

$$(\mathcal{L}_0(\zeta)p)(x) = \frac{a_1}{\pi} \int_{-1}^{1} \ln \zeta |s-x| p(s) ds + \frac{a_2}{\pi} \int_{-1}^{1} p(s) ds,$$

$$(\mathcal{L}_1(\zeta)p)(x) = \int_{-1}^{1} L(x,s;\zeta) p(s) ds, \tag{24}$$

$$\mathcal{L}(\zeta)p = \mathcal{L}_0(\zeta)p + \mathcal{L}_1(\zeta)p.$$

It is known (see [11, 12]) that if the condition

$$a_1 \ln(\zeta/2) + a_2 \neq 0 \tag{25}$$

holds and the function $f(x)$ is continuous on the segment $[-1, 1]$, and its derivative belongs to the Hoelder space H^α for a certain $\alpha \in (0, 1)$ (i.e. the function $f \in H_1^\alpha$), then the equation $\mathcal{L}_0(\zeta)p = f$ possesses a solution,

$$\begin{aligned} p(x) &= (\mathcal{L}_0^{-1}(\zeta)f)(x) \equiv \\ &\equiv \frac{1}{\pi r(x)} \left\{ \frac{1}{a_1} \int_{-1}^{1} \frac{r(x)f'(s)}{s-x} ds + \frac{1}{a_1 \ln(\zeta/2) + a_2} \int_{-1}^{1} \frac{f(s)}{r(s)} ds \right\}. \end{aligned} \tag{26}$$

This solution is unique in the class of functions $p(x)$ that belong to the whole Hoelder space $H^\alpha(r)$,

$$p(x) = \frac{g(x)}{r(x)}, \quad g(x) \in H^\alpha.$$

Let us introduce the integration operator J and the singular Cauchy operator S,

$$(Jp)(x) = \int_{-1}^{x} p(s) ds, \quad (Sp)(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{p(s)}{s-x} ds.$$

The following formula hold [11] (§ 55)

$$\frac{1}{\pi} \int_{-1}^{1} \ln|s-x| p(s) dt = -(JSp)(s) + \frac{1}{\pi} \int_{-1}^{1} \ln(1+s) p(s) ds, \quad p \in H^\alpha(r). \tag{27}$$

Here, the operator S is a Noether operator in the space $L_q[-1, 1]$ for any $q \in (1, 2)$ and has the index $\kappa(S) = 1$ [10] (§ 11, 31). Since the operator $J \in \mathcal{R}(L_q, W_q^1)$ is Noether and has the index $\kappa(J) = -1$, it follows from (27) that the operator $\mathcal{L}_0(\zeta) \in \mathcal{R}(L_q, W_q^1)$ is Fredholm for any $q \in (1, 2)$. On the other hand, since (26) gives unique solvability of the equation $\mathcal{L}_0(\zeta)p = f$, $f \in H_1^\alpha$, by virtue of density of inclusion of the space $H^\alpha(r)$ in the Banach space $L_q[-1, 1]$, we get that the integral operator $\mathcal{L}_0(\zeta)$, if the conditions (25) and $q \in (1, 2)$ are fulfilled, defines an isomorphism between the Banach spaces $L_q[-1, 1]$ and $W_q^1[-1, 1]$.

On the base of the above considerations and using standard methods of the theory of linear operators the following theorem is proved.

Theorem 1. *Let $q \in (1, 2)$ and $L_{1,0}(\zeta) \to 0$, $\zeta \to 0$. Then there is $\zeta_0 > 0$ such that, for any $\zeta \in (0, \zeta_0)$, the operators $\mathcal{L}_0(\zeta) \in \mathcal{R}(L_q, W_q^1)$ and $\mathcal{L}(\zeta) \in \mathcal{R}(L_q, W_q^1)$ are continuously invertible with*

$$\mathcal{L}^{-1}(\zeta) = \mathcal{L}_0^{-1}(\zeta)(I + \mathcal{T}(\zeta)), \quad \|\mathcal{T}(\zeta)\|_{\mathcal{R}(W_q^1)} \leq c L_{1,0}(\zeta), \quad \zeta \in (0, \zeta_0). \tag{28}$$

Proof. For sufficiently small ζ (so that the condition (25) is trivially fulfilled) an inverse operator $\mathcal{L}_0^{-1}(\zeta) \in \mathcal{R}(W_q^1, L_q)$ exists, determined by the expression (26). Hence it follows that, for a certain $\zeta_1 > 0$, the uniform estimate

$$\|\mathcal{L}_0^{-1}(\zeta)\|_{\mathcal{R}(W_q^1, L_q)} \leq l_0 < \infty, \quad \zeta \in (0, \zeta_1)$$

holds. Further, from the condition (23) it follows that the operators $\mathcal{L}_1(\zeta)$ continuously act from the space L_q into the space W_q^1, so that their norms are subject to the following estimate:

$$\|\mathcal{L}_1(\zeta)\|_{\mathcal{R}(L_q, W_q^1)} \leq L_{1,0}(\zeta) \to 0, \quad \zeta \to 0.$$

Thus, by choosing the number $\zeta_0 \in (0, \zeta_1)$ from the condition $l_0 L_{1,0}(\zeta) \leq 1/2$, $\zeta \in (0, \zeta_0)$, we get that for $\zeta \in (0, \zeta_0)$ the operators $\mathcal{L}(\zeta) \in \mathcal{R}(L_q, W_q^1)$ are continuously invertible and the following equation holds:

$$\mathcal{L}^{-1}(\zeta) = \mathcal{L}_0^{-1}(\zeta)(I + \mathcal{L}(\zeta)\mathcal{L}_0^{-1}(\zeta))^{-1}.$$

Hence we get, that (28) is fulfilled with the operator

$$\mathcal{T}(\zeta) = -(I + \mathcal{L}_1(\zeta)\mathcal{L}_0^{-1}(\zeta))^{-1}\mathcal{L}_1(\zeta)\mathcal{L}_0^{-1}(\zeta).$$

The theorem is proved.

Corollary. *Under conditions of Theorem 1, the equation $\mathcal{L}(\zeta)p = f$, for any function $f \in W_q^1$, has a unique solution $p \in L_q$. Also, in the space L_q, solutions $p = p(x;\zeta)$ satisfy the asymptotic representation*

$$p(x;\zeta) = (\mathcal{L}_0^{-1}(\zeta)f)(x) + O(L_{1,0}(\zeta)), \quad \zeta \to 0, \tag{29}$$

where the expression $(\mathcal{L}_0^{-1}(\zeta)f)(x)$ is defined by the formula (26). If the function $f \in H_1^\alpha$ $(\alpha \in (0,1))$, then the asymptotic formula (29) is valid also in the space $H_\alpha(r)$ and, in particular, in the weight space of continuous functions $C(r)$,

$$\sup_{|x|<1} |(p(x;\zeta) - (\mathcal{L}_0^{-1}(\zeta)f)(x))r(x)| = O(L_{1,0}(\zeta)), \quad \zeta \to 0. \tag{30}$$

If $f(x) = f_0 = \mathrm{const}$, then for solution of the equation $\mathcal{L}(\zeta)p = f(x)$, the following asymptotic estimate hold:

$$\sup_{|x|<1} \left| p(x;\zeta)r(x) - \frac{f_0}{a_1 \ln(\zeta/2) + a_2} \right| = O(L_{1,0}(\zeta)), \quad \zeta \to 0. \tag{31}$$

6. A low-frequency analysis of the solution of the problem of vertical vibrations of a permeable stamp. In this section for $\zeta \to 0$ an asymptotic analysis of solutions of the integral equation (21) is conducted. It is shown that the use of the structure of the functions $F_1(\xi)$, $F(\xi)$ and the asymptotic formulas of Section 2 allows, for $\zeta \to 0$, to reduce (21) to a consideration of the equation $\mathcal{L}(\zeta)q = f$ with the operator of the type (24) and some constants a_1, a_2. This, in its turn, enables to apply the results from the Section 5. Next, we will need the relations from [13] (Chapter 6),

$$\int_1^\infty \frac{\cos(x\xi)}{\xi} d\xi = -C - \ln|x| + S_1(x),$$

$$S_1(x) = \int_0^x \frac{1-\cos\xi}{\xi} d\xi = -\sum_{k=1}^\infty \frac{(-1)^k x^{2k}}{2k(2k)!}, \tag{32}$$

where $C \approx 0,577$ is the Euler's constant.

Let us use the notations for the constants

$$a_{12} = 2(\overline{n}_2\overline{m}_1 - \overline{n}_1\overline{m}_2), \quad b_{12} = \overline{n}_2 - \overline{n}_1, \quad d_1 = a_{12} - b_{12},$$

and for the functions

$$D_1(s) = D\left(\frac{\beta_1^2}{s^2}, \frac{\beta_2^2}{s^2}, \frac{\beta_3^2}{s^2}\right), \quad s > 0, \quad D_2(\tau) = D(\beta_1^2\tau^2, \beta_2^2\tau^2, \beta_3^2\tau^2), \quad \tau > 0.$$

Let us examine the equation (21) in the absence of dissipation ($B = 0$). Then the functions $D_1(s)$, $D_2(\tau)$ don't depend of the frequency ζ. Let us represent the function defining the kernel of the equation (21) in the form

$$K(x;\zeta) = K_1(x;\zeta) + K_2(x;\zeta) \equiv$$

$$\equiv \int_\zeta^\infty \frac{\beta_3^2\zeta^2 F_1^{(0)}(x_1,x_2)}{\xi^3 D(x_1,x_2,x_3)} \cos(x\xi) d\xi + \int_0^\zeta \frac{\beta_3^2\zeta^2 F_1^{(0)}(x_1,x_2)}{\xi^3 D(x_1,x_2,x_3)} \cos(x\xi) d\xi, \tag{33}$$

where the values $x_j = k_j^2/\xi^2 \equiv (\beta_j^2\zeta^2)/\xi^2$.

For $K_1(x;\zeta)$, after the change of the integration variable, $\xi = s\zeta$, we obtain the expression

$$K_1(x;\zeta) = \beta_3^2 \int_1^\infty \frac{F_1^{(0)}(\beta_1^2/s^2,\, \beta_2^2/s^2)}{s^3 D_1(s)} \cos(x\zeta s) ds.$$

Here, according Section 4 (taking into account the assumption $B = 0$), we have the relation

$$\beta_3^2 \frac{F_1^{(0)}(\beta_1^2/s^2,\, \beta_2^2/s^2)}{s^3 D_1(s)} = R_0 s^{-1} + R_1(s) s^{-3},$$

where the coefficient

$$R_0 = \beta_3^2 \frac{(\overline{n}_1\beta_1^2 - \overline{n}_2\beta_2^2)}{2\beta_1^2\beta_2^2 d_1} = \frac{\Omega^2(\overline{n}_1\Omega_1^2 - \overline{n}_2)}{2\Omega_1^2 d_1} \tag{34}$$

and the function $R_1(s) = O(1)$, $s \to \infty$. Thus, the integral

$$K_1(x;\zeta) = R_0 \int_1^\infty \frac{\cos(x\zeta s)}{s} ds + \int_1^\infty R_1(s) \frac{\cos(x\zeta s)}{s^3} ds.$$

It is not difficult to determine for $\zeta > 0$ a uniform with respect to $x \in [-1,1]$ estimate,

$$\int_1^\infty \frac{|R_1(s)|}{s^3} \{|\cos(x\zeta s) - 1| + \zeta |x| |\sin(x\zeta s)|\} ds \le c_0 \zeta^2 (|\ln \zeta| + 1), \tag{35}$$

with a certain constant $c_0 > 0$.

Further we will use c_0 to denote various constants whose values are not essential. Using (32) we get that there exists a constant K_1 such that, in the space $C^1 = C^1[-1,1]$ of continuously differentiable on the segment $[-1,1]$ functions, the following estimate holds:

$$\|K_1(x;\zeta) + R_0 \ln|\zeta x| - K_1\|_{C^1} \le c_0 \zeta^2 (|\ln \zeta| + 1), \quad \zeta > 0.$$

Let us consider the integral $K_2(x;\zeta)$, which after the substitution $\xi = \zeta \tau^{-1}$, is reduced to the form

$$K_2(x;\zeta) = \int_1^\infty R_2(\tau) \cos(x\zeta\tau^{-1}) d\tau,$$

$$R_2(\tau) = \beta_3^2 \frac{\tau F_1^{(0)}(\beta_1^2 \tau^2,\, \beta_2^2 \tau^2)}{D_2(\tau)}.$$

Using the expressions for the functions $F_1^{(0)}(x_1, x_2)$ and $D(x_1, x_2, x_3)$ from Section 4, we obtain the estimates

$$F_1^{(0)}(\beta_1^2 \tau^2,\, \beta_2^2 \tau^2) = O(\tau^3),$$

$$D_2(\tau) = -\beta_1^2 \beta_2^2 \beta_3^2 a_{12} \tau^6 + O(\tau^4), \quad \tau \to \infty,$$

and so, $R_2(\tau) = O(\tau^{-2})$, $\tau \to \infty$. Hence it follows that the function $K_2(x;\zeta)$ is infinitely differentiable with respect to $x \in [-1,1]$ and

$$\|K_2(x,\zeta) - K_2\|_{C^1} \le c_0 \zeta^2, \quad K_2 = \int_0^\infty R_2(\tau) d\tau.$$

Thus, taking into account (33) – (35), we have the following proposition.

Proposition 1. *The function $K(x;\zeta)$ can be represented as*

$$K(x;\zeta) = -R_0 \ln|\zeta x| + K_0 + \hat{K}(x;\zeta), \tag{36}$$

with the constant R_0 from (34) and a certain constant K_0 independent of $\zeta > 0$, and the function $\hat{K}(x;\zeta)$, for any $\zeta > 0$, is continuously differentiable and satisfies the estimate

$$\|\hat{K}(x,\zeta)\|_{C^1} \le c_0\zeta^2(|\ln\zeta| + 1), \quad \zeta > 0.$$

From Proposition 1, the following statement follows.

Lemma 1. *Let $B = 0$. Then one can find constants R_0, K_0 such that*

$$\|K(x;\zeta) + R_0 \ln|\zeta x| - K_0\|_{C^1} \le c_0\zeta^2(|\ln\zeta| + 1), \quad \zeta > 0.$$

Assume that the linear integral operator $\mathcal{L}_0(\zeta)$ is defined by the expression (24) with the constants $a_1 = -R_0$, $a_2 = K_0$, $\mathcal{L}(\zeta) = \mathcal{K}(\zeta)$ is the integral operator from the equation (21). Lemma 1 shows that the family of integral operators $\mathcal{K}(\zeta)$, for $\zeta \to 0$, tends in the operator norm of $\mathcal{R}(L_q, W_q^1)$ to the operator $\mathcal{L}_0(\zeta)$ and one can apply Corollary 1 from the Section 5 to the equation (21) with $L_{1,0}(\zeta) = \zeta^2(|\ln\zeta| + 1)$ (see also Remark). Thus we have the following theorem.

Theorem 2. *Let $B = 0$ and $q \in (1, 2)$. Then for sufficiently small ζ, the equation (21) for any function $f(x) \in W_q^1$ has a unique solution $p(x) = p(x;\zeta) \in L_q$ and*

$$p(x) = (\mathcal{L}_0^{-1}(\zeta)f)(x) + O(\zeta^2|\ln\zeta|), \quad \zeta \to 0$$

in the norm of the space L_q, where $(\mathcal{L}_0^{-1}(\zeta)f)(x)$ is given by the expression (26). In the case of a given uniform displacement of the stamp, $f(x) = w/\bar{a} = W$, the solution $p(x)$ satisfies the asymptotic representation

$$p(x) = -\frac{W}{R_0 \ln\zeta/2 - K_0}\,\frac{1}{\sqrt{1-x^2}} + O(\zeta^2|\ln\zeta|), \quad \zeta \to 0, \tag{37}$$

in the sense of the norm of the weight space $C(r)$.

Remark. From the foregoing considerations it follows that for the constant K_0, in (37) one can give a rather bulky expression. On the other hand, simplifying (37) we get an asymptotic representation for the solution

$$p(x) = -\frac{W}{R_0 \ln\zeta}\,\frac{1}{\sqrt{1-x^2}} + O\left(\frac{1}{\ln^2\zeta}\right), \quad \zeta \to 0. \tag{38}$$

Now let us consider a more complex case of the equation (21), when $B \ne 0$. Here the values β_j^2 depend on the frequency ζ and, for $\zeta \to 0$,

$$\beta^2 := \zeta\beta_1^2 = -\frac{ic_2^2 b_0}{c^2 b_1} + O(\zeta), \quad \beta_2^2 = c_2^2/c^2 + O(\zeta), \quad \beta_3^2 = c_2^2\rho/N + O(\zeta). \tag{39}$$

Set the integrals in the representation of the function $K(x;\zeta) = K_1(x;\zeta) + K_2(x;\zeta)$ to be

$$K_1(x;\zeta) = \int\limits_{\zeta^{1/2}}^{\infty} \frac{\beta_3^2 F_1(\xi)}{F(\xi)} \cos(x\xi) d\xi,$$

$$K_2(x;\zeta) = \int_0^{\zeta^{1/2}} \frac{\beta_3^2 F_1(\xi)}{F(\xi)} \cos(x\xi) d\xi.$$

Applying the substitution $\xi = \zeta^{1/2} s$ in the first integral and using the relations of the Section 4 for the integrand we obtain

$$K_1(x;\zeta) = \zeta\beta_3^2 \int\limits_1^{\infty} \frac{F_1^{(0)}(\beta^2/s^2,\, \zeta\beta_2^2/s^2)}{s^3 D_1(s/\zeta^{1/2})} \cos(x\zeta^{1/2}s) ds.$$

The integrand admits the representation

$$\zeta \frac{\beta_3^2 F_1^{(0)}(\beta^2/s^2,\, \zeta\beta_2^2/s^2)}{s^3 D_1(s/\zeta^{1/2})} = R_{0,1}(\zeta)s^{-1} + R_1(\zeta)\zeta s^{-1} + \zeta s^{-3} R_2(s;\zeta)$$

with the coefficients

$$R_{0,1}(\zeta) = \frac{\overline{n}_1 \beta_3^2}{2\beta_2^2 d_1}, \quad R_1(\zeta) = -\frac{\overline{n}_2 \beta_2^2}{\overline{n}_1 \beta^2} R_{0,1}$$

and the finite, in respect to $\zeta > 0$ and $s > 0$, function $R_2(s;\zeta)$. Thus,

$$K_1(x;\zeta) = (R_{0,1}(\zeta) + R_1(\zeta)\zeta) \int\limits_1^{\infty} \frac{\cos(x\zeta^{1/2}s)}{s} ds + \zeta \int\limits_1^{\infty} R_2(s;\zeta) \frac{\cos(x\zeta^{1/2}s)}{s^3} ds.$$

Using the relationships (32) we get, for a certain constant K_1, the estimate for $K_1(x;\zeta)$:

$$\|K_1(x,\zeta) + (R_{0,1}(\zeta) + R_1(\zeta)\zeta) \ln|\zeta^{1/2}x| - K_1\|_{C^1} \le c_0\zeta(|\ln\zeta| + 1), \quad \zeta > 0.$$

Let us show that the function $K_2(x;\zeta)$ admits the estimate

$$\|K_2(x;\zeta) - R_{0,2}(\zeta)\ln\zeta - R_{1,2}(\zeta)\|_{C^1} \le c_0\zeta^{1/2}(\ln|\zeta| + 1), \quad \zeta > 0, \tag{40}$$

with certain functions $R_{j,2}(\zeta)$ continuous with respect to $\zeta \ge 0$. In the integral representation for $K_2(x;\zeta)$, we substitute $\xi = \zeta\tau^{-1}$. Then

$$K_2(x;\zeta) = \beta_3^2 \int\limits_{\zeta^{1/2}}^{\infty} \frac{G(\tau;\zeta)\tau^3}{\zeta D_2(\tau)} \cos(x\zeta\tau^{-1}) d\tau,$$

where

$$G(\tau;\zeta) = \overline{n}_1\beta^2(1-\beta_2^2\tau^2)^{1/2} - \overline{n}_2\beta_2^2\zeta^{1/2}(\zeta-\beta^2\tau^2)^{1/2}.$$

For $\zeta D_2(\tau)$ we have the equation

$$\zeta D_2(\tau) = D(\zeta^{-1}\beta^2\tau^2, \beta_2^2\tau^2, \beta_3^2\tau^2) = \tau^4(a_1(\zeta)\tau^2 + a_2(\zeta) + g_1(\tau;\zeta)) + g_2(\tau;\zeta)),$$

where the continuously differentiable with respect to $\zeta \geq 0$ coefficients are

$$a_1(\zeta) = \beta^2\beta_2^2\beta_3^2 a_{12},$$

$$a_2(\zeta) = 2\overline{n}_2\beta_2^2(\zeta\beta_3^2 + 2\overline{m}_1\beta^2) - 2\overline{n}_1\beta^2(\beta_3^2 + 2\overline{m}_2\beta_2^2),$$

and the continuous, uniformly bounded with respect to $\tau \geq 0$ and $\zeta \in (0, \zeta_1)$, $\zeta_1 < \infty$, functions are

$$g_1(\tau;\zeta) = 4\overline{n}_1\beta^2\tau^{-2}(1-(1-\beta_3^2\tau^2)^{1/2}(1-\beta_2^2\tau^2)^{1/2}),$$

$$g_1(0;\zeta) = 2\overline{n}_1\beta^2(\beta_2^2+\beta_3^2),$$

$$g_2(\tau,\zeta) = -4\overline{n}_2\beta_2^2\tau^{-2}\zeta(1-(1-\beta_3^2\tau^2)^{1/2}(1-\beta^2\tau^2/\zeta)^{1/2}).$$

In particular, there is a constant $c > 0$ such that

$$|\zeta D_2(\tau)| \geq c_0\tau(\tau+1), \quad \tau > 0. \tag{41}$$

Further we have:

$$|\cos(x\zeta\tau^{-1}) - 1| \leq \zeta, \quad \tau \geq \zeta^{1/2}, \quad |x| \leq 1. \tag{42}$$

The use of (41), (42) gives the following expression for $K_2(x;\zeta)$:

$$K_2(x;\zeta) = \overline{n}_1\beta^2\beta_3^2\int\limits_{\zeta^{1/2}}^{\infty}\frac{\overline{n}_1\beta^2\beta_3^2(1-\beta_2^2\tau^2)^{1/2}d\tau}{\tau(a_1(\zeta)\tau^2+a_2(\zeta)+g_1(\tau;\zeta)+g_2(\tau;\zeta))}+$$

$$+\,O(\zeta^{1/2})|\ln\zeta|, \quad \zeta\to 0, \quad |x|\leq 1.$$

Next, without loss of generality, we assume that $\zeta \in (0,1]$. For $\tau \geq \zeta^{1/4}$, the estimate $|g_2(\tau;\zeta)| \leq c\zeta^{1/4}, \quad \tau > 0$, is true. Using this estimate and (41) we get

$$K_2^{(1)}(x;\zeta) \equiv \int\limits_{\zeta^{1/4}}^{\infty}\frac{(1-\beta_2^2\tau^2)^{1/2}d\tau}{\tau(a_1(\zeta)\tau^2+a_2(\zeta)+g_1(\tau;\zeta)+g_2(\tau;\zeta))} =$$

$$= \int\limits_{\zeta^{1/4}}^{\infty}\tau^{-1}Q(\tau;\zeta)d\tau + O(\zeta^{1/4}|\ln\zeta|), \quad \zeta\to 0, \quad |x|\leq 1,$$

where the function

$$Q(\tau;\zeta) = \frac{(1-\beta_2^2\tau^2)^{1/2}}{a_1(\zeta)\tau^2 + a_2(\zeta) + g_1(\tau;\zeta)} = O(\tau^{-1}), \quad \tau \to \infty \text{ (uniform for } \zeta > 0).$$

Integration by parts gives the equation

$$\int\limits_{\zeta^{1/4}}^{\infty} \tau^{-1} Q(\tau;\zeta) d\tau = -Q(\zeta^{1/4};\zeta)\ln\zeta^{1/4} + a_3(\zeta), \quad a_3(\zeta) = -\int\limits_0^{\infty} Q'(\tau)\ln\tau d\tau,$$

and the uniform, with respect to $\zeta > 0$, estimates hold,

$$Q'(\tau;\zeta) = O(1), \quad \tau \to 0, \quad Q'(\tau;\zeta) = O(\tau^{-2}), \quad \tau \to \infty.$$

For $\zeta \to 0$ we have $Q(\zeta^{1/4};\zeta) = 1/a_4(\zeta) + O(\zeta^{1/2})$, with the coefficient

$$a_4(\zeta) = a_2(\zeta) + g_1(0,\zeta) = 4\overline{n}_2\overline{m}_1\beta^2\beta_2^2 - 4\overline{n}_1\overline{m}_2\beta^2\beta_2^2 + 2\overline{n}_1\beta^2\beta_2^2 + O(\zeta), \quad \zeta \to 0.$$

So,

$$K_2^{(1)}(x;\zeta) = -\frac{\ln\zeta}{4a_4(\zeta)} + a_3(\zeta) + O(\zeta^{1/4}|\ln\zeta|), \quad \zeta \to 0, \quad |x| \le 1.$$

Let us examine the integral

$$K_2^{(2)}(x;\zeta) \equiv \int\limits_{\zeta^{1/2}}^{\zeta^{1/4}} \frac{(1-\beta_2^2\tau^2)^{1/2}d\tau}{\tau(a_1(\zeta)\tau^2 + a_2(\zeta) + g_1(\tau;\zeta) + g_2(\tau;\zeta))}.$$

Here, using the estimate (41) and the estimate

$$|a_{12}(\zeta)\tau^2| + |g_1(\tau,\zeta) - g_1(0,\zeta)| + |g_2(\tau,\zeta) - g_{2,0}(0,\zeta)| \le c_0\zeta^{1/2}, \quad \tau \in (\zeta^{1/2}, \zeta^{1/4}),$$

where $g_{2,0}(\tau;\zeta) = 4\overline{n}_2\beta_2^2\zeta\tau^{-2}(1-\beta^2\tau^2/\zeta)^{1/2}$, we obtain, for $\zeta \to 0$,

$$K_2^{(2)}(x;\zeta) = \int\limits_{\zeta^{1/2}}^{\zeta^{1/4}} \frac{d\tau}{\tau(a_4(\zeta) + g_{2,0}(\tau;\zeta))} + O(\zeta^{1/2}) =$$

$$= \int\limits_1^{\zeta^{-1/2}} \frac{ds}{2(a_4(\zeta)s + a_5(\zeta)(1-\beta^2 s)^{1/2})} + O(\zeta^{1/2}),$$

$$a_5(\zeta) = 4\overline{n}_2\beta_2^2.$$

The foregoing considerations give the following asymptotic formula:

$$K_2(x;\zeta) = R_{0,2}(\zeta)\ln\zeta^{1/2} + R_{1,2} + O(\zeta^{1/4}\ln|\zeta|), \quad R_{0,2}(\zeta) = -a_4(\zeta), \quad \zeta \to 0,$$

with a certain constant $R_{1,2}$. Then, taking into account (39), we get the estimate (40) for $K_2(x;\zeta)$ in the space of continuous functions. Repeating the considerations for the integral $K_2(x;\zeta)$ and replacing the expression $\cos(x\zeta\tau^{-1})$ with $\zeta\tau^{-1}\sin(x\zeta\tau^{-1})$ we get the estimate (40) for the derivative of the function $K_2(x;\zeta)$.

So, in the case of non-zero dissipation coefficient B we obtain the following proposition.

Proposition 2. *If $B \neq 0$, then for $K_2(x;\zeta)$ the following asymptotic representation holds:*

$$K(x;\zeta) = -(R_{0,1}(\zeta) + R_1(\zeta)\zeta)\ln|\zeta^{1/2}x| + R_{0,2}(\zeta)\ln\zeta^{1/2} + K_1 + \hat{K}_1(x;\zeta),$$
$$\|\hat{K}_1(x;\zeta)\|_{C^1} \leq c\zeta^{1/4}(|\ln\zeta| + 1), \quad \zeta > 0. \tag{43}$$

Note, that the use of (39) gives the equation

$$R_0(\zeta) = R_{0,1}(\zeta) - R_{0,2}(\zeta)/2 = \frac{\overline{n}_1\beta_3^2}{2}\left\{\frac{1}{2\beta_2^2 d_1} + \frac{\beta^2}{a_4(\zeta)}\right\}. \tag{44}$$

Thus, according to Proposition 2 and Section 5, the following statement is true.

Theorem 3. *Let $B \neq 0$ and $q \in (1,2)$. Then for sufficiently small $\zeta > 0$ the equation (21), for any function $f(x) \in W_q^1$, posseses unique solution $p(x) \in L_p$ and*

$$p(x) = (\mathcal{L}_0^{-1}(\zeta)f)(x) + O(\zeta^{1/4}|\ln\zeta|), \quad \zeta \to 0, \tag{45}$$

with respect to the norm of the space L_q. In the case $f(x) = W = \mathrm{const}$, in the spaces L_q and $C(r)$, the following asymptotic representation is true:

$$p(x) = -\frac{W}{R_0(0)\ln\zeta}\frac{1}{\sqrt{1-x^2}} + O(1/\ln^2\zeta), \quad \zeta \to 0. \tag{46}$$

Proof. According to (43) the equation (21) can be written in the form $(\mathcal{L}_0(\zeta^{1/2}) + \mathcal{L}_1(\zeta))p = f$, where in the definition (24) of the operator $\mathcal{L}_0(\zeta)$, the constants $a_j = a_j(\zeta)$ can be taken to be the coefficients

$$a_1 = -(R_{0,1}(0) + R_1(0)\zeta), \quad a_2 = R_{0,2}(0)\ln\zeta + K_1.$$

We also have the following estimate:

$$\|\mathcal{L}_1(\zeta))\|_{\mathcal{R}(L_q, W_q^1)} \leq c\zeta^{1/4}(|\ln\zeta| + 1), \quad \zeta > 0.$$

Then, using Corollary 1 (see Remark) we obtain, for the solutions of the equation (21) with $f(x) \in W_q^1$ and sufficiently small $\zeta > 0$ the asymptotic representations (45) and (46). The theorem is proved.

Thus, for $\zeta \to 0$, no matter what the value of the dissipation coefficient B is, for a given plate displacement law w, the contact stress behaves similar to the contact stress of the corresponding elasticity problem (see [14], § $2-4$), namely it has a root singularity near the stamp edges and the amplitude tending to zero as the inverse logarithm. In the case of $B = 0$, this result, on a physical level of rigor, was obtained in [15].

Let us consider the integral equation (21) corresponding to the original statement of the problem of stamp vibrations, i.e., when the displacement of the plate w satisfies the equation of motion (18). Set the constants

$$P_1 = \frac{\pi}{M_0(c_2/\bar{a})^2}, \qquad P_0 = \frac{\pi P_1}{\bar{a}} P.$$

Then the corresponding integral equation assumes the form

$$\frac{1}{\pi}\int_{-1}^{1} K(s-x;\zeta)p(s)ds - \frac{P_1}{\pi\zeta^2}\int_{-1}^{1} p(s)ds = f(x;\zeta),$$
$$f(x;\zeta) = \frac{P_0}{\zeta^2}. \tag{47}$$

The equation (47) differs from the above integral equation (21) by the presence of a one-dimension perturbation with the coefficient increasing as ζ^{-2} for $\zeta \to 0$ and a similar behavior of the right side $f(x) = f(x;\zeta)$ for $\zeta \to 0$. Thus, one can use the foregoing asymptotic analysis of the kernel $K(s-x;\zeta)$. Then the equation (47) can be written as $(\mathcal{L}_0(\zeta)+\mathcal{L}_1(\zeta))p(x) = f(x;\zeta)$, with the constants $a_1 = -R_0, \quad a_2 = K_0 - P_1\zeta^{-2}$. So, for small values of $\zeta > 0$ and $B = 0$, this equation has a unique solution and, in the space $C(r)$, we have the asymptotic representation

$$p(x) = -\frac{P_0}{(R_0 \ln\zeta/2 - K_0 + P_1\zeta^{-2})\zeta^2}\frac{1}{\sqrt{1-x^2}} + O(\zeta^2|\ln\zeta|) =$$

$$= -\frac{P}{\pi\bar{a}}\frac{1}{\sqrt{1-x^2}} + O(\zeta^2|\ln\zeta|), \quad \zeta \to 0.$$

It $B \neq 0$, a similar asymptotic relation holds for the solution of the equation (47),

$$p(x) = -\frac{P}{\pi\bar{a}}\frac{1}{\sqrt{1-x^2}} + O(\zeta^{1/4}|\ln\zeta|), \quad \zeta \to 0.$$

Thus, in the case when the plate displacement w satisfies the equation of motion (18), the contact stress $p(x) = p(x;\zeta)$, for $\zeta \to 0$, no matter what the value of the dissipation coefficient B is, approaches the limit value

$$\lim_{\zeta\to 0} p(x) = -\frac{P}{\pi\bar{a}}\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1. \tag{48}$$

Note that the limit value (48) coincides with the static distribution of the contact stress obtained by solving a 2D problem of pressure of a rigid plate on an elastic half-space [16, 17].

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