Finite temperature effects in the Fermi liquid theory of the diffusion of ^{4}He in ^{3}He

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The diffusion coefficient D and the thermal diffusion ratio k_T, for dilute ^{4}He in liquid ^{3}He, are calculated from Fermi liquid theory. The collision integral assumes a scattering amplitude a^{34} expanded in scalar combinations of the quasiparticle momenta. As $T \to 0$, D varies as $1/T$ and k_T/c, where c is the concentration, approaches a constant. As shown previously, the limits for DT and k_T/c are determined by thermodynamic properties, the ^{4}He effective mass and partial volume, and properties of pure ^{3}He. We have decreased k_T/c by a few percent, by including the effect of ∇T on the ^{3}He distribution function. The temperature dependence of DT and k_T/c is linear and related to the coefficients in the expansion of a^{34}. Two coefficients can be found from thermodynamics. A conjecture about the remainder suggests that DT may have a maximum between 0 and 0.5 K.

1. Introduction

In this paper we calculate the diffusion coefficient D and the thermal diffusion ratio k_T for very dilute solutions of ^{4}He in normal (non-superfluid) liquid ^{3}He. The quantities D and k_T are defined [1] by the equation for the impurity mass current i in terms of the gradients of the ^{4}He mass concentration, $c = N_4 m_4/(N_4 m_4 + N_3 m_3)$, and the temperature T:

$$i = -\rho\, D\, [\nabla c + (k_T/T)\nabla T] . \qquad (1.1)$$

When the total mass current ρv is zero, i is simply the ^{4}He mass current; otherwise [1] the ^{4}He current is $\rho c v + i$ and the ^{3}He current is $(1-c)\rho v - i$. When there are no currents, $(\nabla c)/c = = (-k_T/c)(\nabla T)/T$.

The low temperature behavior of D and k_T is determined by Fermi liquid theory from the Boltzmann transport equation of Zharkov and Silin [2]. In the dilute limit, the diffusion coefficient D is proportional [3] to $1/T$ and k_T/c tends to a constant number when $T \to 0$. As shown in Ref. 4, k_T/c and DT tend to values determined solely by the $l = 0$ forward scattering amplitude [5,6] a_0^{34} which can be obtained from thermodynamic measurements. The limit for DT is [4]

$$DT = D_0 \frac{\hbar}{m_4^*} \frac{m_3^*}{m_4^*} \left(\frac{v_3^*}{v_4^*}\right)^2 T_F ; \quad D_0 = 0.4461... \quad (1.2)$$

Here m_4^* and m_3^* are the ^{4}He and ^{3}He quasiparticle effective masses and T_F is the ^{3}He Fermi temperature, given by $k_B T_F = p_F^2/2m_3^*$. Equation (1.2) uses the relation between a_0^{34} and thermodynamic properties derived by Saam and Laheurte [5,6]:

$$a_0^{34} = (v_4^*/v_3)/v(0) . \qquad (1.3)$$

In Eqs. (1.2) and (1.3) v_4^* is the partial volume of a ^{4}He atom dissolved in liquid ^{3}He. The ^{3}He atomic volume is v_3 and $v(0) = 3/(2v_3 k_B T)$ is the ^{3}He density of states.

The limit for k_T/c was given as 0.3823 in Ref. 4, but we find a value a few percent smaller. We include the effect of the temperature gradient on the ^{3}He distribution function which was neglected in Ref. 4. The correction, which depends on pressure, can be calculated from m_4^* and v_4^* and known properties of pure liquid ^{3}He. The limit for DT is unchanged.

Equation (1.2) implies a ^{4}He collision time [2], $t = 3m_4 D/(2k_B T)$, that varies as $1/T^2$. For Fermi liquid theory to be valid, the energies of the ^{4}He quasiparticle states must be well defined; this implies $\hbar/t << k_B T$ which is equivalent to

$D \gg 2\hbar/(3m_4)$. This criterion is well satisfied provided $T \ll T_F$.

As remarked in Ref. 4, a measurement of k_T/c or a comparison between measurements of the limit for DT and Eq. (1.2) would provide a stringent test of the underlying Fermi liquid theory. In this respect, ^{4}He in ^{3}He is qualitatively different from other applications of the theory. For example, to relate the kinetic coefficients to the thermodynamic properties in pure ^{3}He or ^{3}He in liquid ^{4}He, some assumptions about the dependence of the ^{3}He–^{3}He scattering amplitude on the momentum transfer are necessary [7]. For ^{4}He in ^{3}He at low T, because the ^{4}He is dilute, only ^{4}He–^{3}He collisions need to be considered. Since the ^{4}He obeys Boltzmann statistics, the ^{4}He momentum and energy are small compared to the Fermi momentum and energy. In addition, the Pauli principle excludes large energy or momentum transfer from the ^{3}He. Therefore, as $T \to 0$, the ^{3}He is restricted to forward scattering, determined by a_0^{34}.

Although measurements [8,9] of D and k_T/c have been made above 0.5 K, experiments to test the theory must be made at temperatures where pure ^{3}He obeys Fermi liquid theory. This is below 0.1 K and above the ^{3}He superfluid transition. Here the solubility $c^{sat}(P, T)$ is very small [10], less than 500 ppm below 0.1 K. The thermodynamic quantities in the expression for D, v_4^* and m_4^*, can be obtained by analyzing measurements of $c^{sat}(P, T)$. Solubility data by Nakamura et al. [10] below 0.1 K and a preliminary measurement [11,12] of D indicate that such experimental tests are feasible.

In the present paper we find the solution of the ^{4}He Boltzmann equation with a momentum-dependent scattering amplitude. We prove the assertion made in Ref. 4, that the momentum dependence produces terms in DT and k_T/c which are of the order of T/T_F, so that only a_0^{34} appears in the limiting values of DT and k_T/c. Since the Fermi temperature T_F of liquid ^{3}He is approximately 1.77 K at zero pressure [13], terms of order T/T_F are expected to be small below 0.1 K, where Fermi liquid theory is valid. However, in Sec. 7, we show that the temperature dependence of DT could be quite measurable. Section 2 of the paper deals with the symmetry and parametrization of the scattering amplitude; Sections 3 to 6 with the solution to the Boltzmann equation. Sections 6 and 7 give the numerical results. Section 7 summarizes the conclusions.

As in Ref. 4, we assume that the ^{4}He quasiparticle spectrum has the conventional, particle-like form $E = -E_{43} + q^2/2m_4^*$, rather than the alternative, roton-like, «bubble spectrum» [2] $E = -E_{43} + (q - q_0)^2/2m_4^*$. Here E_{43} is the binding energy of one ^{4}He atom in liquid ^{3}He in the ground state. A microscopic calculation [14] shows that the conventional spectrum is correct, with $m_4^*/m_4 = 1.21$ at zero pressure. In Ref. 15, a fit to the $c^{sat}(P, T)$ data of Nakamura et al. [10] gave $m_4^*/m_4 = (1.1 + 0.4/-0.1)$.

2. Momentum dependent scattering amplitude

In general the scattering amplitude depends on $\mathbf{p}$ and $\mathbf{q}$, the initial momenta of the ^{3}He and ^{4}He quasiparticles, and the momentum transfer $\mathbf{k} = \mathbf{p}' - \mathbf{p} = \mathbf{q} - \mathbf{q}'$. It is also true that the inverse collision must have the same amplitude to within a phase factor [16]

$$|a^{34}(\mathbf{p}, \mathbf{q}, \mathbf{k})| = |a^{34}(\mathbf{p}', \mathbf{q}', -\mathbf{k})| . \quad (2.1)$$

This symmetry has a simpler form when a^{34} is written as a function of the mean momenta $\mathbf{q}_m = \mathbf{q} - \mathbf{k}/2$ and $\mathbf{p}_m = \mathbf{p} + \mathbf{k}/2$

$$|a^{34}(\mathbf{p}_m, \mathbf{q}_m, \mathbf{k})| = |a^{34}(\mathbf{p}_m, \mathbf{q}_m, -\mathbf{k})| . \quad (2.2)$$

We expand a^{34} in terms of the lowest order scalar functions of $\mathbf{q}_m$, $\mathbf{p}_m$ and $\mathbf{k}$, using the symmetry in Eq. (2.2). Keeping terms up to and including the second power in the small momenta $\mathbf{q}_m$ and $\mathbf{k}$, the results are equivalent to expanding $|a^{34}|$:

$$|a^{34}| = a_0^{34}\left[1 + \alpha_1\left(\frac{p_m^2}{p_F^2} - 1\right) + \alpha_2\frac{\mathbf{p}_m \cdot \mathbf{q}_m}{p_F^2} + \alpha_3\frac{q_m^2}{p_F^2} + \right.$$
$$\left. + \alpha_4\frac{(\mathbf{p}_m \cdot \mathbf{q}_m)^2}{p_F^4} + \dots + \alpha_5\frac{k^2}{p_F^2} + \alpha_6\frac{(\mathbf{p}_m \cdot \mathbf{k})^2}{p_F^4} + \dots \right].$$
$$(2.3)$$

The real coefficients α_i, like a_0^{34}, depend on the pressure. We shall find that α_6 only produces terms of higher order than T/T_F so that it does not appear in our results for DT or k_T/c.

The forward scattering amplitude corresponds to a^{34} with zero momentum transfer $\mathbf{k}$. In this case, $\mathbf{p}_m = \mathbf{p}$ and $\mathbf{q}_m = \mathbf{q}$. Saam derived a relation [5] between the forward scattering amplitude and the thermodynamic Landau interaction function f^{34}. The forward scattering amplitude and the Landau function differ because the latter includes successive or multiple scattering events. Saam's equations, which are analogous to a similar set for the ^{3}He–^{3}He amplitude [7], link the two functions through the coefficients of their expansions in Legendre polynomials of $\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$,

$$\{a^{34}(\mathbf{k}=0)\}_l = \{f^{34}\}_l \, / [1 + F_l^s/(2l+1)] \ . \quad (2.4)$$

The F_l^s are the symmetric Fermi liquid factors [13] for pure liquid ^{3}He.

Using Galilean invariance [17] and invariance with respect to the reference system [5,6,15], some properties of the Landau function f^{34} have been directly related to thermodynamic quantities such as v_4^* and m_4^*. As a result one finds the following relations between a_0^{34} and the α_i and the corresponding known coefficients f_0^{34}, a_1' and a_1'' in a similar expansion for f^{34} in Ref. 15:

$$a_0^{34} = \frac{f_0^{34}}{(1+F_0^s)} = \frac{v_4^*}{v_3 \, v(0)} = \frac{2}{3} \, k_B \, T_F \, v_4^* \ ,$$

$$\alpha_2 = \frac{3a_1'}{(3+F_1^s)} \frac{p_F^2}{a_0^{34}} = 3 \, \frac{v_3}{v_4^*} \left(1 - \frac{m_4}{m_4^*}\right), \quad (2.5)$$

$$\alpha_3 = \frac{a_1''}{(1+F_0^s)} \frac{p_F^2}{a_0^{34}} = \frac{3}{2} \frac{v_3}{v_4^*} \frac{m_3^*}{m_4^*} \frac{1}{(1+F_0^s)} \frac{d \log m_4^*}{d \log v_3} \ .$$

The first relation is the same as Eq. (1.3). Since $(1+F_0^s) \approx 10.6$ in liquid ^{3}He at zero pressure [13], and $m_4 \approx m_4^*$ [14,15], both α_2 and α_3 are smaller than one. The rest of the α_i are undetermined. In Sec. 7 we find that the $\alpha_5 \, k^2$ term in a^{34} has the largest effect on the temperature dependence of DT and k_T/c.

3. The ^{4}He Boltzmann equation

In setting up and solving the Boltzmann equation, we follow the work of Zharkov and Silin [2], Leggett and ter Haar [3], Dandache et al. [18] and Geilikman and Chechetkin [19]. The notation is nearly the same as in Ref. 4. According to Zharkov and Silin, the linearized ^{4}He Boltzmann equation in the dilute limit (negligible ^{4}He–^{4}He scattering) for small temperature and concentration gradients and $\mathbf{v} = 0$ has the form

$$n_{i0}\left(\frac{\mathbf{q}}{m_4^*}\right) \cdot \left[\frac{\nabla c}{c} + \left(\frac{q^2}{2m_4^* k_B T} - \frac{3}{2}\right)\frac{\nabla T}{T}\right] = J_{if}(\mathbf{q}) \ . \quad (3.1)$$

Here n_{i0} is the equilibrium ^{4}He quasiparticle occupation number:

$$n_{i0} = \left(\frac{\rho c}{m_4}\right)\left(\frac{2\pi\hbar}{m_4^* k_B T}\right)^{3/2} \exp\left(-\frac{q^2}{2m_4^* k_B T}\right) \quad (3.2)$$

where $\rho c/m_4$ is the ^{4}He number density. The collision integral is [2,18]

$$J_{if}(\mathbf{q}) = (2\pi/\hbar) \times \int |a^{34}|^2 n_{f0}(1-n_{f0}')n_{i0}(\psi_i - \psi_i' + \varphi_f - \varphi_f') \, \delta(\varepsilon + E - \varepsilon' - E') \frac{2}{h^6} \, dp \, dq' . \quad (3.3)$$

The factor of 2 multiplying dp allows for the sum over the initial ^{3}He spin states. Refs. 18 and 19 have a factor of 4, which implies that the ^{3}He quasiparticle may change its spin orientation during a ^{4}He–^{3}He collision. As explained in Ref. 4, we think this is incorrect. Our equation agrees with the original formulation by Zharkov and Silin.

In the collision integral, $\mathbf{p}'$ is related to the other momenta by conservation of momentum:

$$\mathbf{p}' - \mathbf{p} = \mathbf{q} - \mathbf{q}' = \mathbf{k} \ . \quad (3.4)$$

The Fermi functions n_{f0} and n_{f0}' are the equilibrium occupation numbers for the initial and final ^{3}He states, that have energies ε and ε'. The initial and final ^{4}He energies are E and E', so that the delta function $\delta(\varepsilon + E - \varepsilon' - E')$ enforces conservation of energy.

In Eq. (3.3), the quantities $\psi_i = -\delta n_i / n_{i0}$ and $\psi_i' = -\delta n_i' / n_{i0}'$ are the negatives of the small frac-

tional deviations from the equilibrium ^{4}He distribution function. Because they are linear in the concentration and temperature gradients, ψ_i and ψ_i' depend on $\mathbf{q}$ and $\mathbf{q}'$ according to the equation [2]

$$\psi_i(\mathbf{q}) = a_c(q)\mathbf{q} \cdot \nabla c + a_T(q)\mathbf{q} \cdot \nabla T \ . \quad (3.5)$$

The functions $a_c(q)$ and $a_T(q)$ are found by solving the ^{4}He Boltzmann equation (3.1).

The deviations from equilibrium of the ^{3}He occupation number are related to φ_f and φ_f' in Eq. (3.3) by $\delta n_f = -\varphi_f \, k_B T \, \partial n_{f0}/\partial \varepsilon$. They are determined from the known solution [7] of the Boltzmann equation for a temperature gradient in pure ^{3}He. The effect of the ^{4}He on φ_f is negligible because, in the dilute limit, the number of ^{3}He at the Fermi surface and available for scattering is much larger than the number of ^{4}He. When $\nabla T = 0$, as in the calculation of the diffusion coefficient, φ_f and φ_f' are negligible. In previous works [2,4,18,19], φ_f and

φ_f' were incorrectly omitted in the calculation of k_T.

We divide the collision integral (3.3) into two contributions. The first, $J_i(\mathbf{q})$, is from the term proportional to $(\psi_i - \psi_i')$, the deviations from equilibrium in the ^{4}He distribution function. The second, $J_f(\mathbf{q})$, is from the term in $(\varphi_f - \varphi_f')$.

4. Calculation of the collision integral $J_i(\mathbf{q})$

We first evaluate $J_i(\mathbf{q})$. This is all that is needed to calculate the diffusion coefficient D. We put $\nabla T = 0$ and, using Eq. (3.5), $J_i(\mathbf{q})$ becomes

$$J_i(\mathbf{q}) = \mathbf{C} \cdot \int [\mathbf{q}a_c(q) - \mathbf{q}'a_c(q')]J(\mathbf{q}, \mathbf{q}') \, d\mathbf{q}' \quad (4.1)$$

where

$$J(\mathbf{q}, \mathbf{q}') = \int |a^{34}|^2 n_{f0}(1 - n_{f0}') \, \delta(\varepsilon + E - \varepsilon' - E') \, d\mathbf{p} \tag{4.2}$$

and

$$\mathbf{C} = (8\pi^2/h^7)n_{i0}\nabla c . \tag{4.3}$$

To calculate k_T/c, we set $\nabla c = 0$ and $J_i(\mathbf{q})$ has the same form as (4.1) except that a_c is replaced by a_T and the vector $\mathbf{C}$ has ∇T instead of ∇c.

In $J(\mathbf{q}, \mathbf{q}')$ the energy transfer $E - E' = (q^2 - q'^2)/2m_4^*$ and momentum transfer $\mathbf{k} = \mathbf{q} - \mathbf{q}'$ are fixed by $\mathbf{q}$ and $\mathbf{q}'$. Since $|a^{34}|$ in Eq. (2.3) is written in terms of $\mathbf{p}_m$, $\mathbf{q}_m$ and $\mathbf{k}$, we transform the integral in Eq. (4.2) by replacing $d\mathbf{p}$ by $d\mathbf{p}_m$. We choose an axis along $\mathbf{k}$ and write $d\mathbf{p}_m = p_m^2 \, dp_m \, d\cos\theta_m \, d\varphi_m$. The $\varphi_m = 0$ plane contains the vector $\mathbf{q}_m$.

The integration with respect to φ_m is elementary and performed first. Using

$$\varepsilon' - \varepsilon = k p_m \cos\theta_m /m_3^* \tag{4.4}$$

in the delta function, we integrate over $\cos\theta_m$:

$$J(\mathbf{q}, \mathbf{q}') = 2\pi m_3^*/k \times$$

$$\times \int \langle |a^{34}|^2 \rangle n_{f0}(s_m - \Delta/2)[1 - n_{f0}(s_m + \Delta/2)] \, p_m \, dp_m . \tag{4.5}$$

Here $\langle |a^{34}|^2 \rangle$ is $|a^{34}|^2$ averaged over all values of φ_m. The Fermi function $n_{f0}(s)$ is $(e^s + 1)^{-1}$, and $s_m = \varepsilon_m/(k_B T)$. The mean energy $\varepsilon_m = (p_m^2 + k^2/4 - p_F^2)/2m_3^*$ is measured from the Fermi energy. To order T, the Fermi energy is the same as the ^{3}He chemical potential. Note that $\Delta =$

$= (E - E')/(k_B T)$ does not depend on p_m. The integration over the delta function gives $\cos\theta_m$ a definite value that depends on p_m:

$$\cos\theta_m = m_3^*(E - E')/(p_m k) . \tag{4.6}$$

Thus the factors of $p_m \cos\theta_m$ in $\langle |a^{34}|^2 \rangle$ are simply functions of $\mathbf{q}$ and $\mathbf{q}'$.

The right-hand side of Eq. (4.6) is of the order of q/p or $(T/T_F)^{1/2}$. This means that, usually, the momentum transfer $\mathbf{k}$ is very nearly perpendicular to $\mathbf{p}$ and $\mathbf{p}_m$. From Eq. (4.6), the condition $-1 \le \cos\theta_m \le 1$ gives the lower limit for p_m in the integral (4.5):

$$p_m \ge |m_3^*(E - E')/k| = |(m_3^*/m_4^*)\mathbf{q}_m \cdot \hat{\mathbf{k}}| . \tag{4.7}$$

When $T \ll T_F$, so that q and k are small compared to p_F, replacing this lower limit by zero has an exponentially small effect on the integral, of order $\exp(-T_F/T)$. Therefore we evaluate $J(\mathbf{q}, \mathbf{q}')$ using the relation

$$\int_{-\infty}^{\infty} n_{f0}(s_m - \Delta/2)[1 - n_{f0}(s_m + \Delta/2)] \, ds_m = \Delta/(1 - e^{-\Delta}). \tag{4.8}$$

As in Ref. 4, the replacement of the lower limit (4.7) by zero is the crucial approximation in finding the collision integral. Since the integrand in Eq. (4.5) is always positive, by using $p_m \ge 0$ (or $s_m \ge -\infty$) instead of Eq. (4.7), we have overestimated the effect of collisions with large k. Because q is small at low T (due to the Boltzmann distribution), large k corresponds to large negative values of Δ, and therefore q' must be large if k is large. The collision rate, which is proportional to (4.8), is very small when $\Delta \ll -1$, so the approximation is self-consistent.

The final result for $J(\mathbf{q}, \mathbf{q}')$, to second order in q and q', is

$$J(\mathbf{q}, \mathbf{q}') = 2\pi (m_3^* a_0^{34})^2 k_B T \frac{\Delta}{k(1 - e^{-\Delta})} \times$$

$$\times \left[1 + \gamma_1 \frac{(q - q')^2}{p_F^2} + \gamma_2 \frac{(q + q')^2}{p_F^2} + \gamma_3 \frac{(q^2 - q'^2)^2}{(q - q')^2 p_F^2} \right] \tag{4.9}$$

where

$$\gamma_1 = 2\alpha_5 - \alpha_1/2 ,$$

$$\gamma_2 = \frac{\alpha_3}{2} + \frac{\alpha_4}{4} + \frac{\alpha_2^2}{8} \ , \quad \gamma_3 = \left(\frac{m_3^*}{m_4^*}\right)\frac{\alpha_2}{2} - \frac{\alpha_4}{4} - \frac{\alpha_2^2}{8} \ .$$

$$(4.10)$$

As noted in Sec. 2, α_6 does not appear in Eq. (4.10); it does not affect DT or k_T/c to order T/T_F.

From symmetry, the vector integral in Eq. (4.1) is parallel to **q**. Therefore we choose a new axis along **q**, so that the collision integral (4.1) becomes

$$J_i(\mathbf{q}) = \mathbf{C} \cdot \hat{\mathbf{q}} \times$$

$$\times \int [qa_c(q) - q'\cos\theta' a_c(q')]J(\mathbf{q}, \mathbf{q}')q'^2 \, dq' \, d\cos\theta' \, d\varphi'$$

$$(4.11)$$

where θ' is the angle between **q** and **q**'.

After integrating over φ', and then $\cos\theta'$, and introducing the dimensionless variables $x \equiv q^2/(2m_4^* k_B T)$, $y \equiv q'^2/(2m_4^* k_B T)$, the result is

$$J_i(\mathbf{q}) = n_{i0}(\mathbf{q}/m_4^*)(\nabla c)/c \int_0^\infty F(x, y) \, dy \qquad (4.12)$$

where

$$F(x, y) = F_0(x, y) + (m_4^*/m_3^*)(T/T_F)\sum_{i=1}^{3} \gamma_i \, F_i(x, y),$$

$$(4.13)$$

$$F_i(x, y) \equiv$$

$$\equiv (x-y)/(1- e^{y-x})[g_i(x, y) \, f(x) - h_i(x, y) \, f(y)] \ ;$$

$$i = 0, \ldots, 3 \ . \qquad (4.14)$$

The function $f(x)$ is a dimensionless form of $a_c(q) = f(x)/cC_1$ where the constant C_1 is

$$C_1 = (m_3^* \, m_4^* \, k_B \, T \, a_0^{34})^2/(2\pi^3\hbar^7) \ . \quad (4.15)$$

The quantities $g_i(x, y)$ and $h_i(x, y)$ are the simple algebraic functions of x and y shown in Table I.

5. Solution of the Boltzmann equation

After substituting (4.12) in (3.1) one arrives at a dimensionless form of the Boltzmann equation for the situation where $\nabla T = 0$,

$$\int_0^\infty F(x, y) \, dy = 1 \ . \qquad (5.1)$$

Equation (5.1) is a one-dimensional integral equation for $f(x) = cC_1 a_c(q)$.

In the same way the integral equation for the situation where $\nabla c = 0$ becomes

$$\int_0^\infty F(x, y) \, dy = x - 3/2 - b_3(x) \ . \qquad (5.2)$$

In this case, $f(x) = TC_1 a_T(q)$. The «driving» terms, $x - 3/2$ and 1 on the right-hand sides of Eqs. (5.2) and (5.1), come from the left-hand side of the Boltzmann equation. The additional driving term $b_3(x)$ in Eq. (5.2) is the dimensionless form of the collision integral $J_f(\mathbf{q})$. The calculation of $J_f(\mathbf{q})$ and the derivation of $b_3(x)$ are described in Sec. 6.

The numerical solution of the integral Eqs. (5.1) and (5.2) at $T = 0$, neglecting the term in $b_3(x)$, is given in Ref. 4. At $T = 0$, substitution from Eqs. (4.13) and (4.14) in (5.1) gives

Table 1

The functions $g_i(x, y)$ and $h_i(x, y)$ in Eq. (4.14), and the integral $\int_x^\infty (x - y)[1 - e^{y-x}]g_i(x, y) \, dy$ for $P_i(x)$ in Eq. (5.4). The Riemann zeta function $\zeta(3) = 1.2020...$

i	$y \leq x$		$y \geq x$		
	$(x/y)^{1/2}g_i(x, y)$	$(x/y)^{1/2}h_i(x, y)$	$g_i(x, y)$	$h_i(x, y)$	$\int_x^\infty (x-y)[1-e^{y-x}]g_i(x, y) \, dy$
0	1	$y/3x$	1	$1/3$	$\pi^2/6$
1	$x + y/3$	$-y/3 + y^2/15x$	$y + x/3$	$-y/3 + x/15$	$2\zeta(3) + 2\pi^2 x/9$
2	$x + 5y/3$	$y + 3y^2/5x$	$y + 5x/3$	$y + 3x/5$	$2\zeta(3) + 4\pi^2 x/9$
3	$x - y$	$y - y^2/x$	$y - x$	$y - x$	$2\zeta(3)$

$$f(x)P_0(x) - \int_0^\infty M_0(x, y)f(y)\, dy = 1 \quad (5.3)$$

·where we have defined the functions

$$P_i(x) \equiv \int_0^\infty \frac{x - y}{1 - e^{y-x}} g_i(x, y)\, dy ,$$
$$i = 0, \ldots, 3 \quad (5.4)$$
$$M_i(x, y) \equiv \frac{x - y}{1 - e^{y-x}} h_i(x, y) .$$

In Ref. 4, Eq. (5.3) and the similar one for $\nabla c = 0$ were solved in terms of the variables t and t' defined by $t = \exp(-x/g)$, $t' = \exp(-y/g)$, where $g \sim 3$ is a dimensionless scale factor. The equations were discretized over a one-dimensional lattice of N points evenly spaced in t. As a result, f and P_0 became vectors and M_0 a square matrix. The integral for P_0 from $y = 0$ to x was evaluated numerically. The other part, from $y = x$ to infinity, is listed in Table 1, with the companion formulas for P_1, P_2 and P_3. Rather than inverting M_0, the equations were more efficiently solved by iteration. An approximate form was used for $f(y)$, giving a new estimate for $f(x)$ and so on. Accurate solutions were obtained with $N = 100$ or 200, iterating up to 15 times.

At all temperatures, the impurity current i, and thus the diffusion coefficient D and thermal diffusion ratio k_T/c, are found from the appropriate integration over the ^{4}He occupation number. As explained in Ref. 4, we calculate i from the ^{4}He particle current:

$$i = m_4 \int \delta n_i \, (\mathbf{q}/m_4^*) \, d\mathbf{q}/h^3 . \quad (5.5)$$

In this equation, $\mathbf{q}/m_4^* = \nabla_q E$ is the ^{4}He quasiparticle velocity. Previous authors [2,18] have used $i = \int \delta n_i \, \mathbf{q}\, d\mathbf{q}/h^3$, the momentum density associated with the ^{4}He. The two formulas differ by a factor of m_4/m_4^*. Using Eqs. (5.5) and (3.2), the expression for D_0 in Eq. (1.2) in terms of the function $f(x)$ for $\nabla T = 0$ is

$$D_0 = 16/(3\pi^{3/2}) \int_0^\infty f(x)x^{3/2} e^{-x}\, dx . \quad (5.6)$$

If the $f(x)$ for $\nabla c = 0$ is called $f_T(x)$, then k_T/c is given by

$$k_T/c = \int_0^\infty f_T(x)x^{3/2} e^{-x}\, dx \Big/ \int_0^\infty f(x)x^{3/2} e^{-x}\, dx . \quad (5.7)$$

To obtain D_0 and k_T/c at finite temperature, we use the fact that $(m_4^*/m_3^*)(T/T_F)$ in Eq. (4.13) is a small quantity. Therefore the solutions to the integral Eqs. (5.1) and (5.2) may be expanded in the form

$$f(x) = f^0(x) + (m_4^*/m_3^*)(T/T_F)f^1(x) + \ldots \quad (5.8)$$

Here, $f^0(x)$ is the solution of the integral equation at $T = 0$. The term in f^1 gives the finite temperature correction to f^0 to order T/T_F. As a result of (5.8) we may write the F_i in (4.14) as

$$F_i(x, y) = F_i^0(x, y) + (m_4^*/m_3^*)(T/T_F)F_i^1(x, y) + \ldots \quad (5.9)$$

where $F_i^0(x, y)$ means $F_i^0(x, y) = (x-y)/(1- e^{y-x})\times \times[g_i(x, y)f^0(x) - h_i(x, y)f^0(y)]$ etc. Substituting in (5.1) and retaining terms up to T/T_F gives, for $\nabla T = 0$.

$$\int_0^\infty \Big\{ F_0^0(x, y) + (m_4^*/m_3^*)(T/T_F)[F_0^1(x, y) + \\ + \sum_{i=1}^3 \gamma_i \ F_i^0(x, y)] \Big\} dy = 1 . \quad (5.10)$$

The integral equation for the situation where $\nabla c = 0$ is the same as (5.10), except that the driving function on the right-hand side is $x - 3/2 - b_3(x)$.

When the solution for $T = 0$ is subtracted from Eq. (5.10), the result is

$$\int_0^\infty \Big[F_0^1(x, y) + \sum_{i=1}^3 \gamma_i \ F_i^0(x, y) \Big] dy = 0 . \quad (5.11)$$

From this equation we see that $f^1(x)$ is the sum of three contributions each linearly proportional to one of the γ_i :

$$f^1(x) = \sum_{i=1}^3 \gamma_i f_i^1(x) . \quad (5.12)$$

Using the functions defined in Eq. (5.4), the $f_i^1(x)$ are the solutions of the three integral equations

$$f_i^1(x)P_0(x) - \int_0^\infty M_0(x, y)f_i^1(y) \, dy =$$

$$= -f^0(x)P_i(x) + \int_0^\infty M_i(x, y)f^0(y) \, dy \; ; \; i = 1, 2, 3 \; .$$

$$(5.13)$$

These equations differ from the $T = 0$ Eq. (5.3) only in the driving term on the right-hand side. This is obtained from the $T = 0$ solution. We have solved (5.13) numerically by the method used for $T = 0$ in Ref. 4. The results are given in Sec. 7.

6. Calculation of the collision integral $J_f(\mathbf{q})$

In this Section we evaluate $J_f(\mathbf{q})$, the part of the collision integral produced by the temperature gradient in the solvent ^{3}He and equal to the term proportional to $(\varphi - \varphi')$ in Eq. (3.3). $J_f(\mathbf{q})$ is needed to calculate the driving term $-b_3(x)$ in the right-hand side of the ^{4}He Boltzmann equation (5.2):

$$J_f(\mathbf{q}) = (2\pi/\hbar) \times$$

$$\times \int |a^{34}|^2 n_{f0}(1 - n'_{f0}) n_{i0}(\varphi_f - \varphi'_f) \delta(\varepsilon + E - \varepsilon' - E') \frac{2}{\hbar^6} \, d\mathbf{p} \, d\mathbf{q}' .$$

$$(6.1)$$

Using the notation of the review by Baym and Pethick [7], $\varphi_f = -\tau \Psi(s)(\mathbf{p} \cdot \nabla T)/(m_3^* T)$, where s is the reduced energy $\varepsilon/k_B T$ and ε is measured from the ^{3}He chemical potential. The quantity τ is a characteristic relaxation time, defined more precisely below. The dimensionless function $\Psi(s)$ is odd [7] in s, $\Psi(s) = -\Psi(-s)$. Although it may be calculated exactly from an infinite series [7,20], it is simpler to use the approximate expression due to Emery and Cheng [7,21]:

$$\Psi(s) = \frac{2s}{\pi^2 + s^2} + \frac{15s\lambda_K}{2\pi^2(3 - \lambda_K)} \; . \qquad (6.2)$$

This formula is accurate enough [21] to give the ^{3}He thermal condctivity to about 1%. The pressure dependent number λ_K in Eq. (6.2) is a measure of the angular dependence of the ^{3}He–^{3}He scattering amplitude [7]. For s-wave scattering, λ_K is unity:

$$\lambda_K = \langle W \rangle^{-1} \times$$

$$\times \int d \cos \theta \, d\varphi \, W(\theta, \varphi)(1 + 2 \cos \theta)/\cos (\theta/2) \; ,$$

$$(6.3)$$

$$\langle W \rangle = \int d \cos \theta \, d\varphi \, W(\theta, \varphi)/\cos (\theta/2) \; .$$

Here $W(\theta, \varphi)$ is the ^{3}He–^{3}He scattering probability at the Fermi surface averaged over spin [7]. The angle θ is between the initial quasiparticle momenta and, thanks to conservation of momentum, between the final momenta as well. The collision rotates the plane containing the momenta by φ. The characteristic relaxation time τ is determined by the mean scattering probability $\langle W \rangle$:

$$\tau = \frac{8\pi^4 \hbar^6}{m_3^{*3} \langle W \rangle (k_B T)^2} \; . \qquad (6.4)$$

The s- and p-wave approximation [22] gives λ_K in terms of the Landau parameters. The value of τ can also be found [22] from this model, however, the exact Eqs. (1.2.113a) and (1.2.113c) in Baym and Pethick [7] relate τ to the thermal conductivity and λ_K. Figure 1 shows λ_K and the product τT^2 in the s- and p-wave approximation, as well as τT^2 calculated from Greywall's thermal conductivity data [23]. The τT^2 from the s- and p-wave approximation agrees quite well the exact formula. The Landau parameters, m_3^*, v_3 and the specific heat needed in these calculations were taken from the review by Halperin and Varoquaux [13].

In integrating (6.1) we neglect the momentum dependence of the ^{4}He–^{3}He scattering amplitude $|a^{34}|$ and replace it with a_0^{34}. This simplification is justified because the effect of $b_3(x)$ on k_T/c turns out to be small, a decrease of about 0.01 or 2.5% of k_T/c. The momentum dependence of a^{34} would
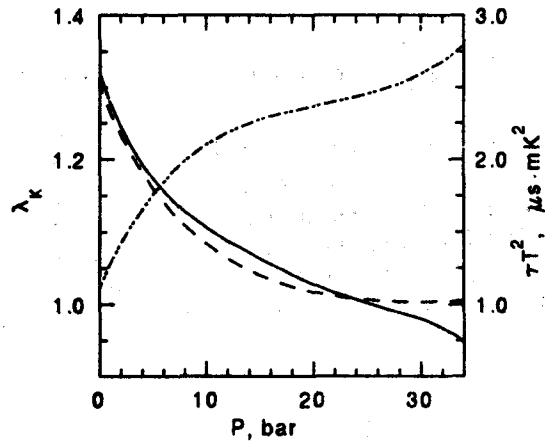

Fig. 1. Plot of the dimensionless factor λ_K and the product τT^2, where τ is the characteristic ^{3}He–^{3}He relaxation time, versus pressure in pure ^{3}He. The dash-dot-dot curve is λ_K, the solid curve is τT^2 calculated from Greywall's thermal conductivity data and the dashed curve is from the s- and p-wave approximation.
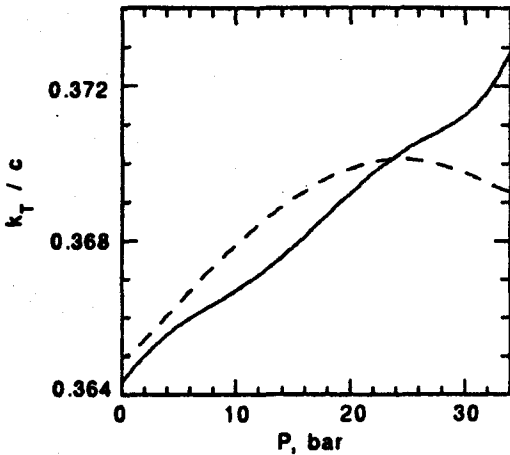
Fig. 2. Plot of the predicted thermal diffusion factor for ^{4}He in liquid ^{3}He, k_T/c, versus pressure as $T \to 0$. The solid curve is calculated using the ^{3}He–^{3}He relaxation time τ from the thermal conductivity data of Greywall, while the less accurate dashed curve uses τ from the s- and p-wave approximation.

produce an effect of the order of T/T_F of this, which is presumably negligible.

To integrate $J_f(q)$, we replace dp by dp_m and choose an axis along **k** with ∇T in the xz plane. The integral over φ_m is done first and then the integral over $\cos \theta_m$. The delta function causes $\cos \theta_m$ to be replaced by $m_3^*(E - E')/(p_m k)$, as in Eq. (4.6). Using the same arguments as in Sec. 4, the lower limit for $s_m = \varepsilon_m/k_B T$ is replaced by $-\infty$. The result is

$$J_f(q) = (8\pi^2/\hbar h^6)|a_0^{34}|^2 n_{i0} \tau m_3^* \times$$

$$\times \int (\nabla k_B T) \cdot \mathbf{k} n_{f0}(1 - n'_{f0})(\Psi' - \Psi)(E - E')/k^3 \, ds_m \, dq'.$$

$$(6.5)$$

Defining a new axis along **q**, and replacing dq' by $q'^2 \, dq' \, d\cos \theta' \, d\varphi'$, one can integrate over φ' and $\cos \theta'$ analytically. We write the final result in terms of $-b_3(x)$) on the right-hand side of Eq. (5.2):

$$b_3(x) = c_3(P) \times$$

$$\times \int_0^\infty \int_{\Delta=0}^x n_{f0}(1 - n'_{f0})(\Psi' - \Psi)(1 - \Delta/x)^{1/2}(\Delta/x) \, d\Delta \, ds_m$$

$$(6.6)$$

where

$$c_3(P) \equiv \frac{4\pi}{\hbar} \frac{m_4^*}{m_3^*} \frac{|a_0^{34}|^2}{\langle W \rangle}.$$

$$(6.7)$$

The ^{4}He–^{3}He scattering amplitude is weak compared to the ^{3}He–^{3}He amplitude. Consequently, $c_3(P)$ is much smaller than one. At zero pressure, $c_3(P) \approx 0.03$.

To calculate $b_3(x)$, we integrated (6.6) numerically, using the Emery–Cheng $\Psi(s)$. The Emery–Cheng formula (6.2) is the sum of two terms; the first which does not depend on λ_K, and the second which is linear in $\lambda_K/(3 - \lambda_K)$. Since $f(x)$ in the integral equation (5.2) is linear in the driving terms, the effect of each term was evaluated separately. The results give the correction to the $T = 0$ value of k_T/c as

$$(k_T/c)_{T=0} =$$

$$= 0.3823 - c_3(P)[0.1045 + 0.391\lambda_K/(3 - \lambda_K)] .$$

$$(6.8)$$

The result of using Eq. (6.7) to calculate k_T/c at $T = 0$ as a function of pressure is shown in Fig. 2. The function $c_3(P)$ is uncertain because it contains m_4^* and v_4^*, which are not accurately known, especially at high pressures. Based on an analysis [15] of the data of Nakamura et al., we assumed in Fig. 2 that $m_4^* \approx m_4$. Laheurte's results [24] for v_4^* at $T = 0$ were extrapolated above 15 atm.

7. Results and conclusions

The finite temperature correction to DT was calculated by solving Eq. (5.13) for $f_1^1(x)$, $f_2^1(x)$, and $f_3^1(x)$. The results are shown in Fig. 3,*b* as a function of the reduced ^{4}He momentum $x^{1/2}$. The physically important parts of these functions are in the region where the Maxwell distribution xe^{-x}, shown in Fig. 3,*a*, is large. From the $f_i^1(x)$ and Eqs. (5.6), (5.8), and (5.12), the diffusion coefficient in the Fermi liquid region below 0.1 K, is given by

$$DT = (\hbar/m_4^*)T_F(v_3/v_4^*)^2\{0.4461 \, (m_3^*/m_4^*) -$$

$$- (T/T_F)[2.127 \, \gamma_1 + 1.479 \, \gamma_2 + 0.337 \, \gamma_3]\} \quad (7.1)$$

where the γ_i are defined in terms of the α_i in Eq. (4.10). As remarked in Sec. 2, α_2 and α_3 are smaller than 1. The α most likely to be important is α_5, the coefficient of k^2 in $|a^{34}|$. This also has by far the largest coefficient in (7.1): it appears as $2\alpha_5$ in γ_1.

If $|a^{34}|$ is regarded as the Fourier transform of a distance-dependent potential $v(r)$, a plausible length scale in $v(r)$ would be $\sim \hbar/p_F$. This implies an α_5 of ~ 1. Putting $\alpha_5 = 1$ in Eq. (7.1) gives a
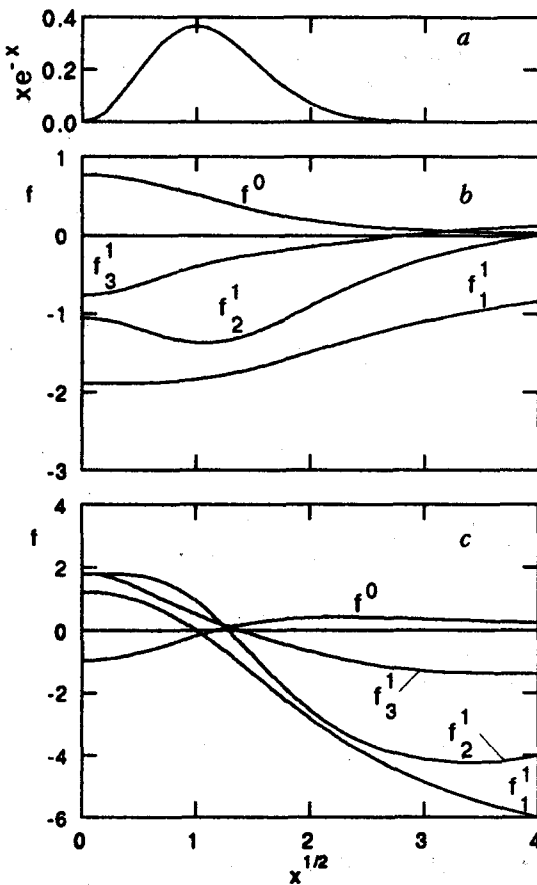
Fig. 3. a) The equilibrium Maxwell-Boltzmann distribution, xe^{-x}, plotted against the reduced ^{4}He quasiparticle momentum, $x^{1/2} = q/(2m_4^* k_B T)^{1/2}$. *b)* The solutions to the integral equations (5.3) and (5.13). The function $f^0(x)$ is the $T = 0$ solution to the reduced ^{4}He Boltzmann equation (5.3) for an isothermal concentration gradient. The functions $f_1^1(x)$, $f_2^1(x)$, and $f_3^1(x)$ give the corrections to $f^0(x)$ at finite temperature. *c)* The same as *b*, but for a temperature gradient at constant concentration.

decrease in DT of about 15% between 0.05 and 0.1 K. Such a small variation would be quite difficult to measure. On the other hand, if the length scale were as large as $2\pi\hbar/p_F$, α_5 would be $4\pi^2$. The T dependence of DT below 0.1 K would then be so large as to require careful measurement in order to extrapolate DT to $T = 0$. Moreover, terms higher than T/T_F would probably be appreciable at 0.1 K.

We note that a large positive value of α_5 is plausible because it would represent a^{34} increasing at large momentum transfer k towards the much larger ^{3}He–^{3}He amplitude. With a positive α_5, DT decreases with temperature towards the measurements of DT by Vvedenskii and Peshkov [8] at 0.5 K. These are ~ 5 times smaller than the $T = 0$ Fermi liquid predictions [4].

On the other hand, we may compare a^{34} to the ^{3}He–^{3}He interaction in dilute solutions of ^{3}He in superfluid ^{4}He. The original «BBP» interaction [25] has the form $V(k) = -V_0 \cos (k\delta/\hbar)$ with the length $\delta \approx 3.16$ Å. The quantity V_0 is similar in magnitude to a_0^{34}, although the interactions are opposite in sign; $V_0 \approx 60$ K·cm^3/mol as compared to $a_0^{34} \approx 30$ K·cm^3/mol, both at zero pressure. If a^{34} had the same dependence on k as $V(k)$, expanding the cosine would give $\alpha_5 = -(p_F \delta/\hbar)^2/2 \approx -3.1$. If α_5 were this negative, the T/T_F term would increase DT by 30% between 0.05 and 0.1 K. Since the measurements of DT by Vvedenskii and Peshkov at 0.5 K are smaller than the $T = 0$ Fermi liquid predictions, there would a maximum in DT between 0 and 0.5 K. This effect would be quite measurable.

Similar conclusions apply to the temperature variation of k_T/c in the Fermi liquid region. The results for $f^0(x)$, $f_1^1(x)$, $f_2^1(x)$, and $f_3^1(x)$ when $\nabla c = 0$ are shown in Fig. 3,*c*. When used to calculate k_T/c, the result is

$$k_T/c = (k_T/c)_{T=0} - (m_3^*/m_4^*)(T/T_F) \times$$

$$\times [2.194\gamma_1 + 1.057\gamma_2 - 0.055\gamma_3] \qquad (7.2)$$

where the value at $T = 0$ is given in Eq. (6.8) and illustrated in Fig. 2. The numerical coefficients in the T/T_F term were calculated without the $-b_3(x)$ term in the integral equation for $f^0(x)$. This means that they are subject to a small pressure-dependent error of ~ 2.5%.

The dominant effect in the T/T_F term in Eq. (7.2) is, again, due to α_5 and γ_1. We expect a decrease in k_T/c with temperature if α_5 is positive. The thermal diffusion ratio has been measured by Dandache and Laheurte [9] between 0.6 to 2 K. This is well outside the Fermi liquid region but k_T/c was negative, about -10 for $c \sim 2\%$. This is consistent with a positive α_5 or a maximum in k_T/c between 0 and 0.6 K. Using arguments from irreversible thermodynamics, Dandache and Laheurte have linked the negative k_T/c at high temperatures with the dependence of the thermal conductivity on the ^{4}He concentration.

In summary, solutions of ^{4}He in liquid ^{3}He have a simple relation between the low temperature limits for the kinetic coefficients D and k_T and the thermodynamic properties v_4^* and m_4^*. The limit for DT and its finite temperature corrections are given in Eq. (7.1) while k_T is predicted in Eqs. (6.8) and (7.2) and shown in Fig. 2. If one can overcome the experimental difficulties in making measurements at sufficiently low temperatures, where the solubility

of ^{4}He approaches a few parts per million [10,12,15,26], these results present a unique opportunity to test Fermi liquid theory.

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