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One exactly solvable random spin-1/2 *XY* chain

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Considering one-dimensional spin-½ isotropic *XY* model with Dzyaloshinskii-Moriya interaction in random Lorentzian transverse field we have calculated exactly thermodynamic quantities of the model. We used the derived average single-particle density of states to examine the validity of coherent potential approximation.

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1. Introduction

Random models that can be solved exactly are of great importance in understanding the effects of disorder because they do not contain uncontrolled errors that are introduced by approximate treatment. One such model has been known for almost thirty years. Considering a model of tight-binding electrons, P. Lloyd was first to recognize that Lorentzian diagonal disorder allows one to perform the averaging of one-electron Green's functions over random variables with the help of contour integrals without making any approximation [1]. This idea was exploited by H. Nishimori for spin-½ isotropic *XY* chain which is related to Lloyd's model via the Jordan-Wigner transformation.

In the present paper our aim is to extend the consideration given in Ref. 2 by introducing additional Dzyaloshinskii-Moriya spin-spin interaction. By analogy with Ref. 2, we obtained exactly average, single-particle density of states and hence the thermodynamic quantities (Sec. 2). Moreover, the average density of states which we obtained enables us to discuss the applicability of coherent potential approximation which is usually used to describe realistic quenched systems for which exact solutions do not exist [3,4] (Sec. 3).

2. Density of states and thermodynamic quantities

We consider N interacting spins ½ in a random transverse field that are governed by the Hamiltonian

$$H = \sum_{j=1}^{N} \left(\Omega_0 + \Omega_j\right) s_j^z + \sum_{j=1}^{N} \left[J(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y) + D(s_j^x s_{j+1}^y - s_j^y s_{j+1}^x)\right] =$$

$$= \sum_{j=1}^{N} \left(\Omega_0 + \Omega_j\right)\left(s_j^+ s_j^- - \frac{1}{2}\right) + \sum_{j=1}^{N} \left(\frac{J+iD}{2} s_j^+ s_{j+1}^- + \frac{J-iD}{2} s_j^- s_{j+1}^+\right), \quad (1)$$

$$s_{N+j}^{\alpha} = s_j^{\alpha} ,$$

with the Lorentzian probability distribution density

$$p(\ldots, \Omega_j, \ldots) = \prod_{j=1}^{N} \frac{1}{\pi} \frac{\Gamma}{\Omega_j^2 + \Gamma^2} .$$

The introduced Hamiltonian (1) contains the interaction terms proportional to D which describe the Dzyaloshinskii-Moriya interactions of the neighbouring spins.

The Jordan-Wigner transformation [5]

$$c_1 = s_1^- , \; c_1^+ = s_1^+ , \; c_j = P_{j-1} s_j^- , \; c_j^+ = P_{j-1} s_j^+ ,$$

$$P_j \equiv \prod_{n=1}^{j} (-2 s_n^z) , \; j = 2, \ldots, N$$

converts (1) into a Hamiltonian of noninteracting spinless fermions

$$H = H^- + BP^+ ,$$

$$H^- = \sum_{j=1}^{N} (\Omega_0 + \Omega_j) \left(c_j^+ c_j - \frac{1}{2} \right) + $$

$$+ \sum_{j=1}^{N} \left(\frac{J + iD}{2} c_j^+ c_{j+1} - \frac{J - iD}{2} c_j c_{j+1}^+ \right),$$

$$c_{N+j} = c_j , \quad c_{N+j}^+ = c_j^+ ,$$

$$B = -\left[(J + iD) c_N^+ c_1 - (J - iD) c_N c_1^+ \right],$$

$$P^+ = \frac{1 + P_N}{2} . \qquad (2)$$

The boundary term B may be omitted since it does not influence thermodynamic quantities [6]. Hence, the thermodynamics of spin model (1) is determined by the average one-fermion Green's functions $\overline{G_{nm}^{\mp}(E)}$, where

$$G_{nm}^{\mp}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE\, e^{-iEt}\, G_{nm}^{\mp}(E \pm i\varepsilon) , \quad \varepsilon \to +0 ,$$

$$G_{nm}^{\mp}(t) \equiv \mp\ i\, \Theta(\pm t) \langle \{ c_n(t), c_m^+ \} \rangle ,$$

$$\overline{(\ldots)} \equiv \int_{-\infty}^{\infty} d\Omega_1 \ldots \int_{-\infty}^{\infty} d\Omega_N p(\ldots, \Omega_j , \ldots)(\ldots) ,$$

via the average density of states

$$\overline{\rho(E)} = \mp \frac{1}{\pi} \operatorname{Im} \overline{G_{jj}^{\mp}(E)} ,$$

and, therefore, the main goal is to find $\overline{G_{nm}^{\mp}(E)}$.

The equation of motion for $G_{nm}^{\mp}(t)$ that follows from (2) leads to the following set of equations for $G_{nm}^{\mp}(E \pm i\varepsilon)$:

$$(E \pm i\varepsilon)\, G_{nm}^{\mp}(E \pm i\varepsilon) = \delta_{nm} + (\Omega_0 + \Omega_n)\, G_{nm}^{\mp}(E \pm i\varepsilon) + $$

$$+ \frac{J - iD}{2} G_{n-1,m}^{\mp}(E \pm i\varepsilon) + \frac{J + iD}{2} G_{n+1,m}^{\mp}(E \pm i\varepsilon) . \qquad (3)$$

To average (3), it can be assumed that Ω_j are complex variables and contour integration can be used in complex planes Ω_j s. Following the paper by John and Schreiber [7], we can rewrite Eq. (3) in the form

$$(\mathbf{A} \pm i\mathbf{B}^{\mp})\, \mathbf{G}^{\mp}(E \pm i\varepsilon) = \mathbf{I} ,$$

where

$$\mathbf{A} \equiv \begin{pmatrix} E - \Omega_0 - \operatorname{Re}\Omega_1 & -\frac{J + iD}{2} & 0 & \ldots \\ -\frac{J - iD}{2} & E - \Omega_0 - \operatorname{Re}\Omega_2 & -\frac{J + iD}{2} & \ldots \\ 0 & -\frac{J - iD}{2} & E - \Omega_0 - \operatorname{Re}\Omega_3 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

$$\mathbf{B}^{\mp} \equiv \begin{pmatrix} \varepsilon \mp \operatorname{Im}\Omega_1 & 0 & 0 & \ldots \\ 0 & \varepsilon \mp \operatorname{Im}\Omega_2 & 0 & \ldots \\ 0 & 0 & \varepsilon \mp \operatorname{Im}\Omega_3 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

$$\mathbf{G}^{\mp}(E \pm i\varepsilon) \equiv \begin{pmatrix} G_{11}^{\mp}(E \pm i\varepsilon) & G_{12}^{\mp}(E \pm i\varepsilon) & G_{13}^{\mp}(E \pm i\varepsilon) & \ldots \\ G_{21}^{\mp}(E \pm i\varepsilon) & G_{22}^{\mp}(E \pm i\varepsilon) & G_{23}^{\mp}(E \pm i\varepsilon) & \ldots \\ G_{31}^{\mp}(E \pm i\varepsilon) & G_{32}^{\mp}(E \pm i\varepsilon) & G_{33}^{\mp}(E \pm i\varepsilon) & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Evidently, the poles of $\mathbf{G}^{\mp}(E \pm i\varepsilon)$ are determined by the zeros of det $(\mathbf{A} \pm i\mathbf{B}^{\mp})$. If all eigenvalues λ_s of $\mathbf{B}^{\mp}$ are positive $\mathbf{B}^{\mp} = (\mathbf{b}^{\mp})^2$, where $\mathbf{b}^{\mp}$ is symmetric, $(\mathbf{b}^{\mp})^{-1}$ is symmetric, $(\mathbf{b}^{\mp})^{-1}\mathbf{A}(\mathbf{b}^{\mp})^{-1}$ is Hermitian, and therefore,

$$\det(\mathbf{A} \pm i\mathbf{B}^{\mp}) = \det \mathbf{B}^{\mp} \det\left[(\mathbf{b}^{\mp})^{-1}\mathbf{A}(\mathbf{b}^{\mp})^{-1} \pm i\mathbf{I}\right] \neq 0.$$

Relying on Gershgorin criterion [8] for the matrix $\mathbf{B}^{\mp}$, we see that at least one of the inequalities

$$|\varepsilon \mp \operatorname{Im} \Omega_j - \lambda| \leq 0 \ , \quad j = 1, \dots, N$$

must be true. Therefore, the retarded Green's function $G^{-}_{nm}(E + i\varepsilon)$ [the advanced Green's function $G^{+}_{nm}(E - i\varepsilon)$] does not have poles for $\operatorname{Im} \Omega_j \leq 0$ ($\operatorname{Im} \Omega_j \geq 0$). While averaging (3) one must close the contours of integration in these planes and compute the residua originating from the Lorentzian probability distribution density at $\mp i\Gamma$, obtaining finally

$$(E - \Omega_0 \pm i\Gamma)\, \overline{G^{\mp}_{nm}(E)} - \frac{J - iD}{2}\, \overline{G^{\mp}_{n-1,m}(E)} - \\ - \frac{J + iD}{2}\, \overline{G^{\mp}_{n+1,m}(E)} = \delta_{nm} \ . \qquad (4)$$

It is worthwhile to note that Eqs (4) may be obtained in a slightly different manner. According to (3), we can rewrite $G^{\mp}_{nm}(E \pm i\varepsilon)$ as a series in degrees of $(J \pm iD)/2$. Due to a magic property of the Lorentzian distribution

$$\overline{\left(E - \Omega_0 - \Omega_j \pm i\varepsilon\right)^{-s}} = \left(E - \Omega_0 \pm i\Gamma \pm i\varepsilon\right)^{-s}$$

the averaging is straightforward and after summation of the series we again obtain (4).

Equations (4) are translationally invariant and can be solved in a standard way with the result

$$\overline{G^{\mp}_{nm}(E)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\kappa\, e^{i(n-m)\kappa}}{E - [\Omega_0 + (J^2 + D^2)^{1/2} \cos(\kappa + \varphi)] \pm i\Gamma} =$$

$$= \frac{e^{i\varphi(n-m)}}{(J^2 + D^2)^{1/2}} \frac{[x - (x^2 - 1)^{1/2}]^{|n-m|}}{(x^2 - 1)^{1/2}} \ , \qquad (5)$$

where

$$\tan \varphi = \frac{D}{J} \ , \quad x = \frac{E}{(J^2 + D^2)^{1/2}} - \omega_0 \pm i\gamma \ ,$$

$$\omega_0 = \frac{\Omega_0}{(J^2 + D^2)^{1/2}} \ , \quad \text{and} \quad \gamma = \frac{\Gamma}{(J^2 + D^2)^{1/2}} \ .$$

From (5) it follows that

$$\overline{\rho(E)} = \mp \frac{1}{\pi} \operatorname{Im} \left((E - \Omega_0 \pm i\Gamma)^2 - (J^2 + D^2)\right)^{-1/2} =$$

$$= \frac{1}{\pi} \left[\frac{(A^2 + B^2)^{1/2} - A}{2(A^2 + B^2)}\right]^{1/2} , \qquad (6)$$

where $A \equiv (E - \Omega_0)^2 - \Gamma^2 - J^2 - D^2$, $B = 2\Gamma(E - \Omega_0)$. Hence, the introduction of the Dzyaloshinskii-Moriya interaction from the viewpoint of thermodynamics results in the renormalization of the spin-spin interaction: $J^2 \to J^2 + D^2$. Thermodynamic quantities of the spin model (1) are determined by the average density of states (6) in a standard way. The corresponding formulas for entropy $\overline{s}$, specific heat $\overline{c}$, transverse magnetization $\overline{m_z} \equiv \overline{\langle (1/N) \sum_{j=1}^{N} s_j^z \rangle}$, and static transverse susceptibility $\chi_{zz} \equiv \partial \overline{m_z} / \partial \Omega_0$ are

$$\overline{s} = \int dE \overline{\rho(E)} \left[\ln\left(2 \cosh \frac{E}{2kT}\right) - \frac{E}{2kT} \tanh \frac{E}{2kT}\right] ,$$

$$\overline{c} = \int dE \overline{\rho(E)} \left(\frac{E/2kT}{\cosh(E/2kT)}\right)^2 ,$$

$$\overline{m_z} = -\frac{1}{2} \int dE \overline{\rho(E)} \tanh \frac{E}{2kT} ,$$

$$\overline{\chi_{zz}} = -\frac{1}{kT} \int dE \overline{\rho(E)} \frac{1}{[2 \cosh(E/2kT)]^2} \ .$$

In the numerical calculations of thermodynamic quantities J was set to 1. Figures 1–4 show the dependences of the entropy $\overline{s}$, specific heat $\overline{c}$, transverse magnetization $\overline{m_z}$, and static transverse susceptibility $\overline{\chi_{zz}}$ on Ω_0 at low temperatures. The temperature dependences of $\overline{m_z}$ and $\overline{\chi_{zz}}$ at $\Omega_0 = 0.5$ are shown in Figs. 5 and 6. The dashed curves correspond to $D = 0$, and the solid curves correspond to $D = 1$; 1 refers to the nonrandom case $\Gamma = 0$, 2 refers to $\Gamma = 0.1$, and 3 refers to $\Gamma = 0.5$. It can be seen how some of the pronounced features of the plotted dependences disappear due to randomness. Dzyaloshinskii-Moriya interaction leads only to quantitative changes in thermodynamic quantities.

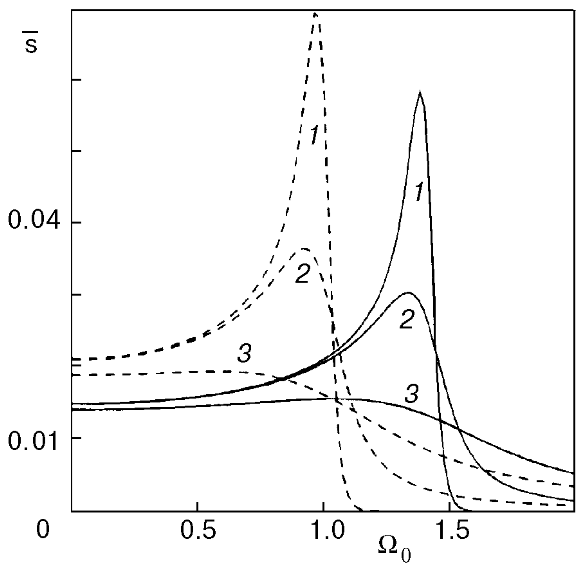

Fig. 1. $\overline{s}$ versus Ω_0 at $T = 0.02$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

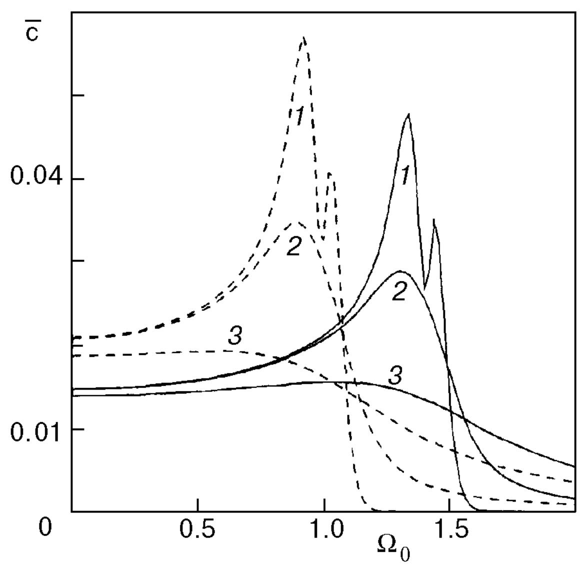

Fig. 2. $\overline{c}$ versus Ω_0 at $T = 0.02$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

The importance of the two-spin correlation functions is obvious; it was recognized by Kontorovich and Tsukernik [9] in connection with a possibility for the appearance of the spiral structure in nonrandom spin-½ isotropic XY chain with Dzyaloshin-

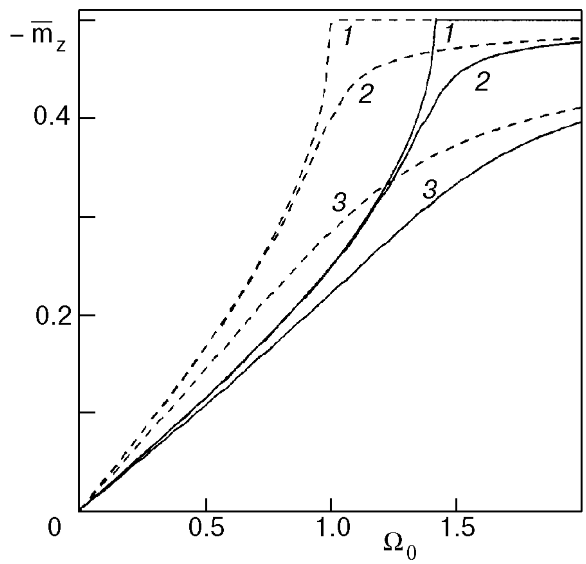

Fig. 3. $\overline{-m_z}$ versus Ω_0 at $T = 0$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

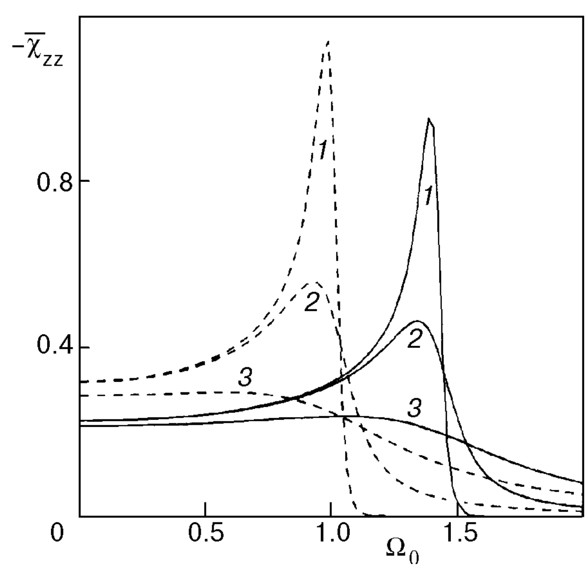

Fig. 4. $\overline{-\chi_{zz}}$ versus Ω_0 at $T = 0.02$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

skii-Moriya interaction. Although the average one-fermion Green's functions (5) yield the average fermion correlation function $\overline{\langle c_m^+ c_n(t)\rangle}$ [10], and at $t = 0$ and $T = 0$ the latter quantity can be calculated explicitly [11]

$$\overline{\langle c_m^+ c_n\rangle} = \frac{1}{\pi|n-m|}\,\mathrm{Im}\left\{e^{i\varphi(n-m)}\left[-\omega_0 + \left(\frac{(\mathcal{C}^2+\mathcal{D}^2)^{1/2}+\mathcal{C}}{2}\right)^{1/2} + i\gamma - i\left(\frac{(\mathcal{C}^2+\mathcal{D}^2)^{1/2}-\mathcal{C}}{2}\right)^{1/2}\right]^{|n-m|}\right\},$$

where $\mathcal{C} = \omega_0^2 - \gamma^2 - 1$, and $\mathcal{D} = 2\omega_0\gamma$, this does not allow us to obtain the spin correlation functions. The simplest equal-time zz spin correlation function in fermion representation has the form

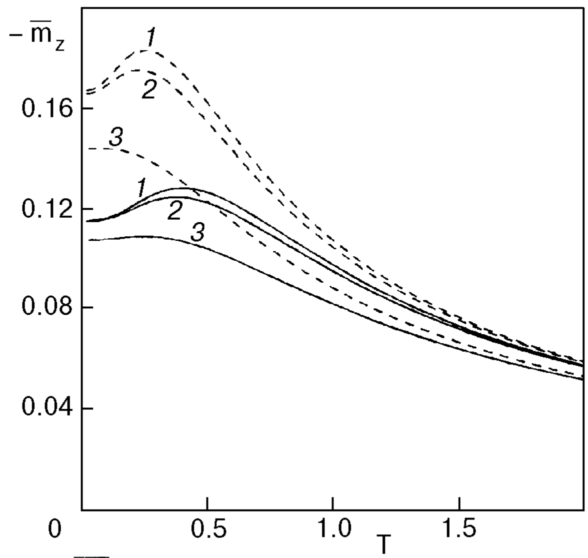

Fig. 5. $\overline{-m_z}$ versus T at $\Omega_0 = 0.5$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

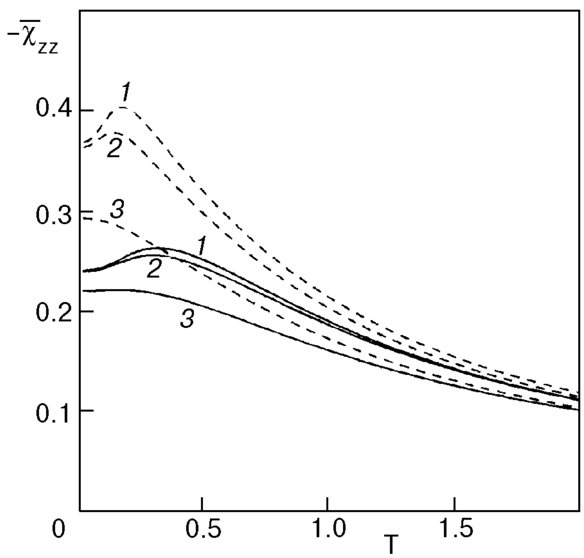

Fig. 6. $\overline{-\chi_{zz}}$ versus T at $\Omega_0 = 0.5$; $D = 0$ (dashed curves), $D = 1$ (solid curves), $\Gamma = 0$ (*1*), 0.1 (*2*), 0.5 (*3*).

$$\overline{\langle s^z_j\, s^z_{j+n}\rangle} = \overline{\langle c^+_j c_j\rangle\langle c^+_{j+n} c_{j+n}\rangle} + \overline{\langle c^+_j c_{j+n}\rangle\langle c_j c^+_{j+n}\rangle} - \overline{\langle c^+_j c_j\rangle} + \frac{1}{4} ,$$

and hence its evaluation requires the knowledge of the average products of two fermion correlation functions. Similar difficulties arose in the calculation of the electric conductivity for Lloyd's model [12].

3. Coherent potential approximation

Consider the spin model (1) with arbitrary (not necessarily Lorentzian) random transverse field in the framework of coherent potential approximation. Choosing the random part of (1) and introducing a coherent transverse field $\hat\Omega$, we rewrite (3) in the form of a propagator expansion

$$G^{\mp}_{gm}(E) = \hat G^{\mp}_{gm}(E) + \hat G^{\mp}_{gn}(E)(\Omega_n - \hat\Omega)\hat G^{\mp}_{nm}(E) + \dots$$

where $\hat G^{\mp}_{gm}(E)$ is determined by (5) with $\Gamma = 0$ and $\Omega_0 + \hat\Omega$ instead of Ω_0 , and then as an expansion in degrees of the $\hat t$-matrix

$$G^{\mp}_{gm}(E) = \hat G^{\mp}_{gm}(E) + \hat G^{\mp}_{gn}(E)\ \hat t_n \hat G^{\mp}_{nm}(E) + \dots .$$

Here $\hat t_n \equiv (\Omega_n - \hat\Omega)/(1 - \hat G^{\mp}_{nn}(E)(\Omega_n - \hat\Omega))$. The coherent field is determined from the condition

$$\overline{\hat t_n} = \int d\Omega_1 \dots \times$$

$$\times \int d\Omega_N p(\dots, \Omega_j , \dots)\, \frac{\Omega_n - \hat\Omega}{1 - \hat G^{\mp}_{nn}(E)(\Omega_n - \hat\Omega)} = 0 , \tag{7}$$

where $\hat G^{\mp}_{nn}(E) = [(E - \Omega_0 - \hat\Omega \pm i\varepsilon)^2 - (J^2 + D^2)]^{-1/2}$ (see Refs. 3 and 4).

In the case of a Lorentzian transverse field Eq. (7) has a solution $\hat\Omega = \mp\, i\Gamma$ and, therefore, $\hat G^{\mp}_{nn}(E)$ coincides with the exact expression (5).

Consider another probability distribution density

$$p(\dots, \Omega_j , \dots) = \prod_{j=1}^{N} \Big[x\delta(\Omega_j) + (1 - x)\, \delta(\Omega_j - \Omega)\Big] ,$$

$$0 \le x \le 1 .$$

Equation (7) will then reduce to a qubic equation for $\hat\Omega$. Its solutions yield the Green's functions and the density of states $\overline{\rho(E)} \approx \mp\, (1/\pi)\ \mathrm{Im}\ \hat G^{\mp}_{nn}(E)$. In Fig. 7 the quantity $R(E^2) = (\overline{\rho(E)} + \overline{\rho(-E)})/2|E|$, which follows from the coherent potential approximation (dashed curves), is compared with the result of exact finite-chain computation of this quantity (solid curves) [13]. A good agreement between approximate and exact results apparently is contingent on the fact that thermodynamic averaging for noninteracting fermions has been performed exactly.

4. Conclusions

We have presented exact calculations of the thermodynamic quantities of spin-½ isotropic XY chain with the Dzyaloshinskii-Moriya interaction in random Lorentzian transverse field. The approach exploits reformulation in terms of fermions and the possibility of averaging exactly the equations for

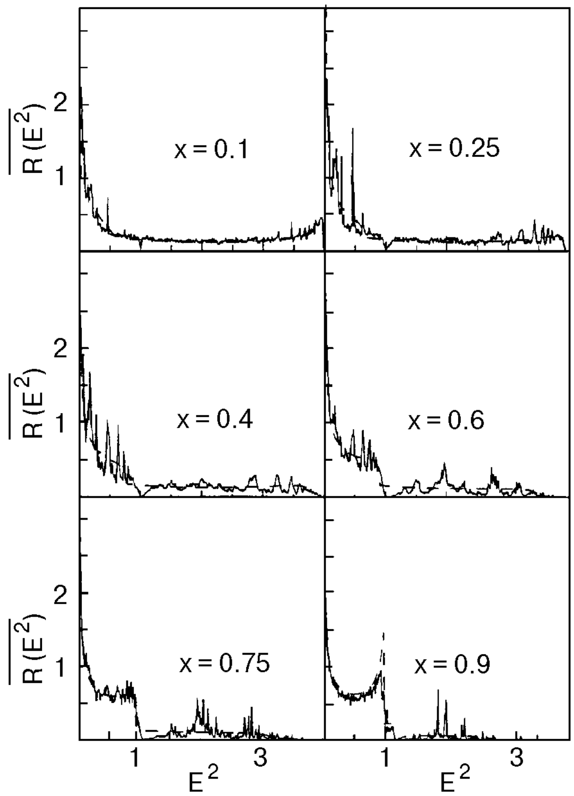

Fig. 7. $\overline{R(E^2)}$ versus E^2: the results of exact calculation (solid curves) and coherent potential approximation (dashed curves).

one-fermion Green's functions that yield thermodynamics. Such scheme is essentially limited by Lorentzian disorder. The results obtained by us supplement to some extent the existing exact analytical results for random spin-½ XY chains [14–20]. The comparison of the density of states obtained within the coherent potential approximation and the exact result illustrates that the region of validity of more sophisticated approaches of disordered spin systems theory can be tested.

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