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Quantum nucleation of cavities in a liquid helium at low temperatures

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The rate of the quantum cavitation in normal fluid ^{3}He and superfluid ^{4}He at temperatures down to absolute zero has been studied. The effect of energy dissipation due to viscosity and the effect of the finite compressibility of a fluid are incorporated into the calculation of the quantum cavitation rate. Because of the dissipative processes, the kinetics of the quantum cavitation in ^{3}He and ^{4}He proves to be qualitatively different. In normal ^{3}He it corresponds to the dissipative tunneling through a potential barrier. In contrast, in superfluid ^{4}He the effect of dissipation is of minor importance. In both liquids the role of the compressibility of a fluid enhances significantly for the small critical nuclei, which have several interatomic distances and can provide us the nucleation rates sufficient for the experimental observation of the homogeneous cavitation in the quantum regime.

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1. Introduction

Considerable theoretical discussion on the macroscopic quantum nucleation has recently been focused on the low-temperature cavitation in liquid helium at negative pressures [1–4]. Some intriguing problems, such as the tensile strength of liquid helium, i.e., the magnitude of the negative pressure required to produce nucleation of cavities, and the critical pressure at which liquid helium becomes thermodynamically unstable against the density fluctuations, have aroused special interest. Various cavitation experiments have also been performed [5–7].

According to the first estimates [8] of the rates at which bubbles nucleate in a liquid ^{4}He, it is expected that quantum nucleation should dominate over the thermally activated nucleation at temperatures below ≈ 0.3 K and that in this temperature range the pressure providing the noticeable nucleation rate or the tensile strength should be about $P \approx -15$ atm. Later, Maris and Xiong [2] pointed out the possibility that, before this pressure can be attained, the liquid ^{4}He is unstable against the long-wavelength fluctuations of density since the square of the sound velocity becomes negative. The extrapolations of the sound velocity into the negative pressure range and some numerical calculations suggest that the sound velocity at pressure P goes to zero as

$$c(P) \propto (P - P_c)^{\nu} . \quad (1)$$

Here the exponent ν is close from $1/3$ to $1/4$. The critical pressure P_c , i.e., the pressure at the lability point, is estimated to be $P_c = -(8–9)$ atm at absolute zero. For liquid ^{3}He, it is expected that $P_c = -(2–3)$ atm [2].

In order to find the tensile strength, one needs a theory on the nucleation of cavities in the liquid. So far all the calculations of the nucleation rate and tensile strength in the region of the quantum tunneling regime have been performed within the framework of the Lifshitz–Kagan theory [9] of first-order phase transitions. However, in this well-known theory there were several assumptions that reduced its general validity. In particular, metastable liquid phase was assumed to be absolutely incompressible or, in other words, sound velocity in the liquid is infinite. Clearly, a more realistic theory of the quantum cavitation should involve the effect of the finite compressibility, especially in the closest vicinity of the instability point at which the sound velocity vanishes.

As follows from the recent studies involving the effect of finite compressibility on the quantum decay rate of a metastable phase, the ratio of the nucleus growth rate $\dot{R}$ to the sound velocity c is a physical parameter which governs the magnitude of the compressibility in the case of three-dimensional nucleation [10]. In turn, for the decay of low-dimensional metastable systems the involvement of nonzero compressibility of a medium in the calculation of the decay rate is of principal importance since the approximation of an incompressible medium has no applicability [11]. Furthermore, compared with the standard theories [9,12] based on the models of an incompressible medium, in which the decay kinetics of a metastable phase has a dissipationless character, the sound retardation due to the finite velocity of sound propagation produces qualitative changes in the quantum decay, which becomes completely analogous to the dissipative quantum tunneling through a potential barrier. The mechanism of energy dissipation is associated with the emission of sound during the growth of the stable phase. On the whole, this leads to the time nonlocality of the effective Euclidean action and, as a result, to the appearance of the explicit temperature dependence for the nucleation rate in the quantum tunneling regime.

The examination of the compressibility effect on the quantum nucleation of cavities in a metastable liquid, which has not been made yet, is the main topic considered in this paper. In order to investigate quantum-mechanical tunneling between the metastable and stable states of a condensed medium and to calculate the rate at which cavities nucleate, we employ the formalism based on the use of the finite-action solutions (instantons) of equations continued to the imaginary time. (For review see, for example, Ref. 13.) This approach [14,15] for describing quantum-mechanical tunneling in the systems with macroscopic number of degrees of freedom was used for incorporating the influence of energy dissipation in a metastable, condensed medium on the quantum kinetics of first-order phase transitions at low temperatures [11,16].

2. Dynamics of a thin-wall bubble in the liquid

The growth of a bubble in the liquid, as well as the formation of a bubble, is a very complex process. The growth of a bubble occurs in a condensed medium representing a system of many particles. As a result, the growth of a bubble is accompanied by nondissipative and dissipative processes, including the nonhomogeneous outflow of the liquid from the bubble, the viscosity, the heat conduction, and sound emission due to the compressibility of the liquid. Thus, even for a spherical bubble that expands uniformly in all directions, the derivation of the general growth equation, which is valid for an arbitrary expansion rate, is a complex problem. We therefore start from a number of simplifying assumptions.

Let us consider a normal fluid, say, ^{3}He held at arbitrary pressure P, either positive or negative. As the next step, we assume that a spherical bubble of radius $R(t)$ has been produced and that its radius is growing at certain rate $\dot{R}(t)$. For simplicity, we disregard the possible presence of the helium vapor inside the bubble, since the density of the vapor is much smaller compared to that of the bulk liquid. We can then consider the bubble within a thin-wall approximation assuming that the bubble has an abrupt boundary between a void and the liquid surrounding the bubble. In other words, we will describe the liquid-vacuum interface in the terms of the surface energy coefficient α. Of course, this is reasonable only if the bubble radius is much larger than the interface thickness.

The total energy of the system will then be

$$\mathcal{E} = \int\limits_{r>R(t)} d^3r \left[\frac{1}{2}\,\rho(\mathbf{r})\,\mathbf{v}^2(\mathbf{r}) + \rho(\mathbf{r})\,\varepsilon(\rho(\mathbf{r}))\right] + 4\pi\alpha R^2(t)\,, \tag{2}$$

where the velocity and density of the liquid at point $\mathbf{r}$ are $\mathbf{v}(\mathbf{r})$ and $\rho(\mathbf{r})$, respectively. The first bulk term represents a sum of the kinetic and internal energies of the liquid, and ε is the internal energy per unit mass. The second term is the surface energy of the bubble. To make the further simplification of the bubble growth, we disregard all the heat effects which, in general, can accompany the growth of a bubble. For this purpose, one should ignore the possible temperature dependence in the coefficient of surface energy α and the heat transfer due to the viscosity of the medium.

Let us now turn to the derivation of the equation which the growth of a bubble obeys. First of all, we note that, according to the conservation of the mass flux across the boundary of a bubble, we have an equality between the fluid velocity and the growth rate at $r = R(t)$, i.e.,

$$v(R) = \dot{R}(t)\,. \tag{3}$$

Next, one possible way to obtain the growth equation is to use the conservation of the momentum density flux across the boundary at $r = R(t)$:

$$P(R) + \tau_{rr}(R) + \frac{2\alpha}{R} = 0\,. \tag{4}$$

Here $P(R)$ is the pressure, and $\tau_{rr}(R)$ is the radial component of the viscous stress tensor at the surface of the bubble. The viscous stress tensor τ_{ik} is defined by the standard expression [17] as

$$\tau_{ik} = -\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\, \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) - \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l} , \quad (5)$$

here η and ζ are the viscosity coefficients, and the subscripts i, k and l run over the values of 1, 2, and 3 corresponding to the components of the radius vector. The last term in Eq. (4) takes into account the existence of the Laplace pressure due to the curvature of the surface.

The boundary condition (4) is essentially an equation of the bubble growth. We must express the pressure $P(R)$ and the viscous stress $\tau_{rr}(R)$ in terms of the variables describing the growth of a bubble, i.e., $\dot{R}(t)$ and $R(t)$. For this purpose one should employ two equations which govern the motion of a fluid. The first is the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (6)$$

and the second is the Navier–Stokes equation [17]

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] =$$

$$= -\nabla P + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v}) . \quad (7)$$

We are now in the position to calculate the unknown quantities $P(R)$ and $\tau_{rr}(R)$, using Eqs. (6) and (7), and the boundary condition (3). However, the derivation of the general analytic solution for an arbitrary dependence of the growth rate $\dot{R}$ on time t is unfeasible and we restrict the analysis to the limit of sufficiently low growth rates, $\dot{R} \to 0$. In what follows, only the quantities of the order not smaller than $\dot{R}/c << 1$ will be kept, where c is the sound velocity. The time derivatives of $R(t)$ to third order are also retained. Each term of the decomposition has its own physical meaning and, in addition, its relative contribution to the bubble growth kinetics depends on several factors, including the bubble radius, growth rate, temperature, and kinetic properties of the liquid near the bubble.

As usual, to solve Eqs. (6) and (7), it is convenient to introduce the velocity potential $\varphi(rt)$ according to

$$v = \nabla \varphi .$$

In the above approximations the motion of a fluid medium can be reduced to the linear equation corresponding to the propagation of sound under damping

$$\nabla^2 \varphi - \frac{\ddot{\varphi}}{c^2} + \frac{(4/3)\eta + \zeta}{\rho c^2} \nabla^2 \dot{\varphi} = 0 . \quad (8)$$

The general solution for the sound that propagates from the bubble and vanishes at infinity can be represented as

$$\varphi(rt) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \frac{e^{-i\lambda_\omega r}}{r} \Phi(\omega) , \quad (9)$$

$$\lambda_\omega = \frac{\omega}{c} \left(1 - 2i\gamma_\omega \frac{c}{\omega} \right)^{-1/2} , \quad \gamma_\omega = \frac{4\eta/3 + \zeta}{2\rho c^3}\, \omega^2 ,$$

where γ_ω is the sound absorption coefficient due to the viscosity of a fluid. The unknown function $\Phi(t)$ must be determined from the boundary condition (3) setting $\partial\varphi/\partial r = \dot{R}(t)$ at $r = R(t)$. The involvement of first-order time derivative alone is sufficient in our approximation; i.e.,

$$\Phi(t) \approx -\frac{\dot{V}(t)}{4\pi} , \quad V(t) = \frac{4\pi}{3} R^3(t) , \quad (10)$$

where $V(t)$ is the volume of an expanding bubble. Using the usual relation for the pressure in a fluid

$$P(rt) - P = -\rho\dot{\varphi} - \rho \frac{(\nabla\varphi)^2}{2} + \left(\frac{4}{3}\eta + \zeta \right) \nabla^2 \varphi$$

and Eq. (5) for the viscous stress tensor, we obtain for the equation of the bubble growth

$$\frac{2\alpha}{R} + P + 4\eta \frac{\dot{R}}{R} + \rho \left(R\ddot{R} + \frac{3}{2} \dot{R}^2 \right) -$$

$$- \frac{\rho}{4\pi c} \dddot{V}(R) + \ldots = 0 , \quad (11)$$

where P is the external pressure. In the absence of the surface, viscous, and sound terms the equation for the radial growth of a bubble was derived for the first time by Lord Rayleigh. Later, the growth equation was generalized by M. Plesset with allowance for the surface tension.

For further analysis, let us rewrite the growth equation in a more general form. Multiplying Eq. (11) by $4\pi R^2$, we obtain

$$U'(R) + \mu_1(R)\dot{R} + \mu_2(R)\left[\ddot{R} + \frac{1}{2}\frac{\mu_2'(R)}{\mu_2(R)}\dot{R}^2\right] - \mu_3(R)\left[\dddot{R} + \frac{3}{2}\frac{\mu_3'(R)}{\mu_3(R)}\dot{R}\ddot{R} + \frac{1}{2}\left(\frac{\mu_3''(R)}{\mu_3(R)} - \frac{\mu_3'^2(R)}{2\mu_3^2(R)}\right)\dot{R}^3\right] + \ldots = 0\,. \quad (12)$$

The expression which we derived essentially represents a general form for lowest terms of the expansion of the equation of bubble growth in a series in the slowness of variation of the bubble radius $R(t)$ in time; i.e., expression (12) is a low-frequency expansion.

The term $U'(R)$ which remains finite at $\dot{R} \equiv 0$ originates from

$$U(R) = \frac{4\pi}{3} PR^3 + 4\pi\alpha R^2\,. \quad (13)$$

Accordingly, $U(R)$ can be treated as a potential energy of the bubble. Note that for negative pressures the bubbles with radii exceeding the critical size

$$R_c = 3\alpha/|P| \quad (14)$$

prove to be energetically favorable and cavitation becomes unavoidable.

The second term with the first derivative represents the drag force, which hinders the growth of a bubble and which is completely analogous to the Stokes force, which is proportional to the growth rate

$$\mu_1(R)\dot{R} = 16\pi\eta R\dot{R}\,. \quad (15)$$

It is obvious that the drag force governs the evolution of a bubble at sufficiently small growth rate when the other terms which depend on the temporal derivatives, can be disregarded. As we shall see below, such situation for the quantum cavitation is possible provided the critical size R_c of a bubble is large enough or, identically, in the limit of small negative pressures $|P| \to 0$.

We would like to make an important remark concerning the behavior of the friction coefficient $\mu(R)$ as a function of the bubble radius and temperature. The point is that in the course of deriving the Rayleigh–Plesset equation (11) we employed the Navier–Stokes equation with the viscous stress tensor (5) in the form of the expansion in the gradients of the fluid velocity. This implies, however, that the hydrodynamic approximation is satisfied; i.e., the bubble radius should be much larger compared with the mean free path $l(T)$ of excitations in the medium surrounding the bubble. Since the mean free path increases rapidly at low temperatures, in particular, $l(T) \propto 1/T^2$ for ^{3}He, the crossover from the hydrodynamic $R \gg l$ regime to the ballistic or Knudsen regime of $R \ll l$ should occur at a certain temperature $T_l(R)$.

In the ballistic regime the friction coefficient is governed by the interaction of excitations with the surface of a bubble and is proportional to the area of the bubble surface. The general expression for the friction coefficient $\mu_1(R)$ can be represented as [16]

$$\mu_1(R) = 16\pi\eta R f(R/l)\,, \quad (16)$$

$$f(x) = \begin{cases} 1, & \text{if } x \gg 1 \\ ax, & \text{if } x \ll 1 \end{cases}.$$

Here $f(x)$ is a dimensionless function of the ratio of the bubble radius to the mean free path of excitations in the liquid. The numerical factor a is of the order of unity and depends on the particular features of the interaction of excitations with the bubble surface. It should be noted that in the ballistic regime the friction coefficient $\mu_1(R)$ is independent of the mean free path $l(T)$ since $\eta \sim \rho c l$.

The terms with the second derivative and with the square of the first derivative in Eq. (12) are standard terms and can be described in terms of the variable mass of a bubble [9]:

$$\mu_2(R) = 4\pi\rho R^3\,. \quad (17)$$

These terms can be attributed to the kinetic energy of the fluid that flows away from the bubble.

The other third-order terms are associated mainly with the finite velocity of the propagation of sound in a medium. The corresponding coefficient $\mu_3(R)$ is given by

$$\mu_3(R) = \frac{4\pi\rho}{c} R^4\,. \quad (18)$$

Clearly, the smaller the sound velocity, the larger the effect of this term on the growth of a bubble and on the cavitation kinetics.

To gain further physical insight, we represent the growth equation in terms of the bubble energy dissipated per unit time,

$$\frac{d}{dt}\left(U(R) + \frac{1}{2}\mu_2(R)\dot{R}^2 - \mu_3(R)\dot{R}\ddot{R} - \frac{1}{2}\mu_3'(R)\dot{R}^3\right) =$$

$$= -\mu_1(R)\dot{R}^2 - \frac{\rho}{4\pi c}\ddot{V}^2 . \quad (19)$$

As one can see, the right-hand side of Eq. (19) is described by the dissipative function. The first term of the dissipative function corresponds to the standard ohmic dissipation with the variable friction coefficient. The second term is exactly equal to the total intensity of the sound emission as the volume of the body immersed in the fluid changes [17] if the wavelength λ of the sound emitted is much larger than the size of the body; i.e., $\lambda >> R$. In our case the latter is identical to the inequality $\dot{R} << c$.

In conclusion, we would like to emphasize two important points. First, the growth equation (12) has a limited region of applicability, which is restricted by the low growth rates so that the growth time of a bubble would be longer than the characteristic times of the relaxation processes in the medium surrounding the bubble. Second, the kinetic coefficients $\mu_n(R)$, in general, are different in various media, for example, in the normal or superfluid liquid.

3. Quantum nucleation rate

In this section we shall estimate the thermal-quantum crossover temperature and calculate the rate at which a bubble nucleates at zero temperature. The quantum cavitation problem is treated within the approach elaborated for describing the decay of a metastable state in the presence of energy dissipation [14,15] and used for the analysis of the quantum nucleation processes during first-order phase transitions [16]. This approach is based on finding the extremum values of the effective Euclidean action determined in imaginary time and on using one-to-one correspondence between the classical equation of growth in real time and the Euler-Lagrange equation for the effective action due to the principle of the analytic continuation ($|\omega_n| \to -i\omega$) into imaginary time.

The rate of the quantum nucleation can be written as

$$\Gamma(T) = \Gamma_0(T) \exp(-S(T)/\hbar) , \quad (20)$$

where the preexponential factor Γ_0 is the rate of cavitation per unit volume and unit time. According to the general theory of the nucleation kinetics, the factor Γ_0 can be evaluated approximately as the attempt frequency ν_0 multiplied by the number of centers at which the independent cavitation events can occur.

In turn, the exponent S is the extremum value of the effective Euclidean action

$$S_{\text{eff}}|R_\tau| = \int_{-\beta\hbar/2}^{\beta\hbar/2} d\tau \left[U(R_\tau) + \frac{1}{2}\mu_2(R_\tau)\left(\frac{dR}{d\tau}\right)^2 \right] + \frac{1}{4\pi}\int\int_{-\beta\hbar/2}^{\beta\hbar/2} d\tau\, d\tau' \left[\gamma_1(R_\tau) - \gamma_1(R_{\tau'})\right]^2 \frac{(\pi T)^2}{\hbar^2 \sin^2 \pi T(\tau - \tau')/\hbar} -$$

$$- \frac{1}{4\pi}\int\int_{-\beta\hbar/2}^{\beta\hbar/2} d\tau\, d\tau' \left[\frac{\partial\gamma_3(R_\tau)}{\partial\tau} - \frac{\partial\gamma_3(R_{\tau'})}{\partial\tau'}\right]^2 \frac{(\pi T)^2}{\hbar^2 \sin^2 \pi T(\tau - \tau')/\hbar} , \quad (21)$$

where $\beta = T^{-1}$ is the inverse temperature. The path $R(\tau)$ which is defined in imaginary time τ satisfies the periodic boundary conditions $R(-\beta\hbar/2) = R(\beta\hbar/2)$. It should be emphasized that all the parameters of the effective action are associated unambiguously with the corresponding parameters in the classical equation of growth (12). The correspondence can readily be settled with the analytic continuation ($|\omega_n| \to -i\omega$) of the Euler-Lagrange ($\delta S_{\text{eff}}/\delta R_\tau = 0$) equation for the effective action to real time, which entails the classical equation of growth. The substitution ($|\omega_n| \to -i\omega$) of the Matsubara frequencies with the real frequencies must be performed in the frequency representation of the corresponding equations.

It is clear that the first two terms in Eq. (21) can be attributed to the potential and kinetic energies of a bubble. The other terms, nonlocal in time, are due to the energy dissipation during the bubble growth. The parameters $\gamma_1(R)$ and $\gamma_3(R)$ are determined by the kinetic coefficients $\mu_1(R)$ and $\mu_3(R)$, respectively,

$$\mu_1(R) = \left(\frac{\partial\gamma_1(R)}{\partial R}\right)^2 , \quad \mu_3(R) = \left(\frac{\partial\gamma_3(R)}{\partial R}\right)^2 . \quad (22)$$

Depending on whether the hydrodynamic or ballistic regime takes place, as it follows from Eq. (16), we obtain

$$\gamma_1(R) = \begin{cases} \frac{2}{3}\sqrt{16\pi\eta}\ R^{3/2}, & \text{if } R >> l \\ \frac{1}{2}\sqrt{16\pi\alpha\eta/l}\ R^2, & \text{if } R << l \end{cases}.$$

Similar effective actions have been studied in the application to the general theory of the quantum kinetics of first-order phase transitions. However, the various authors [9,10,16] used the kinetic terms separately. It is interesting to note that, in contrast to the term with the ohmic dissipation which is related to the dissipative function proportional to the square of the first-order time derivative, the contribution due to the finite compressibility of a fluid to the effective action is negative. The latter results in enhancing the quantum nucleation rate compared with the one calculated in the framework of the Lifshitz–Kagan model of an incompressible fluid. Some hints for such conclusion can be seen from the fact that the finiteness of the velocity of the sound restricts the region of the bubble environment that can be disturbed and set in motion. The size of this region is approximately $\Lambda = c\tau$, where τ is a typical growth time. In a sense, one can say that the total kinetic energy of the fluid flowing away from the expanding bubble becomes smaller than in the case of an incompressible fluid where the perturbation induced by the formation of the bubble extends instantaneously to infinity.

To be closer to what can actually be observed in low-temperature experiment, we consider only the case of the ballistic $R_c >> l$ regime, when the critical radius is much larger than the mean free path of excitations. In fact, at low temperatures, $T < 1$ K, the mean free path increases drastically in the normal ^{3}He and in the superfluid ^{4}He. Hence, the opposite case of the hydrodynamic regime requires large values for the critical bubble radius, which increases progressively as the temperature decreases. In addition, the large critical radius of a bubble results in such negligible nucleation rates that the homogeneous cavitation becomes unobservable on the scale of the reasonable experimental times. In the quantitative manner, the impossibility of the hydrodynamic quantum regime is expressed by the inequality of $T_l(R_c) > T_0(R_c)$, where $T_0(R_c)$ is the thermal-quantum crossover temperature.

Eventually, it is convenient to represent the effective action in the following way:

$$S_{\text{eff}}[R_\tau] = \int_{-\hbar/2T}^{\hbar/2T} d\tau \left[\frac{4\pi}{3} PR_\tau^3 + 4\pi\alpha R_\tau^2 + 2\pi\rho R_\tau^3 \dot{R}_\tau^2\right] +$$

$$+ \frac{1}{4\pi} \int\int_{-\hbar/2T}^{\hbar/2T} d\tau\, d\tau' \left\{\frac{\rho}{4\pi} u[A(R_\tau) - A(R_{\tau'})]^2 - \right.$$

$$\left. - \frac{\rho}{4\pi c} [\dot{V}(R_\tau) - \dot{V}(R_{\tau'})]^2 \right\} \frac{(\pi T)^2}{\hbar^2 \sin^2 \pi T(\tau - \tau')/\hbar}, \tag{23}$$

where $A = 4\pi R^2$ is the area of the surface, and $V = 4\pi R^3/3$ is the volume of a bubble. The quantity $u \sim \eta/\rho l$ is approximately the characteristic velocity of excitations in a medium. For a normal liquid like ^{3}He, the order of magnitude of the velocity u is the Fermi velocity and the possible temperature corrections to the zero temperature are associated with the quantities of about $(T/T_F)^2$, where T_F is the degeneration temperature of the Fermi-like excitations.

In the superfluid ^{4}He where the energy dissipation of the ohmic type is due to the presence of the normal component alone, we have a different behavior of the quantity u:

$$u(T) \approx c\rho_n(T)/\rho\ . \tag{24}$$

Here $\rho_n(T)$ is the density of the normal component and at low temperatures $T < 0.5$ K the normal density is determined mainly by phonons [18]

$$\rho_n(T) = \frac{2\pi^2}{45} \frac{T^4}{\hbar^3 c^5}\ .$$

It should be noted that since $u \sim c^{-4}$, the relative role of this ohmic term increases in the vicinity of the lability point because of the reduction of the sound velocity.

First, we consider the high-temperature region in which there is only a classical extremum path. The path which satisfies the condition $R(\tau) \equiv R_0 = 2R_c/3$ goes through the maximum U_0 of the potential energy (13) and yields the action $S = \hbar U_0/T$ resulting in the standard Arrhenius law for the nucleation rate,

$$\Gamma = \Gamma_0 \exp(-U_0/T)\ ;\ \ U_0 = \frac{16\pi\alpha^2}{3|P|^2}\ . \tag{25}$$

We begin the study of the low-temperature quantum behavior of the nucleation rate by analyzing the classical $R(\tau) \equiv R_0$ extremum path with respect to small oscillations about the maximum of the potential energy. For this purpose, we represent an arbitrary path as

$$R(\tau) = R_0 + r(\tau) .$$

Next we expand the effective action $S_{\rm eff}[R(\tau)]$ in a series in small powers of deviation of $r(\tau)$. Truncating a series in $r(\tau)$ at second order and turning to the Fourier representation

$$r(\tau) = \frac{T}{\hbar} \sum_n r_n \exp(-i\omega_n \tau) ,$$

$$r_n^* = r_{-n} \; ; \quad \omega_n = 2\pi T n/\hbar, \quad n = 0, \pm 1, \pm 2, \ldots ,$$

we obtain after some calculations the expression

$$S_{\rm eff} = \frac{\hbar U_0}{T} + \frac{T}{2\hbar} \sum_n \alpha_n |r_n|^2. \tag{26}$$

Here the coefficients α_n are given by

$$\alpha_n = U_0'' + 16\pi\rho u R_0^2 |\omega_n| + 4\pi\rho R_0^3 \omega_n^2 - \frac{4\pi\rho R_0^4}{c} |\omega_n|^3 . \tag{27}$$

As the temperature is lowered, the coefficients $\alpha_{\pm 1}$ vanish first at $T = T_1$, which is determined by the equation

$$-\alpha + 4\rho u R_0^2 \omega_1 + \rho R_0^3 \omega_1^2 - \frac{\rho R_0^4}{c} \omega_1^3 = 0 , \quad T_1 = \frac{\hbar\omega_1}{2\pi} . \tag{28}$$

Below the temperature T_1 the classical path $R(\tau) = R_0$ becomes absolutely unstable against the oscillations of mode $r_{\pm 1}$.

Depending on the type of the quantum-classical path transition [16], the genuine thermal-quantum crossover temperature T_0 coincides with the temperature T_1 if the effective action matches smoothly the exponent of the Arrhenius law or lies at a temperature slightly higher than the temperature T_1 if the quantum-classical path transition has a discontinuous, jump-like character, i.e., $T_0 \le T_1$. Although the action (23) we are concerned with refers to the last case and although the crossover temperature T_0 should be found from $S(T_0) = \hbar U_0/T_0$, the approximate estimate of $T_0 \approx T_1$ is fully sufficient for our purpose.

According to Eq. (28), in the limit of sufficiently large radius $R_0 \to \infty$ or, correspondingly, small negative pressures $|P| \to 0$ we obtain the following estimate of the crossover temperature:

$$T_0 \approx \frac{\hbar\alpha}{8\pi\rho u R_0^2} = \frac{\hbar}{32\pi\alpha\rho u} |P|^2 . \tag{29}$$

For the above formula to be correct, it is necessary that the growth rate $\dot{R}$ be smaller than the velocity of excitations and the sound velocity. Since the characteristic time of the underbarrier evolution of a bubble is $(2\pi\omega_1)^{-1}$, we have

$$\frac{2\pi\omega_1 R_0}{u} = \frac{\pi\alpha}{2\rho u^2 R_0} << 1 . \tag{30}$$

Obviously, this inequality restricts the magnitude of the pressure

$$|P| << \rho u^2 , \tag{31}$$

for which our approximations hold true. If the strong inequality (30) breaks down, in Eq. (28) we must use terms of higher orders in ω_1, and the estimate (29) of the thermal-quantum crossover temperature ceases to be valid.

In contrast with normal ^{3}He, in superfluid ^{4}He the density of the normal component $\rho_n(T)$ vanishes as $T \to 0$ and therefore the contribution of the ohmic term in Eq. (28) decreases. In order to analyze all the facts of the case, let us rewrite Eq. (28) for temperature T_1, taking into account Eq. (24) for $\rho_n(T)$

$$-\alpha + \frac{R_0^2}{90\pi^2 c^4} \hbar\omega_1^5 + \rho R_0^3 \omega_1^2 - \frac{\rho R_0^4}{c} \omega_1^3 = 0 . \tag{32}$$

Of course, the condition $\omega_1 R_0/c << 1$ is assumed to be satisfied.

As one can see, the dissipative ohmic term linear in ω_1 has no significant influence on the thermal-quantum crossover temperature T_0 provided $R_0 >> R_*$ where the radius R_* is given by

$$R_* \approx \left(\frac{\hbar^2\alpha^3}{8100\, \pi^4 c^8 \rho^5} \right)^{1/11} . \tag{33}$$

For the radius $R_0 >> R_*$, the thermal-quantum crossover temperature is found to be approximately the same, as it follows from the nondissipative model of the quantum cavitation [1,4]

$$T_0 \approx \frac{\hbar}{2\pi} \sqrt{\frac{\alpha}{\rho R_0^3}} = \frac{\hbar}{4\pi} \frac{|P|^{3/2}}{\sqrt{\alpha\rho}} . \tag{34}$$

To satisfy the approximation of the low growth rate $\omega_1 R_0 << c$, we must impose a restriction on the radius R_0 or on the negative pressure P:

$$R_0 >> \frac{\alpha}{\rho c^2} \text{ or } |P| << \rho c^2 . \qquad (35)$$

Numerically, if the physical parameters of ^{4}He are measured at zero pressure $P = 0$, we find that $R_* \approx \alpha/\rho c^2$. Since $\alpha/\rho c^2 \approx 0.5$ Å, the validity of the estimate (34) is connected with the applicability of the macroscopic description, which is correct for large bubble radii compared with the interface thickness. Note that the condition (35) can be satisfied only in the range of pressures far enough from the lability point at which the sound velocity vanishes.

Let us now focus our attention on the low-temperature $T << T_0$ behavior of the nucleation rate. First, we consider the case of $u(T) = const$, which corresponds to the normal ^{3}He. Since we should remain within the approximation of the low growth rate, the main contribution to the effective action originates from the dissipative ohmic term which is nonlocal in time. The other two kinetic terms can therefore be treated as perturbations. Accordingly, for temperature $T = 0$ we have approximately

$$S(T = 0) \approx$$

$$\approx 4\pi\rho u R_c^4 \left[1 + \frac{\alpha}{2\rho u^2 R_c} - \frac{u}{9c} \left(\frac{\alpha}{\rho u^2 R_c} \right)^2 \right] \propto |P|^{-4} . \qquad (36)$$

This result represents the decomposition of the effective action in $\dot{R}/u << 1$ if we take into account that the typical time of the bubble growth or, identically, the transit time along the extremum underbarrier path is about

$$\tau_c = \frac{\rho u R_c^2}{\alpha} . \qquad (37)$$

In contrast with the dissipationless kinetics [1,4,9] the energy dissipation during the bubble growth leads to the effective action in which the kinetic terms depend on temperature in an explicit form. It is natural therefore to expect a temperature-dependent behavior of the nucleation rate in the quantum tunneling regime below the crossover temperature T_0. We thus can expect [16]

$$\Delta S(T) = S(T) - S(0) \approx -S(0)(T/T_0)^2 . \qquad (38)$$

It is obvious that the temperature correction affects essentially the nucleation rate, while $|\Delta S(T)| > \hbar$. Introducing the temperature T_2 at which $|\Delta S(T_2)| \approx \hbar$, i.e.,

$$T_2 \approx \frac{\alpha}{R_c^4} \left(\frac{\hbar}{4\pi\rho u} \right)^{3/2} \propto P^4 , \qquad (39)$$

we obtain a noticeable range of temperatures $T_2 < T << T_0$, for which the enhancement of nucleation rate $\Gamma(T)$ follows the law of $\log [\Gamma(T)/\Gamma(0)] \propto T^2$.

Let us turn now to the case of the cavitation in a superfluid ^{4}He. In contrast to a normal fluid, where the density of excitations remains finite down to zero temperature, the density of the normal component in superfluid ^{4}He vanishes at zero temperature and the nucleation kinetics is governed mainly by the well-known nondissipative term , which is related to the kinetic energy of the liquid [1,4,9]. Using the correction due to the finite velocity of the sound propagation, we can describe the effective action at $T = 0$ approximately by

$$S(T = 0) \approx \frac{5\sqrt{2}\,\pi^2}{16} (\alpha\rho)^{1/2} R_c^{7/2} \left(1 - \frac{4}{9c} \sqrt{\frac{2\alpha}{\rho R_c}} \right) . \qquad (40)$$

The order of magnitude of the second term is a ratio of the underbarrier growth rate to the sound velocity. On the whole, the model of an incompressible liquid [4,9], as one can see, underrates the cavitation rate in the quantum regime. As the pressure decreases, the underestimate of the cavitation rate increases due to the reduction of the critical radius and the sound velocity. For large critical radii, although the relative correction to the quantum nucleation rate is small, the absolute value of the correction is very large because of an exponential dependence of the nucleation rate on the effective action.

To conclude the section, we shall analyze the low $T << T_0$ temperature behavior of the nucleation rate. The temperature-dependent behavior for the nucleation rate is entirely due to the terms in the effective action (23), which are nonlocal in time and which describe the energy dissipation processes occurring in the superfluid ^{4}He during the bubble growth.

The temperature correction from the ohmic dissipation term is governed by the temperature behavior of the normal density $\rho_n(T)$ [16]

$$\Delta S_{\text{ohm}}(T) \approx 4\pi\rho u(T) R_c^4 = 4\pi c \rho_n(T) R_c^4 .$$

This contribution reduces the nucleation rate. In contrast, the temperature correction resulting from the sound emission term has a negative sign and increases the nucleation rate [10]

$$\Delta S_s(T) \approx \frac{\rho^2 R_c^9}{c\alpha}\left(\frac{T}{\hbar}\right)^4 .$$

The temperature dependence of the correction is the same as for $\rho_n(T)$. The total temperature correction is determined by a sum

$$\Delta S(T) = S(T) - S(0) \approx \left(\frac{8\pi^3\hbar}{45c^3} - \frac{\rho^2 R_c^5}{\alpha}\right)\frac{R_c^4}{c}\left(\frac{T}{\hbar}\right)^4 . \quad (41)$$

Note that at least in the immediate vicinity of the lability point, when $P \to P_c$ and $c(P) \to 0$, the correction associated with the existence of the normal component will dominate over the sound emission mechanism. In contrast, in the range of the small negative pressures of $p < -3\alpha/R_*$ or large critical radii,

$$R_c > R_* \approx \left(\frac{8\pi^3}{45}\frac{\alpha\hbar}{\rho^2 c^3}\right)^{1/5} , \quad (42)$$

the sound emission mechanism governs the temperature behavior of the nucleation rate. If we take the parameters of ^4He at zero pressure, the numerical estimate gives the value of about 2.4 Å for radius R_*, which is comparable with the interatomic distance a. In the whole region of the macroscopic $R_c >> a$ approximation the contribution from the ohmic dissipation is therefore negligible and the nucleation rate $\Gamma(T)$ should increase with increasing temperature.

Let us now evaluate the temperature T_2 at which the temperature correction for the exponent becomes significant, i.e., if $|\Delta S(T_2)| \approx \hbar$. Using Eq. (41), we obtain

$$T_2 \approx \hbar\left(\frac{\hbar\alpha c}{\rho^2 R_c^9}\right)^{1/4} . \quad (43)$$

However, the temperature T_2 is smaller than the temperature T_0 of the thermal-quantum crossover only for the sufficiently large critical radii which exceed a certain radius R_2

$$R_c > R_2 \approx \frac{4\pi}{3}\left(\frac{16\pi\hbar c}{3\alpha}\right)^{1/3} . \quad (44)$$

The estimate for pressures $|P| \approx 0$ yields $R_2 \approx 40$ Å. Thus, only for macroscopically large bubbles of radius $R_c >> a$ there is a noticeable range of temperatures $T_2 < T < T_0$ where $|\Delta S(T)| > \hbar$. For $R_c < R_2$, the scale of the $\log \Gamma(T)/\Gamma(0) \propto T^4$ variation is not large. Note that the radius R_2 decreases near the lability point $c(P_c) = 0$.

4. Summary

In this paper we have examined the effect of the dissipative processes and finite compressibility on the rate at which bubbles can nucleate via quantum tunneling in the normal ^3He and superfluid ^4He at negative pressures and sufficiently low temperatures. In conclusion, we would like to emphasize several important points common and distinct for the kinetics of the quantum cavitation in the normal and superfluid liquids.

The common feature of quantum kinetics is that the dissipative processes, which are associated with the viscosity of a fluid, hinder the quantum nucleation of bubbles. The viscous phenomena have an origin entirely in the spatially nonuniform flow of the fluid which has to spread in the radial directions away from the expanding bubble.

In contrast, the finite compressibility of a fluid facilitates the quantum nucleation of the bubbles since it is easier to push the fluid out from the cavity if the medium surrounding it is light-compressible. This phenomenon is accompanied by the excitation and emission of the sound waves induced by an expanding sphere.

This effect is essential for the negative pressures of about several atmospheres when the critical sizes of the bubbles should be approximately equal to several interatomic distances and the rate of tunneling is comparable with the sound velocity. On the whole, these two processes result in the appearance of the explicit temperature-dependent behavior of the cavitation rate in the quantum regime.

On the other hand, it is the dissipative processes that make the quantum cavitation kinetics diverse in the normal and superfluid liquids. In the normal fluid ^3He, where the density of excitations does not vanish at low temperatures, the quantum cavitation kinetics corresponds entirely to the dissipative tunneling through a potential barrier in the overdamped regime. Compared with the calculations [1,2,4,9] performed on the basis of the dissipationless models of quantum cavitation, the quantum cavitation rate for the bubbles of the large critical sizes proves to be significantly smaller and, correspondingly, the tensile strength should be also somewhat smaller.

In addition, the temperature necessary for observing the quantum tunneling regime instead of thermal activation decreases and should be below about 70 mK. The $\log \Gamma(T)/\Gamma(0) \propto T^2$ behavior for the nucleation rate is expected in the low-temperature limit.

In contrast with the normal ^3He, in superfluid ^4He, where all excitations are frozen out as the temperature tends to absolute zero, the dissipative processes do not play an essential role with the exception of the range of small negative pressures. This range of pressures of about $P > -1$ atm refers to the sufficiently large critical sizes of the bubbles which have an astronomically large lifetime and thereby do not determine the tensile strength of ^{4}He under ordinary experimental conditions.

Although the compressibility and sound excitation effects during the nucleation must undoubtedly be involved in the cavitation kinetics of the bubbles of small critical sizes, the quantum cavitation rate $\Gamma(T)$ and therefore the tensile strength of ^{4}He remain, as in the case of the incompressible liquid models, nearly independent of the temperature. The involvement of the compressibility of superfluid ^{4}He leads to the thermal-quantum crossover temperature which is somewhat higher than that calculated on the basis of the incompressible liquid model. The last two consequences for the quantum cavitation in the homogeneous ^{4}He together with the estimate of the crossover temperature of about $T_0 \approx 0.3$ K for the small critical bubbles do not contradict the recent low-temperature cavitation experiments [19].

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