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CLASSICAL SOLUTIONS OF HYPERBOLIC IBVPS WITH STATE DEPENDENT DELAYS

КЛАСИЧНІ РОЗВ'ЯЗКИ ГІПЕРБОЛІЧНИХ ГРАНИЧНИХ ЗАДАЧ ІЗ ПОЧАТКОВИМИ ДАНИМИ ТА ЗАПІЗНЕННЯМ, ЩО ЗАЛЕЖИТЬ ВІД СТАНУ

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We consider the initial boundary-value problem for a system of quasilinear partial functional differential equations of the first order,

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,V(z;t,x))\partial_{x_j} z_i(t,x) = G_i(t,x,V(z;t,x)), \quad 1 \le i \le m,$$

where V is a nonlinear operator of Volterra type, mapping bounded (w.r.t. seminorm) subsets of the space of Lipschitz-continuously differentiable functions, into bounded subsets of this space.

Using the method of bicharacteristics and the fixed-point theorem we prove the local existence, uniqueness and continuous dependence on data of classical solutions of the problem.

This approach covers systems of the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi(t,x,z_{(t,x)})})\partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \quad 1 \le i \le m,$$

where $(t,x) \mapsto z_{(t,x)}$ is the Hale operator, and all the components of ψ may depend on $(t,x,z_{(t,x)})$. More specifically, problems with deviating arguments and integro-differential systems are included.

Розглядається гранична задача з початковими даними для системи квазілінійних функціонально-диференціальних рівнянь з частинними похідними першого порядку

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,V(z;t,x))\partial_{x_j} z_i(t,x) = G_i(t,x,V(z;t,x)), \quad 1 \le i \le m,$$

де V — нелінійний оператор типу Вольтерра, що відображає обмежені відносно семінорми підмножини простору диференційовних функцій, похідна яких задовольняє умову Ліпшиця, в обмежені підмножини цього простору.

З допомогою методу біфуркацій та теореми про нерухому точку доведено локальне існування класичних розв'язків задачі, їх єдиність та неперервну залежність від даних.

Цей підхід можна застосувати до систем типу

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi(t,x,z_{(t,x)})})\partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \quad 1 \le i \le m,$$

де $(t,x) \mapsto z_{(t,x)}$ *— оператор Хейла і всі компоненти* ψ *можуть залежати від* $(t,x,z_{(t,x)})$*. Зокрема, він може бути застосовним до всіх задач з аргументами, що відхиляються, та інтегро-диференціальних задач.*

1. Introduction. We formulate the functional differential problem. Let $a > 0$, $h_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, and $b = (b_1,\ldots,b_n) \in \mathbb{R}_+^n$, $h = (h_1,\ldots,h_n) \in \mathbb{R}_+^n$ be given. We use the sets

$$E = [0,a] \times [-b,b], \quad D = [-h_0,0] \times [-h,h].$$

Let $\bar{c} = (c_1,\ldots,c_n) = b + h$ and

$$E_0 = [-h_0,0] \times [-\bar{c},\bar{c}], \quad \partial_0 E = [0,a] \times ([-\bar{c},\bar{c}] \setminus (-b,b)), \quad \Omega = E_0 \cup E \cup \partial_0 E.$$

When it does not lead to misunderstading, we write $U_t = U \cap ([-\infty,t] \times \mathbb{R}^n)$ for $U \subset \mathbb{R}^{1+n}$ and $t \in [0,a]$. The symbol U° denotes the interior of U. For k, l being arbitrary positive integers, we denote by $M_{k\times l}$ the class of all $k \times l$ matrices with real elements, and we choose the norms in $\mathbb{R}^k$ and $M_{k\times l}$ to be ∞-norms: $\|y\| = \|y\|_\infty = \max_{1\le i\le k} |y_i|$ and $\|A\| = \|A\|_\infty = \max_{1\le i\le k} \sum_{j=1}^{l} |a_{ij}|$, respectively, where $A = [a_{ij}]_{i=1,\ldots,k,j=1,\ldots,l}$. The product of two matrices is denoted by "$*$". For $U \subset \mathbb{R}^{1+n}$ and a normed space Y, equipped with the norm $\|\cdot\|_Y$, we define $C(U,Y)$ to be the set of all continuous functions $w : U \to Y$; this space is equipped with the usual supremum norm $\|w\|_{C(U,Y)} = \sup_{P\in U} \|w(P)\|_Y$. We write it simply $C(U)$ when no confusion can arise.

Put $X = C(D,\mathbb{R}^m)$. Let $V : C(\Omega,\mathbb{R}^m) \times E \to X$, in variables $(z;t,x)$, be a nonlinear Volterra operator. By the Volterra property we mean that for $z, \bar{z} \in C(\Omega,\mathbb{R}^m)$ and $t \in [0,a]$,

$$z\Big|_{\Omega_t} \equiv \bar{z}\Big|_{\Omega_t} \qquad \text{implies} \qquad V(z;\tau,x) \equiv V(\bar{z};\tau,x) \quad \text{for} \quad (\tau,x) \in E_t.$$

Let

$$\rho_{ij},\ G_i\colon E \times X \to \mathbb{R}, \quad 1 \le i \le m,\ 1 \le j \le n, \quad \text{and} \quad \varphi\colon E_0 \cup \partial_0 E \to \mathbb{R}^m$$

be given. We consider the hyperbolic functional differential system

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,V(z;t,x))\ \partial_{x_j} z_i(t,x) = G_i(t,x,V(z;t,x)), \quad 1 \le i \le m, \tag{1}$$

augmented with the initial boundary condition

$$z(t,x) = \varphi(t,x) \tag{2}$$

on $E_0 \cup \partial_0 E$. A function $\tilde{z} \in C^1(\Omega_c,\mathbb{R})$, where $0 < c \le a$, is a classical solution of (1), (2) if it satisfies (1) on E_c and condition (2) holds on $E_0 \cup \partial_0 E_c$.

Note that different models of the functional dependence in partial equations are used in literature. The first group of results is connected with initial problems for equations

$$\partial_t z(t,x) = G(t,x,z,\partial_x z(t,x)) \tag{3}$$

where the variable z represents the functional argument. This model is suitable for differential functional inequalities generated by initial problems considered on the Haar pyramid. Existence results for (3) can be characterized as follows: theorems have simple assumptions and their proofs are very natural (see [1, 2]). Unfortunately, a small class of differential functional problems is covered by this theory. There are a lot of papers concerning initial value problems for equations

$$\partial_t z(t,x) = H(t,x,W[z](t,x),\partial_x z(t,x)) \tag{4}$$

where W is an operator of Volterra type and H is defined on finite-dimensional Euclidean space. The main assumptions in existence theorems for (4) concern the operator W. They are formulated [3, 4] in terms of inequalities for norms in some functional spaces.

A new model of a functional dependence is proposed in [5, 6]. Partial equations have the form

$$\partial_t z(t,x) = F(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{5}$$

where $z_{(t,x)}$ is a functional variable. This model is well-known for ordinary functional differential equations (see, for example, [7–9]). It is also very general since equations with deviating variables, integral differential equations, and equations of forms (3) and (4) can be obtained from (5) by specifying the operator F. In the paper we use the model (5). In existence results, concerning partial differential equations with state dependent delays [10–12], Carathéodory type or semiclassical solutions were considered and the functional variable was

$$z_{(\psi_0(t),\psi'(t,x,z_{(t,x)}))}.$$

We deal in this paper with a slightly wider class of deviating functions, admitting functional variable of the form

$$z_{(\psi_0(t,x,z_{(t,x)}),\psi'(t,x,z_{(t,x)}))}$$

and we consider classical solutions of the respective problem. Cases of more (or less) complicated deviating functions are also covered by our operator formulation.

Delay systems with state dependent delays occur as models for the dynamics of diseases when the mechanism of infection is such that the infectious dosage received by an individual has to reach a threshold before the resistance of the individual is broken down and as a result the individual becomes infectious. A prototype of such model was proposed in [13].

The aim of this paper is to prove a theorem on the existence and continuous dependence of classical solutions to (1), (2). The paper is organized as follows. In Section 2 we prove a result on the existence and regularity of bicharacteristics, having assumed our conditions on the operator V. In the next section, the method of bicharacteristics is used to transform the Cauchy problem into a system of integral equations. A fixed-point equation is constructed. The Section 4 contains the main result. Application of our approach to the systems with state dependent delays is described in the last section.

2. Bicharacteristics. Let $U \subset \mathbb{R}^{1+n}$ and k be a positive integer. For $z : U \to \mathbb{R}^k$ and $(t,x) \in U$, denote

$$\partial_x z(t,x) = [\partial_{x_j} z_i(t,x)]_{i=1,\ldots,k, j=1,\ldots,n} \in M_{k\times n}$$

and

$$\partial z(t,x) = [\partial_{x_j} z_i(t,x)]_{i=1,\ldots,k, j=0,\ldots,n} \in M_{k\times (n+1)},$$

where $\partial_{x_0} \equiv \partial_t$.

For a fixed $p \in \mathbb{R}_+$, we consider the space

$$C^1[p](U,\mathbb{R}^k) = \left\{ w \in C(U,\mathbb{R}) : w \text{ is continuously differentiable on } U^\circ \text{ and } \|\partial w\|_{C(U^\circ)} \le p \right\}.$$

Similarly, for $p = (p_1, p_2) \in \mathbb{R}^2_+$, we define

$$C^{1,L}[p](U,\mathbb{R}^k) = \left\{ w \in C^1[p_1](U,\mathbb{R}^k) : |\partial w|_{C^{0,L}(U^\circ)} \le p_2 \right\},$$

where $|z|_{C^{0,L}(U)} = \sup_{P\neq \bar P; P,\bar P\in U} \|z(P) - z(\bar P)\| \cdot \|P - \bar P\|^{-1}$. We denote

$$C^{1,L}(U,\mathbb{R}^k) = \bigcup_{p\in\mathbb{R}^2_+} C^{1,L}[p](U,\mathbb{R}^k).$$

We are now able to define the function space, in which we seek the solutions to (1), (2). The symbol $C^{1,L}_\partial[p]$ is short for $C^{1,L}[p](E_0 \cup \partial_0 E, \mathbb{R}^m)$. Given $p \in \mathbb{R}^2_+$, $\varphi \in C^{1,L}_\partial[p]$, and $d \in \mathbb{R}^2_+$ such that $d_j \ge p_j$, $j = 1, 2$, we set

$$C^{1,L}_{\varphi,c}[d] = \left\{ z \in C^{1,L}[d](\Omega_c, \mathbb{R}^m) : z \equiv \varphi \text{ on } E_0 \cup \partial_0 E_c \right\}.$$

We prove that under suitable assumptions on ρ, G, V, φ, on the parameters p, d, and for sufficiently small $c \in (0,a]$, there exists a solution $\bar z$ of problem (1), (2) such that $\bar z \in C^{1,L}_{\varphi,c}[d]$.

Let Y stand for $C(D, M_{m\times(n+1)})$. Write $\rho_i = (\rho_{i1}, \ldots, \rho_{in})$, $1 \le i \le m$. For the convenience of calculations, we consider m Fréchet derivatives $\partial_w \rho_i(t,x,w) \in L(X, \mathbb{R}^n)$, $1 \le i \le m$, rather than mn Fréchet derivatives $\partial_w \rho_{ij}(t,x,w)$, $1 \le i \le m$, $1 \le j \le n$. We are interested in estimating it in the norm $\|\cdot\|_{L(Y, M_{n\times(n+1)})}$, since we use the notation

$$\partial_w \rho_i(t,x,w)\delta = \Big(\partial_w \rho_i(t,x,w)\delta_0, \ldots, \partial_w \rho_i(t,x,w)\delta_n\Big) \in M_{n\times(n+1)} \tag{6}$$

for $\delta \in Y$, $\delta = (\delta_0, \ldots, \delta_n)$, $\delta_j \in X$, $0 \le j \le n$.

Assumption H [ρ]. Suppose that $\rho\colon E\times X \to M_{m\times n}$, in the variables (t,x,w), is continuous and

1) the derivatives: $\partial_x \rho_i(t,x,w)$ and the Fréchet derivative $\partial_w \rho_i(t,x,w)$ exist for $(t,x,w) \in E \times C^{1,L}(D, \mathbb{R}^m)$, $1 \le i \le m$,

2) for $1 \le i \le m$, $\partial_x \rho_i$ and $\partial_w \rho_i$ are continuous in t on $E \times C^{1,L}(D, \mathbb{R}^m)$,

3) there is a non-negative constant A such that, for $1 \le i \le m$,

$$\|\rho_i(t,x,w)\|, \ \|\partial_x \rho_i(t,x,w)\|, \ \|\partial_w \rho_i(t,x,w)\|_{L(Y, M_{n\times(n+1)})} \le A \quad \text{on} \quad E \times C^{1,L}(D, \mathbb{R}^m)$$

and

$$\|\partial_x \rho_i(t,x,w) - \partial_x \rho_i(t,\bar{x},\bar{w})\|,$$

$$\|\partial_w \rho_i(t,x,w) - \partial_w \rho_i(t,\bar{x},\bar{w})\|_{L(Y,M_{n\times(n+1)})} \leq A\big(\|x-\bar{x}\| + \|w-\bar{w}\|_X\big)$$

for $(t,x,w), (t,\bar{x},\bar{w}) \in E \times C^{1,L}(D,\mathbb{R}^m)$,

4) there is $\kappa > 0$ such that, for $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\rho_{ij}(t,x,w) < -\kappa \quad \text{on} \quad [0,a] \times \{b_j\} \times C^{1,L}(D,\mathbb{R}^m),$$

$$\rho_{ij}(t,x,w) > \quad \kappa \quad \text{on} \quad [0,a] \times \{-b_j\} \times C^{1,L}(D,\mathbb{R}^m).$$

Assumption H [V]. The operator $V : C(\Omega,\mathbb{R}^m) \times E \to X$ is such that for every $d \in \mathbb{R}^2_+$ there are $\bar{d} \in \mathbb{R}^2_+$, $L \in \mathbb{R}_+$ such that:

1) for $z \in C^1[d_1](\Omega,\mathbb{R}^m)$, $(t,x) \in E$,

$$\|\partial V(z;t,x)\|_Y \leq \bar{d}_1,$$

2) for $z \in C^{1,L}[d](\Omega,\mathbb{R}^m)$ and (t,x), $(t,\bar{x})$ in E,

$$\|\partial V(z;t,x)\|_Y \leq \bar{d}_1, \quad \|\partial V(z;t,x) - \partial V(z;t,\bar{x})\|_Y \leq \bar{d}_2 \|x-\bar{x}\|,$$

3) for every $z, \bar{z} \in C^1[d_1](\Omega,\mathbb{R}^m)$ and $(t,x) \in E$,

$$\|V(z;t,x) - V(\bar{z};t,x)\|_X \leq L\|z-\bar{z}\|_{C(E_t)}.$$

Suppose that $\varphi \in C^{1,L}_\partial[p]$ and $z \in C^{1,L}_{\varphi,c}[d]$. For $1 \leq i \leq m$, and a point $(t,x) \in E_c$, we consider the Cauchy problem

$$\eta'(\tau) = \rho_i(\tau,\eta(\tau),V(z;\tau,\eta(\tau))), \quad \eta(t) = x, \tag{7}$$

and denote by $g_i[z](\cdot,t,x) = (g_{i1}[z](\cdot,t,x),\ldots,g_{in}[z](\cdot,t,x))$ its classical solution. This function is the bicharacteristic of the i-th equation of (1), corresponding to z. Let $\delta_i[z](t,x)$ be the left end of the maximal interval on which the bicharacteristic $g_i[z](\cdot,t,x)$ is defined. Write

$$Q_i[z](\tau,t,x) = (\tau, g_i[z](\tau,t,x), V(z;\tau,g_i[z](\tau,t,x))).$$

We prove a lemma on bicharacteristics.

Lemma 2.1. *Suppose that Assumptions $H[\rho]$, $H[V]$ are satisfied and let φ, $\bar{\varphi} \in C^{1,L}_\partial[p]$ and $z \in C^{1,L}_{\varphi,c}[d]$, $\bar{z} \in C^{1,L}_{\bar{\varphi}.c}[d]$, be given. Then, for $1 \leq i \leq m$, the solutions $g_i[z](\cdot,t,x)$ and $g_i[\bar{z}](\cdot,t,x)$ exist on intervals $[\delta_i[z](t,x),c]$ and $[\delta_i[\bar{z}](t,x),c]$, respectively, and are unique. If $\xi = \delta_i[z](t,x) > 0$ then $g_i[z](\xi,t,x) \in \partial_0 E \cap E$. Moreover, the estimates*

$$\|\partial g_i[z](\tau,t,x)\| \leq C, \qquad \|\partial g_i[z](\tau,t,x) - \partial g_i[z](\tau,\bar{t},\bar{x})\| \leq Q\max\{|t-\bar{t}|, \|x-\bar{x}\|\} \tag{8}$$

and

$$\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \le \bar{A}\left|\int_t^\tau \|z-\bar{z}\|_{C(E_s)}\,ds\right|, \tag{9}$$

$$|\delta_i[z](t,x) - \delta_i[\bar{z}](t,x)|, \le 2\bar{A}\kappa^{-1}\left|\int_0^t \|z-\bar{z}\|_{C(E_s)}\,ds\right| \tag{10}$$

hold with constants depending only on data and on c, d, p :

$$C = (A+1)e^{cB}, \quad Q = [(1+C)B + \widetilde{C}]e^{cB}, \quad \bar{A} = ALe^{cB}, \tag{11}$$

where

$$B = A(1+\bar{d}_1), \quad \widetilde{C} = C^2 cA[(1+\bar{d}_1)^2 + \bar{d}_2] \tag{12}$$

and $\bar{d} = (\bar{d}_1, \bar{d}_2) \in \mathbb{R}^2_+$ *is the parameter from Assumption* $H[V]$, *corresponding to* d.

Proof. Let $z \in C^{1,L}_{\varphi,c}[d]$. The existence and uniqueness of solutions of (7) follow from the theorem on classical solutions of initial problems. From another classical theorem on differentiation of solutions with respect to the initial data it follows that the derivative $\partial g_i[z]$ exists and fulfils the integral equations

$$\partial g_i[z](\tau,t,x) = \big[\,-\rho_i(t,x,V(z;t,x))\,\big|\,I\,\big] + \\ + \int_t^\tau \Big[\partial_x\rho_i(Q_i[z](s,t,x)) + \partial_w\rho_i(Q_i[z](s,t,x))\partial V(z;s,g_i[z](s,t,x))\Big] * \\ * \,\partial g_i[z](s,t,x)ds \tag{13}$$

where $\big[\,-\rho_i(t,x,V(z;t,x))\,\big|\,I\,\big]$ denotes concatenation of the matrix $-\rho_i(t,x,V(z;t,x))$ with the identity matrix. It follows from (13), from Assumptions H[ρ], H[V] that $\partial g_i[z](\cdot,t,x)$ satisfy the integral inequality

$$\|\partial g_i[z](\tau,t,x)\| \le A + 1 + B\left|\int_t^\tau \|\partial g_i[z](s,t,x)\|ds\right|,$$

and from the Gronwall lemma we get the first estimate in (8). Hence we derive the inequality

$$\|\partial g_i[z](\tau,t,x) - \partial g_i[z](\tau,\bar{t},\bar{x})\| \le (B+\widetilde{C})\max\{|t-\bar{t}|, \|x-\bar{x}\|\} + CB|t-\bar{t}| + \\ + B\left|\int_t^\tau \|\partial g_i[z](s,t,x) - \partial g_i[z](s,\bar{t},\bar{x})\|ds\right|$$

which, by the Gronwall lemma, implies that

$$\|\partial g_i[z](\tau,t,x) - \partial g_i[z](\tau,\bar t,\bar x)\| \le Q_1 \max\{|t-\bar t|, \|x-\bar x\|\} + Q_0|t-\bar t| \le$$
$$\le (Q_0+Q_1)\max\{|t-\bar t|, \|x-\bar x\|\},$$

with $Q_0 = CB\exp(cB)$ and $Q_1 = (\widetilde{C}+B)\exp(cB)$, yielding the second estimate in (8).

We now prove (9). The function $g_i[z](\tau,t,x)$ satisfies the following relation:

$$g_i[z](\tau,t,x) = x + \int_t^\tau \rho_i(s, g_i[z](s,t,x), V(z;s,g_i[z](s,t,x)))ds.$$

This leads to

$$\|g_i[z](\tau,t,x) - g_i[\bar z](\tau,t,x)\| \le B\left|\int_t^\tau \|g_i[z](s,t,x) - g_i[\bar z](s,t,x)\|ds\right| +$$
$$+ AL\left|\int_t^\tau \|z-\bar z\|_{C(E_s)}\, ds\right|.$$

Again from the Gronwall inequality we obtain

$$\|g_i[z](\tau,t,x) - g_i[\bar z](\tau,t,x)\| \le AL\exp(cB)\left|\int_t^\tau \|z-\bar z\|_{C(E_s)}\, ds\right|,$$

and hence (9).

Now we proceed to the proof of (10), fixing $(t,x) \in E_c$ and beginning by a local version of the estimate, that is, under the condition that

$$\|z-\bar z\|_{C(E_t)} \le \frac{\kappa}{2(cB\bar A + AL)}. \tag{14}$$

Since (10) is obvious if both values of δ are zero, we may assume that $0 \le \delta_i[z](t,x) < \delta_i[\bar z](t,x)$. Denoting $\zeta = \delta_i[\bar z](t,x)$, we then have $|g_{ij}[\bar z](\zeta,t,x)| = b_j$ for some coordinate j. Let us focus on the case $g_{ij}[\bar z](\zeta,t,x) = b_j$; the opposite case $g_{ij}[\bar z](\zeta,t,x) = -b_j$ is then treated analogously. By virtue of (9) and (14),

$$|\rho_{ij}(Q_i[z](\zeta,t,x)) - \rho_{ij}(Q_i[\bar z](\zeta,t,x))| \le B\|g_i[z](\zeta,t,x) - g_i[\bar z](\zeta,t,x)\| + AL\|z-\bar z\|_{C(E_t)} \le$$
$$\le B\bar A\int_0^t \|z-\bar z\|_{C(E_s)}\, ds + AL\|z-\bar z\|_{C(E_t)} \le \frac{\kappa}{2}.$$

This, together with the condition 4 of Assumption H $[\rho]$ applied to $\rho_{ij}(Q_i[\bar{z}](\zeta,t,x))$, gives

$$\rho_{ij}(Q_i[z](\zeta,t,x)) \le \rho_{ij}(Q_i[\bar{z}](\zeta,t,x)) +$$
$$+ |\rho_{ij}(Q_i[z](\zeta,t,x)) - \rho_{ij}(Q_i[\bar{z}](\zeta,t,x))| < -\kappa + \frac{\kappa}{2} = -\frac{\kappa}{2}.$$

Consequently,

$$\partial_t g_{ij}[z](\zeta,t,x) < -\frac{\kappa}{2} < 0,$$

and hence $g_{ij}[z](\cdot,t,x)$ is decreasing on the interval (ξ,ζ) for some $\xi \in [\delta_i[z](t,x),\zeta)$. This fact, and the estimate

$$b_j - g_{ij}[z](\zeta,t,x) = g_{ij}[\bar{z}](\zeta,t,x) - g_{ij}[z](\zeta,t,x) \le$$
$$\le c\bar{A}\|z-\bar{z}\|_{C(E_t)} \le \frac{\kappa c\bar{A}}{2(cB\bar{A}+AL)} \le \frac{\kappa}{2B} \le \frac{\kappa}{2A},$$

imply

$$b_j - g_{ij}[z](s,t,x) \le \frac{\kappa}{2A} \quad \text{for} \quad s \in (\xi,\zeta].$$

Let us now define $\beta : \mathbb{R}^n \to \mathbb{R}^n$ by $\beta(x) = (x_1,\ldots,x_{j-1},b_j,x_{j+1},\ldots,x_n)$ and notice that the condition 4 of Assumption H $[\rho]$ may be applied to give

$$\rho_{ij}(s,\beta(g_i[z](s,t,x)),V(z;s,g_i[z](s,t,x))) < -\kappa, \qquad s \in (\xi,\zeta].$$

Then, for $s \in (\xi,\zeta]$,

$$\rho_{ij}(Q_i[z](s,t,x)) = \rho_{ij}(s,g_i[z](s,t,x),V(z;s,g_i[z](s,t,x))) \le$$
$$\le \rho_{ij}(s,\beta(g_i[z](s,t,x)),V(z;s,g_i[z](s,t,x))) +$$
$$+ A\big(b_j - g_{ij}[z](s,t,x)\big) \le$$
$$\le -\kappa + A\frac{\kappa}{2A} = -\frac{\kappa}{2}.$$

Note that the last inequality implies $\xi = \delta_i[z](t,x)$, otherwise it would be $0 = \partial_t g_{ij}[z](\xi,t,x) = \rho_{ij}(Q_i[z](\xi,t,x))$ for some $\xi \in (\delta_i[z](t,x),\zeta)$. Hence this inequality holds for $s \in [\delta_i[z](t,x), \delta_i[\bar{z}](t,x)]$, yielding

$$-\frac{\kappa}{2}\big(\delta_i[\bar z](t,x)-\delta_i[z](t,x)\big) \geq$$

$$\geq \int\limits_{\delta_i[z](t,x)}^{\delta_i[\bar z](t,x)} \rho_{ij}(Q_i[z](s,t,x))\,ds =$$

$$= g_{ij}[z](\delta_i[\bar z](t,x),t,x) - g_{ij}[z](\delta_i[z](t,x),t,x) \geq$$

$$\geq g_{ij}[z](\delta_i[\bar z](t,x),t,x) - g_{ij}[\bar z](\delta_i[\bar z](t,x),t,x) \geq$$

$$\geq -\bar A \left| \int\limits_{t}^{\delta_i[\bar z](t,x)} \|z-\bar z\|_{C(E_s)}\,ds \right| \geq$$

$$\geq -\bar A \int\limits_0^t \|z-\bar z\|_{C(E_s)}\,ds.$$

Since this proves (10) for z, $\bar z$ satisfying (14), the argument of convexity of $C^{1,L}_{\partial}[p]$ and $C^{1,L}[d'](\Omega_c,\mathbb{R}^m)$ completes the proof.

Lemma 2.2. *Suppose that Assumptions $H[\rho]$, $H[V]$ are satisfied and let $\varphi \in C^{1,L}_{\partial}[p]$, $z \in$ $\in C^{1,L}_{\varphi,c}[d]$ be given. Then, for $1 \leq i \leq m$,*

$$\delta_i[z] \in C(E_c) \cap C^1[\widetilde B](E_c^{(i,\partial)}[z],\mathbb{R}),$$

where

$$E_c^{(i,\partial)}[z] = \{(t,x) \in E_c : \delta_i[z](t,x) > 0\}^\circ \quad and \quad \widetilde B = C\kappa^{-1}.$$

Proof. Fix $z \in C^{1,L}_{\varphi,c}[d]$ and i, $1 \leq i \leq m$. Once it is done, we may introduce the notation: $f = \delta_i[z]$, U for the set $E_c^{(i,\partial)}[z]$, and W for the set

$$E_c^{(i,0)}[z] = \{(t,x) \in E_c : \delta_i[z](t,x) = 0\}.$$

We first prove that $f \in C^1[\widetilde B](U,\mathbb{R})$. Let us temporarily fix $(t,x) \in U$ and set $\xi = f(t,x)$. Since by the Lemma 2.1, $g_i[z]$ is of class C^1 in all variables, and by Assumption H[ρ], $\partial_\tau g_i[z](\tau,t,x) \neq$ $\neq 0$ at (ξ,t,x), the existence and continuity of the gradient ∂f at the point (t,x) follow from the implicit function theorem applied to $\pm b_j - g_{ij}[z](\xi,t,x) = 0$, $1 \leq j \leq n$. By the same token, fixing j, we may calculate the gradient with the help of the formula

$$\partial_{x_k} f(t,x) = -\frac{\partial_{x_k} g_{ij}[z](\xi,t,x)}{\rho_{ij}(Q_i[z](\xi,t,x))}, \qquad 0 \leq k \leq n,$$

and estimate it by

$$\|\partial f\|_{C(U)} \leq \frac{C}{\kappa},$$

independently of i, j.

Remark that

$$E_c = U \cup W \cup \{c\} \times (-b, b) \cup \bigcup_{k=1}^{n} \Delta^{(k)},$$

where $\Delta^{(k)} = \{(t, x) \in E_c : |x_k| = b_k\}$, $1 \le k \le n$. The continuity of f on W is obvious, and on U follows from differentiability. Extending ρ, V and z in a natural way, we could replace E_c with $[-\varepsilon, c] \times [-b, b]$ in the formulation of (7) and in the consequent results on bicharacteristics — including differentiability of the new function $\tilde{f} = \tilde{\delta}_i[z]$ on the set analogous to $E_c^{(i,\partial)}[z]$ and containing $\tilde{f}^{-1}(\{0\})$. By the uniqueness of bicharacteristics, $\tilde{f}$ is identical with f on $\overline{U}$. Hence, for $(t, x) \in \tilde{f}^{-1}(\{0\}) = \overline{U} \cap W$, the one-sided limit vanishes:

$$\lim_{\substack{(\bar{t},\bar{x}) \to (t,x) \\ (\bar{t},\bar{x}) \in U}} f(\bar{t}, \bar{x}) = 0.$$

The continuity of f on $U \cup W$ follows now from the fact that f vanishes on the other side, i.e., on $W = \overline{W}$. Then, by extending the data in the opposite direction, we get the continuity on $U \cup W \cup \{c\} \times (-b, b)$. Finally, from the condition 4 of the Assumption H[ρ] follows easily that f satisfies the local Lipschitz condition, with the uniform constant $\max\{1, C\kappa^{-1}\}$, at each point of $\Delta^{(k)}$, $1 \le k \le n$. This shows that $f \in C(E_c)$ and completes the proof.

3. Functional integral system. Let W stand for $L(C(D, M_{m\times n}), M_{1\times n})$. The expression $\partial_w G_i(t, x, w)\delta$, for $\delta \in C(D, M_{m\times n})$, is to be interpreted in a way analogous to (6); for the sake of simplicity of calculations, we use $\|\cdot\|_W$ (rather than $\|\cdot\|_{L(X,\mathbb{R})}$) for measuring the values of $\partial_w G_i$.

Assumption H [ρ, G]. The Assumption H[ρ] is fulfilled, $G\colon E \times X \to \mathbb{R}^m$, in the variables (t, x, w), is continuous and, for $1 \le i \le m$,

1) the derivative $\partial_x G(t, x, w)$ and the Fréchet derivative $\partial_w G(t, x, w)$ exist for $(t, x, w) \in E \times C^{1,L}(D, \mathbb{R}^m)$,

2) for $(t, x, w), (t, \bar{x}, \bar{w}) \in E \times C^{1,L}(D, \mathbb{R}^m)$,

$$\|G(t, x, w)\|, \quad \|\partial_x G(t, x, w)\|, \quad \|\partial_w G_i(t, x, w)\|_W \le A,$$

$$\|G_i(t, x, w) - G_i(\bar{t}, x, w)\| \le A|t - \bar{t}|,$$

$$\|\partial_x G_i(t, x, w) - \partial_x G_i(t, \bar{x}, \bar{w})\|, \quad \|\partial_w G_i(t, x, w) - \partial_w G_i(t, \bar{x}, \bar{w})\|_W \le A\big(\|x - \bar{x}\| + \|w - \bar{w}\|_X\big)$$

with the same constant A as in the Assumption H[ρ].

Write

$$S_i[z](t, x) = (\, \delta_i[z](t, x),\ g_i[z](\delta_i[z](t, x), t, x)\,).$$

We define the operator $F = (F_1, \ldots, F_m)$ on $C^{1,L}_{\varphi,c}[d]$ by the formula

$$F_i z(t, x) = \varphi_i(S_i[z](t, x)) + \int_{\delta_i[z](t,x)}^{t} G_i(Q_i[z](s, t, x))ds \quad \text{on} \quad E_c, \quad 1 \le i \le m, \tag{15}$$

$$Fz \equiv \varphi \quad \text{on} \quad E_0 \cup \partial_0 E_c.$$

Remark 3.1. The right-hand side of (15) is obtained in the following way. We consider each equation of (1) along its bicharacteristic:

$$\partial_t z_i(\tau, g_i[z](\tau,t,x)) + \partial_x z_i(\tau, g_i[z](\tau,t,x)) * \rho_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x))) = \\ = G_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x)))$$

from which, using (7), we get

$$\frac{d}{d\tau} z_i(\tau, g_i[z](\tau,t,x)) = G_i(\tau, g_i[z](\tau,t,x), V(z;\tau, g_i[z](\tau,t,x))).$$

By integrating the latter equation with respect to τ, and adding the initial value, we get the right-hand side of (15).

Assumption H[c, d, V]. The Assumption H[V] is fulfilled, and the constant $c \in (0, a]$ is small enough so to satisfy, together with d and p,

$$d_1 \geq p_1 C + A + cCB, \tag{16}$$

$$d_2 \geq p_1(Q + B\widehat{C}) + p_2(C^2 + \widehat{A}\widehat{C}) + B + cBQ + \widetilde{C} + 2B\widehat{C}, \tag{17}$$

with constants $B, C, \widetilde{C}, Q$ defined in (11), (12), and with $\widehat{C} = C\max\{1, C\kappa^{-1}\}$, $\widehat{A} = \max\{1, A\}$.

Write $\mathcal{I} = \{0 \leq i \leq n : h_i > 0\}$. The following compatibility condition for the problem (1), (2) will be needed in our considerations.

Assumption H$_c$[G, φ]. The equivalence

$$G(t, x, V(z;t,x)) = G(t, x, V(\bar{z};t,x)) \quad \text{on} \quad \partial_0 E \cap E$$

holds for any $z, \bar{z} \in C^{1,L}_{\varphi,a}[d]$. Moreover, there is $\psi \in C(\partial_0 E, M_{m\times n})$ such that, for each k, $1 \leq k \leq n$, the system of equations

$$\partial_t \varphi_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,V(z;t,x))\psi_{ij}(t,x) = G_i(t,x,V(z;t,x)), \quad 1 \leq i \leq m, \tag{18}$$

holds on $\Delta^{(k)} = \{(t,x) \in E : |x_k| = b_k\}$ with

$$\psi_{ij} \equiv \partial_{x_j}\varphi_i \quad \text{on} \quad \Delta^{(k)} \quad \text{whenever} \quad j \in \mathcal{I}.$$

Additionally, if $0 \in \mathcal{I}$, then on $\{0\} \times (-b, b)$ holds

$$\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x) * \rho_i(t,x,V(z;t,x)) = G_i(t,x,V(z;t,x)), \quad 1 \leq i \leq m. \tag{19}$$

Remark 3.2. Relation (18) may be considered as an assumption on φ on $\bigcup_{k\in\mathcal{I}} \Delta^{(k)}$ and (18) defines the number $\partial_{x_j}\varphi_i(t,x)$ at the points where there is not enough space to define the partial derivative.

Remark 3.3. Let us explain our application of the chain rule to the term $\varphi_i(S_i[z](t,x))$, made in the next proof. We write

$$E_c^{(ij,\partial)}[z] = \left\{(t,x) \in E_c^{(i,\partial)}[z] : S_i[z](t,x) \in (\Delta^{(j)})^{\circ}\right\}, \qquad 1 \le j \le n.$$

Note that for $(t,x) \in E_c^{(ij,\partial)}[z]$ we shall not use the partial derivative $\partial_{x_j}\varphi_i$ in the expansion of the differential

$$\frac{d}{dx_k}\varphi_i(S_i[z](t,x)) \qquad \big(k \text{ is fixed},\ 0 \le k \le n\big).$$

Fortunately, for those (t,x) the differential

$$\frac{d}{dx_k}g_{ij}[z](\delta_i[z](t,x),t,x) = 0, \qquad 0 \le k \le n,$$

and the number $\partial_{x_j}\varphi_i(S_i[z](t,x))$ is defined even for $j \notin \mathcal{I}$ (by compatibility condition), thus it is justified to write

$$\frac{d}{dx_k}\varphi_i(S_i[z](t,x)) = \partial_t\varphi_i(S_i[z](t,x))\partial_{x_k}\delta_i[z](t,x)+$$

$$+\sum_{j=1}^{n}\partial_{x_j}\varphi_i(S_i[z](t,x))\frac{d}{dx_k}g_{ij}[z](\delta_i[z](t,x),t,x).$$

Lemma 3.1. *Suppose that Assumptions $H[\rho,G]$, $H[c,d,V]$, $H_c[G,\varphi]$ are satisfied. Then the operator F maps $C^{1,L}_{\varphi,c}[d]$ into itself.*

Proof. Let $z \in C^{1,L}_{\varphi,c}[d]$. Write

$$\Phi_i[z](s,t,x) = \partial_x G_i(Q_i[z](s,t,x)) + \partial_w G_i(Q_i[z](s,t,x))\ \partial_x V(z;s,g_i[z](s,t,x)),$$

where $\partial_w G_i(Q_i)\partial_x V(z;\tau,y)$ is to be interpreted column-wise. Fix $(t,x) \in E_c$. Once it is done, we may introduce the notation $g_i = g_i[z](\cdot,t,x)$ and $\delta_i = \delta_i[z](t,x)$. From (15) and by the Remark 3.3, for $(t,x) \in E_c^{(i,\partial)}[z]$,

$$\partial F_i z(t,x) =$$

$$= \Big[\partial_t\varphi_i(\delta_i,g_i(\delta_i)) + \partial_x\varphi_i(\delta_i,g_i(\delta_i)) * \rho_i(Q_i[z](\delta_i,t,x)) - G_i(Q_i[z](\delta_i,t,x))\Big]\partial\delta_i[z](t,x)+$$

$$+\,\partial_x\varphi_i(\delta_i,g_i(\delta_i)) * \partial g_i[z](\delta_i,t,x) + \big[\,G_i(t,x,V(z;t,x))\,\big|\,0\,\big]+$$

$$+\int_{\delta_i[z](t,x)}^{t}\Phi_i[z](s,t,x) * \partial g_i[z](s,t,x)\,ds, \tag{20}$$

where $\left[\ G_i(t,x,V(z;t,x))\ \middle|\ 0\ \right] = \big(G_i(t,x,V(z;t,x)),\ 0,\ \ldots,\ 0\big) \in \mathbb{R}^{1+n}$. Moreover, on $E_c^{(i,0)}[z] \cap E_c^\circ$,

$$\partial F_i z(t,x) = \partial_x \varphi_i(0, g_i(0)) * \partial g_i[z](0,t,x) + \left[\ G_i(t,x,V(z;t,x))\ \middle|\ 0\ \right] + \\ + \int_0^t \Phi_i[z](s,t,x) * \partial g_i[z](s,t,x)\, ds. \tag{21}$$

Due to the compatibility condition, and by the continuity of $\delta_i[z]$,

$$\partial F_i z(t,x) = \partial_x \varphi_i(\delta_i, g_i(\delta_i)) * \partial g_i[z](\delta_i,t,x) + \left[\ G_i(t,x,V(z;t,x))\ \middle|\ 0\ \right] + \\ + \int_{\delta_i[z](t,x)}^t \Phi_i[z](s,t,x) * \partial g_i[z](s,t,x)\, ds, \tag{22}$$

on E_c°.

It follows that $\|\partial F z(t,x)\| \le p_1 C + CBc + A$ on E_c°, which, by the Assumption H$[c,d,V]$, implies $\|\partial F z\|_{C(E_c^\circ)} \le d_1$. Furthermore, for $1 \le i \le m$ and for $(t,x), (\bar t, \bar x) \in E_c^\circ$,

$$\|\partial F_i z(t,x) - \partial F_i z(\bar t,\bar x)\| \le \\ \le \Big\| \partial_x \varphi_i(S_i[z](t,x)) * \partial g_i[z](\delta_i[z](t,x),t,x) - \partial_x \varphi_i(S_i[z](\bar t,\bar x)) * \partial g_i[z](\delta_i[z](\bar t,\bar x),\bar t,\bar x)\Big\| + \\ + \Big| G_i(t,x,V(z;t,x)) - G_i(\bar t,\bar x,V(z;\bar t,\bar x))\Big| + \\ + \int_{\delta_i[z](t,x)}^t \Big\| \Phi_i[z](s,t,x) * \partial g_i[z](s,t,x) - \Phi_i[z](s,\bar t,\bar x) * \partial g_i[z](s,\bar t,\bar x)\Big\| ds + \\ + \left| \int_t^{\bar t} \Big\| \Phi_i[z](s,\bar t,\bar x) * \partial g_i[z](s,\bar t,\bar x)\Big\| ds \right| + \left| \int_{\delta_i[z](t,x)}^{\delta_i[z](\bar t,\bar x)} \Big\| \Phi_i[z](s,\bar t,\bar x) * \partial g_i[z](s,\bar t,\bar x)\Big\| ds \right|.$$

Note that the Lemma 2.2 gives

$$|\delta_i[z](t,x) - \delta_i[z](\bar t,\bar x)| \le C\kappa^{-1} \max\{|t-\bar t|, \|x-\bar x\|\}.$$

From the above inequalities, Assumption H$[\rho, G]$ and Lemma 2.1 it follows that

$$\|\partial F_i z(t,x) - \partial F_i z(\bar t,\bar x)\| \le \\ \le \big(p_1(Q + B\widehat{C}) + p_2(C^2 + \widehat{A}\widehat{C}) + B + cBQ + \widetilde{C} + B\widehat{C}\big) \max\{|t-\bar t|, \|x-\bar x\|\} + BC|t-\bar t|,$$

for $(t,x), (\bar{t},\bar{x}) \in E_c^\circ$, which, in view of the second inequality from the Assumption H$[c,d,V]$, gives $|\partial Fz|_{C^{0,L}(E_c^\circ)} \leq d_2$.

The fact that F_iz are continuous extensions of φ_i, is a simple consequence of the definition (15); it remains to prove that this extension is of class C^1. From (13), (22), and from the compatibility condition (18) we obtain for $(t,x) \in \Delta^{(k)}$, $1 \leq k \leq n$,

$$\lim_{\substack{(\bar{t},\bar{x})\to(t,x)\\(\bar{t},\bar{x})\in E_c^\circ}} \partial F_iz(\bar{t},\bar{x}) = \partial_x\varphi_i(t,x) * \partial g_i[z](t,t,x) + \left[\, G_i(t,x,V(z;t,x)) \,\middle|\, 0 \,\right] =$$

$$= \partial_x\varphi_i(t,x) * \left[\, -\rho_i(t,x,V(z;t,x)) \,\middle|\, I \,\right] + \left[\, G_i(t,x,V(z;t,x)) \,\middle|\, 0 \,\right] =$$

$$= \partial\varphi_i(t,x), \quad 1 \leq i \leq m.$$

If $0 \in \mathcal{I}$, then similar arguments, incurring (19), apply to the case $(t,x) \in \{0\} \times (-b,b)$.

Lemma 3.1 is proved.

4. Existence of solutions.

Theorem 4.1. *Suppose that $\varphi \in C_\partial^{1,L}[p]$, and Assumptions $H[\rho,G]$, $H[c,d,V]$, $H_c[G,\varphi]$ are satisfied. Then there exists exactly one solution $\bar{z} \in C_{\varphi,c}^{1,L}[d]$ of problem (1), (2). Moreover, there is $\Lambda_c \in \mathbb{R}_+$ such that*

$$\|\bar{z} - v\|_{C(E_t)} \leq \Lambda_c \|\varphi - \psi\|_{C(E_0\cup\partial_0E_t)}, \quad 0 \leq t \leq c, \tag{23}$$

for $v \in C_{\bar{\varphi},c}^{1,L}[d]$ being a solution of (1) with the initial boundary condition (2) with φ replaced by $\psi \in C_\partial^{1,L}[p]$.

Proof. We prove that there exists exactly one $\bar{z} \in C_{\varphi,c}^{1,L}[d]$ satisfying the equation $z = F[z]$. Lemma 3.1 shows that $F: C_{\varphi,c}^{1,L}[d] \to C_{\varphi,c}^{1,L}[d]$. From the definition (15) of F, from the Lipschitz continuity of φ_i, and from the Lipschitz continuity (see (9), (10)) of g_i and δ_i with respect to z, follows easily the existence of an $L^* > 0$ such that

$$\|F_iz(t,x) - F_i\tilde{z}(t,x)\| \leq L^* \int_0^t \|z - \tilde{z}\|_{C(E_s)}\, ds \tag{24}$$

for $z, \tilde{z} \in C_{\varphi,c}^{1,L}[d]$, $(t,x) \in E_c$, $1 \leq i \leq m$. Let $\lambda > L^*$. We define a metric in $C_{\varphi,c}^{1,L}[d]$ by

$$d_\lambda(z,\tilde{z}) = \sup\left\{ \|(z-\tilde{z})(t,x)\| e^{-\lambda t} : (t,x) \in E_c \right\}.$$

We now prove that there exists $q \in [0,1)$ such that

$$d_\lambda(Fz, F\tilde{z}) \leq q d_\lambda(z,\tilde{z}). \tag{25}$$

According to (24),

$$\|Fz(t,x) - F\tilde{z}(t,x)\| \leq L^* \int\limits_0^t \|z-\tilde{z}\|_{C(E_s)}\, ds = L^* \int\limits_0^t \|z-\tilde{z}\|_{C(E_s)}\, e^{-\lambda s} e^{\lambda s} ds \leq$$

$$\leq L^* d_\lambda(z,\tilde{z}) \int\limits_0^t e^{\lambda s} ds = \frac{L^*}{\lambda} d_\lambda(z,\tilde{z})(e^{\lambda t} - 1) \leq \frac{L^*}{\lambda} d_\lambda(z,\tilde{z}) e^{\lambda t}$$

for $(t,x) \in E_c$. Then

$$\|Fz(t,x) - F\tilde{z}(t,x)\| e^{-\lambda t} \leq \frac{L^*}{\lambda} d_\lambda(z,\tilde{z}) \quad \text{for all} \quad (t,x) \in E_c,$$

which gives (25) with $q = L^* \lambda^{-1}$. By the Banach fixed point theorem, there exists a unique fixed point of F. Denoting this fixed point by $\bar{z}$, we have for $(t,x) \in E_c$

$$\bar{z}_i(t,x) = \varphi_i(\delta_i[\bar{z}](t,x), g_i[\bar{z}](\delta_i[\bar{z}](t,x),t,x)) +$$

$$+ \int\limits_{\delta_i[\bar{z}](t,x)}^t G_i(s, g_i[\bar{z}](s,t,x), V(\bar{z}; s, g_i[\bar{z}](s,t,x))) ds, \quad 1 \leq i \leq m.$$

Now put $\zeta = \delta_i[\bar{z}](t,x)$. For a given $x \in [-b,b]$, let us denote $y = g_i[\bar{z}](\zeta,t,x)$. It follows from Lemma 2.1 that $g_i[\bar{z}](s,t,x) = g_i[\bar{z}](s,\zeta,y)$ for $s,t \in [\zeta,c]$ and $x = g_i[\bar{z}](t,\zeta,y)$. Then we get

$$\bar{z}_i(t, g_i[\bar{z}](t,\zeta,y)) = \varphi_i(\zeta,y) + \int\limits_\zeta^t G_i(s, g_i[\bar{z}](s,\zeta,y), V(\bar{z}; s, g_i[\bar{z}](s,\zeta,y))) ds, \quad 1 \leq i \leq m. \tag{26}$$

Relations $y = g_i[\bar{z}](\zeta,t,x)$ and $x = g_i[\bar{z}](t,\zeta,y)$ are equivalent for $x,y \in [-b,b]$. By differentiating (26) with respect to t and putting again $x = g_i[\bar{z}](t,\zeta,y)$ we conclude that $\bar{z}$ satisfies (1). Since $\bar{z}$ satisfies initial boundary condition (2), it is a solution of our problem.

We now prove the relation (23). The function v satisfies the integral functional system

$$z(t,x) = Fz(t,x)$$

and initial boundary condition (2) with ψ instead of φ. It follows easily that there is $\Lambda \in \mathbb{R}_+$ such that the integral inequality

$$\|\bar{z} - v\|_{C(E_t)} \leq \|\varphi - \psi\|_{C(E_0 \cup \partial_0 E_t)} + \Lambda \int\limits_0^t \|\bar{z} - v\|_{C(E_s)}\, ds, \quad 0 \leq t \leq c,$$

is satisfied. Using the Gronwall inequality, we obtain (23) with $\Lambda_c = \exp(\Lambda c)$.

Theorem 4.1 is proved.

Remark 4.1. Inequalities (16), (17), given in the Assumption H$[c, d, V]$, have the following impact on the conditions on the operator V.

We indicate, how to solve those inequalities. Put, for example, $d_1 = A + 2(1 + A)(1 + p_1)$; the condition 1 of Assumption H$[V]$ produces then a corresponding constant $\bar{d}_1 \geq 0$. Having performed easy calculations, one can see that condition $c \leq A^{-1}(1 + \bar{d}_1)^{-1} \log 2$, on c, assures the fulfilment of (16).

After the construction of $\bar{d}_1$, an example of a suitable value of d_2, **in terms of $\bar{\mathbf{d}}_1$** and of given constants, may be found using (17) (we shall assume that $c\bar{d}_2$ is appropriately bounded). Since at this stage d_1 and d_2 are fixed, the condition 2 of Assumption H$[V]$ gives $\bar{d}_2$. This leads to one more constraint on c, of which we assume the stronger one.

The above explained dependence of choice of d_2 on $\bar{d}_1$ shows that the condition 2 of the considered Assumption does not suffice for solvability of inequalities from Assumption H $[c, d, V]$, but that the condition 1 has to be added.

5. Systems with state dependent delays. Suppose that $z : \Omega \to \mathbb{R}$ and $(t, x) \in E$ are fixed. We define the function $z_{(t,x)} : D \to \mathbb{R}$ as follows:

$$z_{(t,x)}(\tau, \xi) = z(t + \tau, x + \xi), \quad (\tau, \xi) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t - h_0, t] \times [x - h, x + h]$ and this restriction is shifted to the set D. For $z : \Omega \to \mathbb{R}^m$, $z = (z_1, \ldots, z_m)$, write $z_{(t,x)} = ((z_1)_{(t,x)}, \ldots, (z_m)_{(t,x)})$.

Let $\psi_{ij} : E \times C^{1,L}(D, \mathbb{R}^m) \to \mathbb{R}$, $1 \leq i \leq m$, $0 \leq j \leq n$, be given. Consider the function

$$\Big((z_1)_{\psi_1(t,x,w)}, \ldots, (z_m)_{\psi_m(t,x,w)}\Big) \in X,$$

where $\psi_i = (\psi_{i0}, \ldots, \psi_{in})$, $1 \leq i \leq m$, and $z : \Omega \to \mathbb{R}^m$. We write it $z_{\psi(t,x,w)}$ for brevity. We show that the operator V, defined by

$$V(z; t, x) = z_{\psi(t,x,z_{(t,x)})} \tag{27}$$

satisfies Assumption H$[V]$, provided that certain regularity conditions on ψ are met.

Assumption H $[\psi]$. Deviating function $\psi : E \times X \to M_{m\times(n+1)}$ is continuous and, for $1 \leq i \leq m$,

1) the relations $\psi_i(t, x, w) \in E_t$ hold on $E \times X$,

2) derivatives: $\partial\psi_i$ and the Fréchet derivative $\partial_w\psi_i$ exist on $E \times C^{1,L}(D, \mathbb{R}^m)$,

3) there is a non-negative constant A_1 independent of i and such that, for $(t, x, w), (t, \bar{x}, \bar{w}) \in \in E \times C^{1,L}(D, \mathbb{R}^m)$,

$$\|\partial\psi_i(t, x, w)\|, \quad \|\partial_w\psi_i(t, x, w)\|_{L(Y, M_{(n+1)\times(n+1)})} \leq A_1$$

and

$$\|\partial\psi_i(t, x, w) - \partial\psi_i(t, \bar{x}, \bar{w})\|, \quad \|\partial_w\psi_i(t, x, w) - \partial_w\psi_i(t, \bar{x}, \bar{w})\|_{L(Y, M_{(n+1)\times(n+1)})}$$

are bounded from above by $A_1\big(\|x - \bar{x}\| + \|w - \bar{w}\|_X\big)$.

In view of the above Assumption, differentiation of (27) gives

$$\partial V_i(z;t,x) \equiv (\partial z_i)_{\psi_i(t,x,z_{(t,x)})} * \left[\partial\psi_i(t,x,z_{(t,x)}) + \partial_w\psi_i(t,x,z_{(t,x)})(\partial z)_{(t,x)}\right] \quad \text{on} \quad D,$$

and, consequently, for $z \in C^{1,L}_{\varphi,c}[d]$ and $(t,x),(t,\bar{x}) \in E$, $\|\partial V_i(z;t,x)\|_{C(D,M_{1\times(n+1)})} \le d_1A_1(1+ +d_1)$ and

$$\|\partial V_i(z;t,x) - \partial V_i(z;t,\bar{x})\|_{C(D,M_{1\times(n+1)})} \le A_1[d_1d_2 + (1+d_1)^2(d_1+d_2A_1)]\|x-\bar{x}\|.$$

Taking maximum (w.r.t. i) on the left-hand sides of these estimates, we obtain the conditions 1, 2 of Assumption H$[V]$ with $\bar{d}_1 = d_1A_1(1+d_1)$ and $\bar{d}_2 = A_1[d_1d_2 + (1+d_1)^2(d_1+d_2A_1)]$. Fulfilment of the condition 3 of that Assumption follows from the estimates

$$\begin{aligned}\|V_i(z;t,x) - V_i(\bar{z};t,x)\|_{C(D)} &\le \|(z_i)_{\psi_i(t,x,z_{(t,x)})} - (z_i)_{\psi_i(t,x,\bar{z}_{(t,x)})}\|_{C(D)} + \\ &\quad + \|(z_i-\bar{z}_i)_{\psi_i(t,x,\bar{z}_{(t,x)})}\|_{C(D)} \le \\ &\le d_1A_1\|z-\bar{z}\|_{C(E_t)} + \|z_i-\bar{z}_i\|_{C(E_t)} \le \\ &\le (d_1A_1+1)\|z-\bar{z}\|_{C(E_t)}, \quad 1 \le i \le m.\end{aligned}$$

Thus we have proved the following theorem.

Theorem 5.1. *Suppose that $\varphi \in C^{1,L}_{\partial}[p]$ and Assumptions $H[\rho,G]$, $H[\psi]$ are satisfied. Furthermore, assume that the inequalities (16), (17) hold, as well as the compatibility conditions (18), (19). Then there exists exactly one solution $\bar{z} \in C^{1,L}_{\varphi,c}[d]$ of the system*

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi(t,x,z_{(t,x)})})\partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x,z_{(t,x)})}), \quad 1 \le i \le m, \tag{28}$$

augmented with the generalized Cauchy condition (2). Moreover, there is $\Lambda_c \in \mathbb{R}_+$ such that the Lipschitz condition (23), with respect to data, holds for $\psi \in C^{1,L}_{\partial}[p]$ and for $v \in C^{1,L}_{\bar{\varphi},c}[d]$ being a solution of (28) with the initial boundary condition $z \equiv \psi$ on $E_0 \cup \partial_0 E_c$.

Assumption H $[\bar{\rho},\bar{G}]$. Functions $\bar{\rho} : E \times \mathbb{R}^m \to M_{m\times n}$, $\bar{G} : E \times \mathbb{R}^m \to \mathbb{R}^m$, in variables (t,x,y), are continuous, uniformly bounded, and

1) $\bar{G}$ is Lipschitz continuous in t,

2) the derivatives $\partial_x\bar{\rho}$, $\partial_y\bar{\rho}$, $\partial_x\bar{G}$, $\partial_y\bar{G}$ exist on $E \times \mathbb{R}^m$, are continuous in t, and uniformly bounded,

3) these derivatives are Lipschitz continuous in x and y.

Example 5.1. Suppose that Assumption $H[\bar{\rho},\bar{G}]$ is satisfied and set

$$\rho(t,x,w) = \bar{\rho}\,(t,x,w(0,0)), \quad G(t,x,w) = \bar{G}\,(t,x,w(0,0)).$$

Then the Assumption $H[\rho,G]$ is fulfilled and the system (28) takes the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \bar{\rho}_{ij}\left(t,x,z(\psi(t,x,z_{(t,x)}))\right)\partial_{x_j} z_i(t,x) = \bar{G}_i\left(t,x,z(\psi(t,x,z_{(t,x)}))\right), \quad 1 \le i \le m,$$

that is, it becomes a system of equations with deviating argument, where the deviation is state dependent.

Example 5.2. Suppose that Assumption $H[\bar{\rho}, \bar{G}]$ is satisfied and set

$$\rho(t,x,w) = \bar{\rho}\left(t, x, \int\limits_D w(\tau,\xi)\,d\tau\,d\xi\right), \quad G(t,x,w) = \bar{G}\left(t, x, \int\limits_D w(\tau,\xi)\,d\tau\,d\xi\right).$$

Then the Assumption $H[\rho, G]$ is fulfilled and the system (28) takes the form

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \bar{\rho}_{ij}\left(t, x, \int\limits_D z_{\psi(t,x,z_{(t,x)})}(\tau,\xi)\,d\tau\,d\xi\right)\partial_{x_j} z_i(t,x) =$$

$$= \bar{G}_i\left(t, x, \int\limits_D z_{\psi(t,x,z_{(t,x)})}(\tau,\xi)\,d\tau\,d\xi\right), \quad 1 \le i \le m,$$

that is, it becomes a system of integro-differential equations, where the domain of integration is state dependent.

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