Nonlinear Oscillations, Vol. 4, No. 4, 2001

ASYMPTOTIC STABILITY FOR A THERMOELECTROMAGNETIC MATERIAL*

G. Amendola

Dipartimento di Matematica Applicata "U.Dini", Facoltà di Ingegneria
via Diotisalvi 2, 56126-Pisa, Italy

In this work we consider a linear thermoelectromagnetic material, whose behaviour is characterized by two rate-type equations for the heat flux and the electric current density. We derive the restrictions imposed by the laws of thermodynamics on the constitutive equations and introduce the free energy which yields the existence of a domain of dependence. Uniqueness, existence and asymptotic stability theorems are then proved.

AMS Subject Classification: 80A05, 74F05, 74F15

1. Introduction

Electromagnetic materials in presence of thermal effects have been studied by Coleman and Dill, who have also derived the restrictions which the laws of thermodynamics place on the constitutive equations [1, 2]. Heat conduction in electromagnetic solids has been considered in [3] to study its effects on the asymptotic behaviour of the solutions when constitutive equations with memory have been supposed for the heat flux and for the electric current density.

In this work we are concerned with a linear thermoelectromagnetic solid, characterized by a rate-type equation both for the heat flux and for the electric current density, while the simple proportionality is assumed between the electric displacement, the magnetic induction and the electric and magnetic fields, respectively.

After introducing in Sect. 2 Maxwell's equations with the energy one, in Sect. 3 we consider the thermodynamics of simple materials [4, 5], in order to derive the restrictions on the constitutive equations and to introduce the free energy, which allows us to obtain, in the following section, a domain of dependence. A theorem of uniqueness, existence and asymptotic stability of solutions is then proved in the last section.

2. Basic Equations

We consider a thermoelectromagnetic solid $\mathcal{B}$, which at time t occupies a bounded and regular domain Ω in the three-dimensional Euclidean space $\mathbf{R}^3$. The position vector in Ω is denoted by $\mathbf{x}$ and $\mathbf{n}$ is the unit outward normal to the smooth boundary $\partial\Omega$.

We restrict our attention to the linear thermoelectromagnetic theory, where $\mathcal{B}$ is homogeneous, isotropic and conducting material characterized, in particular, by two rate-type equations both for the electric current density and for the heat flux. Thus, we assume for the electric displacement $\mathbf{D}$, the magnetic induction $\mathbf{B}$, the rate at which heat is absorbed per unit volume

* Work perfomed under the auspices of C.N.R. and M.U.R.S.T..

h the following constitutive equations:

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t) + \vartheta(\mathbf{x},t)\mathbf{a}, \quad \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t), \tag{2.1}$$

$$h(\mathbf{x},t) = c\dot{\vartheta}(\mathbf{x},t) + \Theta_0\left[\mathbf{A}_1\cdot\dot{\mathbf{D}}(\mathbf{x},t)/\varepsilon + \mathbf{A}_2\cdot\dot{\mathbf{B}}(\mathbf{x},t)/\mu\right], \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, ϑ is the temperature relative to the uniform absolute reference temperature Θ_0 in Ω, while for the electric current density $\mathbf{J}$ and for the heat flux $\mathbf{q}$ [6] we suppose that the following equations

$$\alpha\dot{\mathbf{J}}(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t) = \sigma\mathbf{E}(\mathbf{x},t), \qquad \tau\dot{\mathbf{q}}(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) = -k\mathbf{g}(\mathbf{x},t), \tag{2.3}$$

hold, where $\mathbf{g} = \nabla\vartheta$ is the temperature gradient.

In these relations the dielectric constant ε, the permeability μ and the specific heat c are positive constants as well as the other two parameters α and τ; moreover, $\mathbf{a}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ denote three constant vectors.

The fundamental system of thermoelectromagnetism consists of Maxwell's equations together with the energy equation, i.e.,

$$\nabla\times\mathbf{H}(\mathbf{x},t) = \dot{\mathbf{D}}(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t) + \mathbf{J}_f(\mathbf{x},t), \quad \nabla\times\mathbf{E}(\mathbf{x},t) = -\dot{\mathbf{B}}(\mathbf{x},t), \tag{2.4}$$

$$\nabla\cdot\mathbf{B}(\mathbf{x},t) = 0, \qquad\qquad \nabla\cdot\mathbf{D}(\mathbf{x},t) = \rho, \tag{2.5}$$

$$h(\mathbf{x},t) = -\nabla\cdot\mathbf{q}(\mathbf{x},t) + r(\mathbf{x},t), \tag{2.6}$$

where ρ is the free charge density, $\mathbf{J}_f$ denotes a forced current density and is a known function of $\mathbf{x}$ and t as well as r , which is the heat sources for unit volume; moreover, we have supposed that the mass density is unitary in (2.6).

Sometimes we will understand the dependence on $\mathbf{x}$.

3. Thermodynamic Restrictions on the Constitutive Equations

The assumed constitutive equations allow us to consider $\mathcal{B}$ as a simple material, whose state is $s = (\mathbf{E},\mathbf{H},\mathbf{J},\mathbf{q},\vartheta)$, while the process of the material element is given by $P(t) = (\dot{\mathbf{E}}(t),\dot{\mathbf{H}}(t),\dot{\vartheta}(t),\mathbf{g}(t))$ and the response function is $U(t) = (\mathbf{D}(t),\mathbf{B}(t),\mathbf{J}(t),h(t),\mathbf{q}(t))$ for every $t\in[0,d_P)\subset\mathbf{R}^+$, d_P being the duration of the process.

The First Law of Thermodynamics for these materials is expressed by this equality

$$\oint[h(t) + \dot{\mathbf{D}}(t)\cdot\mathbf{E}(t) + \dot{\mathbf{B}}(t)\cdot\mathbf{H}(t) + \mathbf{J}(t)\cdot\mathbf{E}(t)]dt = 0, \tag{3.1}$$

which must hold for any cyclic process [1, 2], and, for smooth processes, yields the existence of the internal energy e which satisfies

$$\dot{e}(t) = h(t) + \dot{\mathbf{D}}(t)\cdot\mathbf{E}(t) + \dot{\mathbf{B}}(t)\cdot\mathbf{H}(t) + \mathbf{J}(t)\cdot\mathbf{E}(t). \tag{3.2}$$

The Second Law states that for any cyclic process the inequality [1, 2]

$$\oint \left\{ h(t)/[\Theta_0 + \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)/[\Theta_0 + \vartheta(t)]^2 \right\} dt \leq 0 \tag{3.3}$$

holds, with the equality sign referring to reversible processes.

Also by (3.3) it is possible to derive, for smooth processes, another local relation,

$$\dot{\eta}(t) \geq h(t)/[\Theta_0 + \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)/[\Theta_0 + \vartheta(t)]^2, \tag{3.4}$$

where η denotes the entropy.

We are concerned with the linear theory, therefore we must consider the approximation of the Second Law; thus, (3.3) and (3.4) become [7]

$$\oint \{h(t)[\Theta_0 - \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)\} \, dt/\Theta_0^2 \leq 0, \tag{3.5}$$

$$\dot{\eta}(t) \geq \{h(t)[\Theta_0 - \vartheta(t)] + \mathbf{q}(t) \cdot \mathbf{g}(t)\} / \Theta_0^2. \tag{3.6}$$

We first examine (3.5), which, eliminating $\Theta_0 h(t)$ by use of (3.1) and integrating on any cycle, yields

$$\oint \left\{ h(t)\vartheta(t) - \Theta_0[\dot{\mathbf{E}}(t) \cdot \mathbf{D}(t) + \dot{\mathbf{H}}(t) \cdot \mathbf{B}(t) - \mathbf{J}(t) \cdot \mathbf{E}(t)] - \mathbf{q}(t) \cdot \mathbf{g}(t) \right\} dt \geq 0. \tag{3.7}$$

If we substitute into (3.7) the constitutive equations (2.1), (2.2) and the two relations, we can derive from (2.3) for $\mathbf{E}$ and $\mathbf{g}$ on supposing $\sigma \neq 0$ and $k \neq 0$, we get an inequality, which, integrated on any cycle, reduces to

$$\oint \left\{ \left[(\mathbf{A}_1 - \mathbf{a}) \cdot \dot{\mathbf{E}}(t) + \mathbf{A}_2 \cdot \dot{\mathbf{H}}(t) \right] \vartheta(t) + \frac{1}{\sigma} \mathbf{J}^2(t) + \frac{1}{k\Theta_0} \mathbf{q}^2(t) \right\} dt \geq 0. \tag{3.8}$$

From this inequality, taking account of the independence of $\dot{\mathbf{E}}$, $\dot{\mathbf{H}}$ and $\mathbf{g}$, it follows that

$$\mathbf{A}_1 = \mathbf{a}, \qquad \mathbf{A}_2 = \mathbf{0}, \qquad \sigma > 0, \qquad k > 0. \tag{3.9}$$

In particular, the expression (2.2), using $(3.9)_{1,2}$ and (2.1), becomes

$$h(\mathbf{x}, t) = \left(c + \Theta_0 \mathbf{a}^2/\varepsilon \right) \dot{\vartheta}(\mathbf{x}, t) + \Theta_0 \mathbf{a} \cdot \dot{\mathbf{E}}(\mathbf{x}, t). \tag{3.10}$$

Let us introduce the pseudo-free energy

$$\psi(\mathbf{x}, t) = e(\mathbf{x}, t) - \Theta_0 \eta(\mathbf{x}, t). \tag{3.11}$$

Substituting into (3.6) the expression of h derived by (3.2) and using (3.11) yields

$$\dot{\psi}(t) \leq [h(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t)]/\Theta_0 + \dot{\mathbf{D}}(t) \cdot \mathbf{E}(t) + \dot{\mathbf{B}}(t) \cdot \mathbf{H}(t) + \mathbf{J}(t) \cdot \mathbf{E}(t). \tag{3.12}$$

We introduce the following functional

$$\tilde{\psi}(\mathbf{x},t) = \frac{1}{2}\left[\frac{1}{\varepsilon}\mathbf{D}^2(\mathbf{x},t) + \frac{1}{\mu}\mathbf{B}^2(\mathbf{x},t) + \frac{\alpha}{\sigma}\mathbf{J}^2(\mathbf{x},t) + \frac{\tau}{k\Theta_0}\mathbf{q}^2(\mathbf{x},t) + \frac{c}{\Theta_0}\vartheta^2(\mathbf{x},t)\right]. \tag{3.13}$$

We observe that this free energy $\tilde{\psi}$ satisfies (3.12). In fact, if we differentiate $\tilde{\psi}$ with respect to time, taking account of (2.1) and (3.10), eliminating $\dot{\mathbf{J}}$ and $\dot{\mathbf{q}}$ with (2.3), we get

$$\begin{aligned}\dot{\tilde{\psi}}(t) &= \frac{1}{\Theta_0}[h(t)\vartheta(t) - \mathbf{q}(t)\cdot\mathbf{g}(t)] + \dot{\mathbf{D}}(t)\cdot\mathbf{E}(t) \\ &\quad + \dot{\mathbf{B}}(t)\cdot\mathbf{H}(t) + \mathbf{J}(t)\cdot\mathbf{E}(t) - \frac{1}{k\Theta_0}\mathbf{q}^2(t) - \frac{1}{\sigma}\mathbf{J}^2(t),\end{aligned} \tag{3.14}$$

which satisfies (3.12).

4. Domain of Dependence Inequality

The system of equations of the evolutive problem of the thermoelectromagnetic solid $\mathcal{B}$ consists of (2.4) and (2.6), which, on account of (2.1) and (3.10), can be put in the following form:

$$\nabla\times\mathbf{H}(\mathbf{x},t) = \varepsilon\dot{\mathbf{E}}(\mathbf{x},t) + \dot{\vartheta}(\mathbf{x},t)\mathbf{a} + \mathbf{J}(\mathbf{x},t) + \mathbf{J}_f(\mathbf{x},t), \tag{4.1}$$

$$\nabla\times\mathbf{E}(\mathbf{x},t) = -\mu\dot{\mathbf{H}}(\mathbf{x},t), \tag{4.2}$$

$$\left(c + \Theta_0\mathbf{a}^2/\varepsilon\right)\dot{\vartheta}(\mathbf{x},t) + \Theta_0\mathbf{a}\cdot\dot{\mathbf{E}}(\mathbf{x},t) = -\nabla\cdot\mathbf{q}(\mathbf{x},t) + r(\mathbf{x},t), \tag{4.3}$$

together with (2.3) and $(2.5)_1$, which becomes

$$\nabla\cdot\mathbf{H}(\mathbf{x},t) = 0 \tag{4.4}$$

for any $(\mathbf{x},t)\in\Omega\times\mathbf{R}^+$. We observe that Eq. $(2.5)_2$, $\varepsilon\nabla\cdot\mathbf{E}+\nabla\vartheta\cdot\mathbf{a} = \rho$, allows us to determine ρ.

To this system we must add the initial conditions

$$\begin{aligned}&\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}),\ \mathbf{J}(\mathbf{x},0) = \mathbf{J}_0(\mathbf{x}),\\ &\mathbf{q}(\mathbf{x},0) = \mathbf{q}_0(\mathbf{x}),\ \vartheta(,0) = \vartheta_0(\mathbf{x}) \qquad \forall\mathbf{x}\in\Omega\end{aligned} \tag{4.5}$$

and the boundary conditions

$$\mathbf{E}(\mathbf{x},t)\times\mathbf{n} = \mathbf{0}, \quad \vartheta(\mathbf{x},t) = 0 \qquad \forall(\mathbf{x},t)\in\partial\Omega\times\mathbf{R}^+. \tag{4.6}$$

Compatibility of the boundary conditions and the initial ones is assumed.

We dente by P this initial-boundary-value problem.

Theorem 4.1. *The free energy defined by (3.13) satisfies*

$$-[\mathbf{E}(t) \times \mathbf{H}(t) + \vartheta(t)\mathbf{q}(t)/\Theta_0] \cdot \mathbf{u} \leq \chi\tilde{\psi}(t), \tag{4.7}$$

where $\mathbf{u}$ *is a unit vector and*

$$\chi = 2\left\{\left(\frac{k}{\tau c}\right)^{1/2} + \frac{1}{\varepsilon\mu}\left[\varepsilon^{1/2} + \mid \mathbf{a} \mid \left(\frac{\Theta_0}{c}\right)^{1/2}\right]\mu^{1/2}\right\}. \tag{4.8}$$

Proof. To prove (4.7) it is enough to consider (3.13), from which it follows that

$$\frac{1}{\varepsilon} \mid \mathbf{D} \mid \leq \left[\frac{2}{\varepsilon}\tilde{\psi}\right]^{1/2}, \ \frac{1}{\mu} \mid \mathbf{B} \mid \leq \left[\frac{2}{\varepsilon}\tilde{\psi}\right]^{1/2}, \ \mid \mathbf{q} \mid \leq \left[\frac{2k}{\tau}\Theta_0\tilde{\psi}\right]^{1/2}, \ \mid \vartheta \mid \leq \left[\frac{2}{c}\Theta_0\tilde{\psi}\right]^{1/2}, \tag{4.9}$$

and to derive from (2.1) the expressions of $\mathbf{E}$ and $\mathbf{H}$, which, taking account of $(4.9)_{1,2}$, yield

$$\mid \mathbf{E} \mid \leq \frac{2^{1/2}}{\varepsilon}\left[\varepsilon^{1/2} + \mid \mathbf{a} \mid \left(\frac{\Theta_0}{c}\right)^{1/2}\right]\tilde{\psi}^{1/2}, \quad \mid \mathbf{H} \mid \leq \left[\frac{2}{\mu}\tilde{\psi}\right]^{1/2}. \tag{4.10}$$

These inequalities give (4.7) at once.

We are now in a position to show the existence of a domain of dependence introducing the total energy

$$E(D,t) = \int_D \tilde{\psi}(\mathbf{x},t)d\mathbf{x} \qquad \forall D \subset \Omega. \tag{4.11}$$

Theorem 4.2. *The total energy* $E(D,t)$ *satisfies the following inequality*

$$\begin{aligned} E(B(\mathbf{x}_0,\rho),T) \leq\ & E(B(\mathbf{x}_0,\rho+\chi T),0) \\ & + \int_0^T \int_{\Omega\cap B(\mathbf{x}_0,\rho+\chi(T-t))} \left[\frac{1}{\Theta_0}r(\mathbf{x},t)\vartheta(\mathbf{x},t) - \mathbf{J}_f(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\right]d\mathbf{x}dt \end{aligned} \tag{4.12}$$

for any fixed $(\mathbf{x}_0,T) \in \Omega \times \mathbf{R}^+$, *with* $B(\mathbf{x}_0,\rho) = \{\mathbf{x} \in \Omega : |\mathbf{x}-\mathbf{x}_0| \leq \rho\}$ *and* χ *given by (4.8).*

Proof. Let

$$E_\phi(\Omega,t) = \int_\Omega \tilde{\psi}(\mathbf{x},t)\phi(\mathbf{x},t)d\mathbf{x}, \tag{4.13}$$

where $\phi(\mathbf{x},t) \in C_0^\infty(\mathbf{R}^3,\mathbf{R}^+)$; its derivative with respect to time, taking account of (3.14) and eliminating h, $\dot{\mathbf{B}}$ and $\dot{\mathbf{D}}$ by the use of (2.6) and (2.4), can be written as

$$\dot{E}_\phi(\Omega,t) = \int_\Omega \left\{ \frac{1}{\Theta_0}\left[r\vartheta - \nabla\cdot(\vartheta\mathbf{q})\right] - \mathbf{J}_f\cdot\mathbf{E} + \nabla\times\mathbf{H}\cdot\mathbf{E} - \nabla\times\mathbf{E}\cdot\mathbf{H} \right\}\phi d\mathbf{x} - \int_\Omega \left(\frac{1}{\sigma}\mathbf{J}^2 + \frac{1}{k\Theta_0}\mathbf{q}^2\right)\phi d\mathbf{x} + \int_\Omega \tilde{\psi}\dot{\phi} d\mathbf{x}. \tag{4.14}$$

This equality, using the identity $\nabla\times\mathbf{E}\cdot\mathbf{H} - \nabla\times\mathbf{H}\cdot\mathbf{E} = \nabla\cdot(\mathbf{E}\times\mathbf{H})$ and taking account of the boundary conditions (4.6) and $(3.9)_{3,4}$, gives the following inequality:

$$\dot{E}_\phi(\Omega,t) \le \int_\Omega (r\vartheta/\Theta_0 - \mathbf{J}_f\cdot\mathbf{E})\,\phi d\mathbf{x} + \int_\Omega \left[(\vartheta\mathbf{q}/\Theta_0 + \mathbf{E}\times\mathbf{H})\cdot\nabla\phi + \tilde{\psi}\dot{\phi}\right] d\mathbf{x} - \int_{\partial\Omega} (\vartheta\mathbf{q}/\Theta_0 + \mathbf{E}\times\mathbf{H})\cdot\mathbf{n}\phi da. \tag{4.15}$$

We now put

$$\phi(\mathbf{x},t) = \phi_\delta(\mathbf{x},t) = \phi_\delta(|\mathbf{x}-\mathbf{x}_0| - \rho - \chi(T-t)) = \phi_\delta(y) = \begin{cases} 1 & \forall\, y \le -\delta, \\ 0 & \forall\, y > \delta, \end{cases} \tag{4.16}$$

where $\phi_\delta \in C^\infty(\mathbf{R})$ is a nonnegative, monotonic decreasing function, $\delta > 0$, $\rho > 0$ and χ is given by (4.8), for any fixed point $\mathbf{x}_0 \in \Omega$ and $T \in \mathbf{R}^+$ with $t \in (0,T)$.

We have

$$\phi_\delta'(y) \le 0, \quad \dot{\phi}_\delta(\mathbf{x},t) = \phi_\delta'(y)\chi, \quad \nabla\phi_\delta(\mathbf{x},t) = \phi_\delta'(y)\mathbf{u}(\mathbf{x}), \tag{4.17}$$

where $\mathbf{u} = \nabla|\mathbf{x}-\mathbf{x}_0|$ is a unit vector.

Substituting $(4.17)_{2,3}$ into (4.15) and using $(4.17)_1$, (4.7) and (4.6) yield

$$4.18\dot{E}_{\phi_\delta}(\Omega,t) \le \int_\Omega \left[r(\mathbf{x},t)\vartheta(\mathbf{x},t)/\Theta_0 - \mathbf{J}_f(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\right]\phi_\delta(\mathbf{x},t) d\mathbf{x}, \tag{4.18}$$

which, integrated over $(0,T)$, gives

$$E_{\phi_\delta}(\Omega,T) - E_{\phi_\delta}(\Omega,0) \le \int_0^T \int_\Omega \left[r(\mathbf{x},t)\vartheta(\mathbf{x},t)/\Theta_0 - \mathbf{J}_f(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\right]\phi_\delta(\mathbf{x},t) d\mathbf{x}dt. \tag{4.19}$$

Whence, the limit as $\delta \to 0$ yields (4.12), because $\phi_\delta(\mathbf{x},t)$ tends to the characteristic function of $B(\mathbf{x}_0, \rho + \chi(T-t))$.

5. Existence, Uniqueness and Asymptotic Stability

The problem P, expressed by the system of equations (4.1) – (4.4) with (2.3) to which are associated the initial-boundary conditions (4.5), (4.6), can be transformed into an equivalent one with zero initial data, we shall denote by P′.

For this purpose, we put

$$\mathbf{E}(\mathbf{x},t) = \tilde{\mathbf{E}}(\mathbf{x},t) + \mathbf{l}(\mathbf{x},t),\ \mathbf{H}(\mathbf{x},t) = \tilde{\mathbf{H}}(\mathbf{x},t) + \mathbf{p}(\mathbf{x},t),\ \mathbf{J}(\mathbf{x},t) = \tilde{\mathbf{J}}(\mathbf{x},t) + \mathbf{w}(\mathbf{x},t),$$

$$\mathbf{q}(\mathbf{x},t) = \tilde{\mathbf{q}}(\mathbf{x},t) + \mathbf{z}(\mathbf{x},t),\ \vartheta(\mathbf{x},t) = \tilde{\vartheta}(\mathbf{x},t) + \gamma(\mathbf{x},t),$$

where $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{J}}, \tilde{\mathbf{q}}, \tilde{\vartheta})$ and $(\mathbf{l}, \mathbf{p}, \mathbf{w}, \mathbf{z}, \gamma)$ belong to the function spaces of the solution $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{q}, \vartheta)$ and hence, in particular, satisfy the boundary conditions (4.6); moreover, taking in mind (2.5), we suppose that $(\mathbf{l}, \mathbf{p}, \gamma)$ satisfies $\nabla \cdot \mathbf{p} = 0$ and $\varepsilon \nabla \cdot \mathbf{l} = -\nabla\gamma \cdot \mathbf{a}$ and we impose the following initial conditions:

$$\mathbf{l}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{p}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}),\ \mathbf{w}(\mathbf{x},0) = \mathbf{J}_0(\mathbf{x}),\ \mathbf{z}(\mathbf{x},0) = \mathbf{q}_0(\mathbf{x}),\ \gamma(\mathbf{x},0) = \vartheta_0(\mathbf{x}).$$

Then, we see that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{J}}, \tilde{\mathbf{q}}, \tilde{\vartheta})$ satisfies the system of equations of P, where the sources $\mathbf{J}_f$ and r must be changed and other sources must be introduced in the other equations; all these sources are expressed by

$$\mathbf{F}(\mathbf{x},t) = \mathbf{J}_f(\mathbf{x},t) + \varepsilon\dot{\mathbf{l}}(\mathbf{x},t) + \dot{\gamma}(\mathbf{x},t)\mathbf{a} + \mathbf{w}(\mathbf{x},t) - \nabla \times \mathbf{p}(\mathbf{x},t),$$

$$\mathbf{G}(\mathbf{x},t) = -\mu\dot{\mathbf{p}}(\mathbf{x},t) - \nabla \times \mathbf{l}(\mathbf{x},t),$$

$$R(\mathbf{x},t) = r(\mathbf{x},t) - \Theta_0\dot{\mathbf{l}}(\mathbf{x},t) \cdot \mathbf{a} - \left(c + \Theta_0\mathbf{a}^2/\varepsilon\right)\dot{\gamma}(\mathbf{x},t) - \nabla \cdot \mathbf{z}(\mathbf{x},t),$$

$$\mathbf{I}(\mathbf{x},t) = -\alpha\dot{\mathbf{w}}(\mathbf{x},t) - \mathbf{w}(\mathbf{x},t) + \sigma\mathbf{l}(\mathbf{x},t),$$

$$\mathbf{Q}(\mathbf{x},t) = -r\dot{\mathbf{z}}(\mathbf{x},t) - \mathbf{z}(\mathbf{x},t) - k\nabla\gamma(\mathbf{x},t).$$

We observe that the hypothesis that $\mathbf{l}$ and γ satisfy $(2.5)_2$ without ρ implies that $\tilde{\mathbf{E}}$ and $\tilde{\vartheta}$ satisfy $(2.5)_2$ and give ρ without any new source.

Using the same notation $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{q}, \vartheta)$, without $\sim$, the problem P′ is the following

$$-\varepsilon\dot{\mathbf{E}}(\mathbf{x},t) - \dot{\vartheta}(\mathbf{x},t)\mathbf{a} + \nabla \times \mathbf{H}(\mathbf{x},t) - \mathbf{J}(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t), \tag{5.1}$$

$$\mu\dot{\mathbf{H}}(\mathbf{x},t) + \nabla \times \mathbf{E}(\mathbf{x},t) = \mathbf{G}(\mathbf{x},t), \tag{5.2}$$

$$\Theta_0\mathbf{a} \cdot \dot{\mathbf{E}}(\mathbf{x},t) + \left(c + \Theta_0\mathbf{a}^2/\varepsilon\right)\dot{\vartheta}(\mathbf{x},t) + \nabla \cdot \mathbf{q}(\mathbf{x},t) = R(\mathbf{x},t), \tag{5.3}$$

$$\alpha\dot{\mathbf{J}}(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t) - \sigma\mathbf{E}(\mathbf{x},t) = \mathbf{I}(\mathbf{x},t), \tag{5.4}$$

$$\tau\dot{\mathbf{q}}(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) + k\nabla\vartheta(\mathbf{x},t) = \mathbf{Q}(\mathbf{x},t), \tag{5.5}$$

with (4.4), (4.5), which now become

$$\mathbf{E}_0(\mathbf{x}) = \mathbf{0},\ \mathbf{H}_0(\mathbf{x}) = \mathbf{0},\ \mathbf{J}_0(\mathbf{x}) = \mathbf{0},\ \mathbf{q}_0(\mathbf{x}) = \mathbf{0},\ \vartheta_0(\mathbf{x}) = 0, \tag{5.6}$$

and the boundary conditions (4.6).

Taking account of the Maxwell equation (4.4) and the boundary conditions (4.6), we introduce the function spaces

$$I(\Omega) = \left\{ \mathbf{H}(\mathbf{x}) \in L^2(\Omega) : \int_\Omega \mathbf{H} \cdot \nabla\varphi \, d\mathbf{x} = 0 \quad \forall\varphi \in C_0^\infty(\Omega) \right\},$$

$$H_E^1(\Omega) = \left\{ \mathbf{E}(\mathbf{x}) \in L^2(\Omega) : \nabla \times \mathbf{E} \in L^2, \quad \mathbf{E}(\mathbf{x}) \times \mathbf{n} \mid_{\partial\Omega} = \mathbf{0} \right\},$$

$$H_H^1(\Omega) = \left\{ \mathbf{H}(\mathbf{x}) \in H^1(\Omega) : \nabla \cdot \mathbf{H} = 0, \quad \mathbf{H}(\mathbf{x}) \cdot \mathbf{n} \mid_{\partial\Omega} = 0 \right\},$$

$$H_\vartheta^1(\Omega) = \left\{ \vartheta(\mathbf{x}) \in H^1(\Omega) : \vartheta(\mathbf{x}) \mid_{\partial\Omega} = 0 \right\},$$

$$\begin{aligned} H(\Omega, \mathbf{R}^+) &= L^2(\mathbf{R}^+; H_E^1(\Omega)) \times L^2(\mathbf{R}^+; H_H^1(\Omega)) \times L^2(\mathbf{R}^+; L^2(\Omega)) \\ &\quad \times L^2(\mathbf{R}^+; L^2(\Omega)) \times L^2(\mathbf{R}^+; H_\vartheta^1(\Omega)), \end{aligned}$$

$$\begin{aligned} W(\Omega, \mathbf{R}^+) &= H^1(\mathbf{R}^+; L^2(\Omega)) \times H^1(\mathbf{R}^+; L^2(\Omega)) \times H^1(\mathbf{R}^+; L^2(\Omega)) \\ &\quad \times H^1(\mathbf{R}^+; L^2(\Omega)) \times \left\{ H^1(\mathbf{R}^+; L^2(\Omega)) \cap L^2(\mathbf{R}^+; H_\vartheta^1(\Omega)) \right\}, \end{aligned}$$

$$\begin{aligned} V(\Omega, \mathbf{R}^+) &= L^2(\mathbf{R}^+; L^2(\Omega)) \times L^2(\mathbf{R}^+; I(\Omega)) \times L^2(\mathbf{R}^+; L^2(\Omega)) \\ &\quad \times L^2(\mathbf{R}^+; L^2(\Omega)) \times L^2(\mathbf{R}^+; L^2(\Omega)), \end{aligned}$$

which are Hilbert's spaces with the usual scalar products.

The sources belong to $V(\Omega, \mathbf{R}^+)$, modified in the following manner

$$\begin{aligned} V'(\Omega, \mathbf{R}^+) = \Bigg\{ &(\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \in V(\Omega, \mathbf{R}^+) : \frac{\partial^{n+1}}{\partial t^{n+1}} (\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \in V(\Omega, \mathbf{R}^+), \\ &\left[\frac{\partial^n}{\partial t^n} (\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \right]_{t=0} = 0 \quad (n = 0, 1, 2, 3) \Bigg\}, \end{aligned}$$

where the last conditions on the initial values of the derivatives of the new sources are satisfied by a suitable choice of the corresponding derivatives of $\mathbf{l}$, $\mathbf{p}$, $\mathbf{w}$, $\mathbf{z}$ and γ.

Definition 5.1. *A 5-tuple* $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{q}, \vartheta) \in H(\Omega, \mathbf{R}^+)$ *is called weak solution of the problem* P' *with sources* $(\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \in V'(\Omega, \mathbf{R}^+)$ *if*

$$\int_0^{+\infty}\int_\Omega [\varepsilon \mathbf{E}(\mathbf{x},t) + \vartheta(\mathbf{x},t)\mathbf{a}] \cdot \dot{\mathbf{e}}(\mathbf{x},t) - \mu \mathbf{H}(\mathbf{x},t) \cdot \dot{\mathbf{h}}(\mathbf{x},t) - \alpha \mathbf{J}(\mathbf{x},t) \cdot \dot{\mathbf{u}}(\mathbf{x},t)$$

$$- \tau \mathbf{q}(\mathbf{x},t) \cdot \dot{\mathbf{v}}(\mathbf{x},t) - \Theta_0 \left[\left(c/\Theta_0 + \mathbf{a}^2/\varepsilon \right) \vartheta(\mathbf{x},t) + \mathbf{a} \cdot \mathbf{E}(\mathbf{x},t) \right] \dot{\beta}(\mathbf{x},t)$$

$$+ \nabla \times \mathbf{E}(\mathbf{x},t) \cdot \mathbf{h}(\mathbf{x},t) + [\nabla \times \mathbf{H}(\mathbf{x},t) - \mathbf{J}(\mathbf{x},t)] \cdot \mathbf{e}(\mathbf{x},t) - \mathbf{q}(\mathbf{x},t) \cdot \nabla \beta(\mathbf{x},t)$$

$$+ [\mathbf{J}(\mathbf{x},t) - \sigma \mathbf{E}(\mathbf{x},t)] \cdot \mathbf{u}(\mathbf{x},t) + [\mathbf{q}(\mathbf{x},t) + k \nabla \vartheta(\mathbf{x},t)] \cdot \mathbf{v}(\mathbf{x},t) \} \, d\mathbf{x} dt$$

$$= \int_0^{+\infty}\int_\Omega [\mathbf{F}(\mathbf{x},t) \cdot \mathbf{e}(\mathbf{x},t) + \mathbf{G}(\mathbf{x},t) \cdot \mathbf{h}(\mathbf{x},t) + \mathbf{I}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t)$$

$$+ \mathbf{Q}(\mathbf{x},t) \cdot \mathbf{v}(\mathbf{x},t) + R(\mathbf{x},t)\beta(\mathbf{x},t)] d\mathbf{x} dt \tag{5.7}$$

for all $(\mathbf{e}, \mathbf{h}, \mathbf{u}, \mathbf{v}, \beta) \in W(\Omega, \mathbf{R}^+)$ *such that* $\mathbf{e}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{h}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$, $\mathbf{v}(\mathbf{x}, 0) = \mathbf{0}$, $\beta(\mathbf{x}, 0) = 0$.

Let us introduce the time-Fourier transform of the causal extension on $\mathbf{R}$ of a function $g : \mathbf{R}^+ \to \mathbf{R}^n$,

$$\hat{g}(\omega) = \int_{-\infty}^{+\infty} g(t) \exp[-i\omega t] dt. \tag{5.8}$$

If g and $g' \in L^2(\mathbf{R}^+)$, then the Fourier transforms $\hat{g}$ and $\hat{g}' \in L^2(\mathbf{R})$ and we have

$$\hat{g}'(\omega) = i\omega \hat{g}(\omega) - g(0), \qquad g(0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \hat{g}(\omega) d\omega. \tag{5.9}$$

We denote by $\hat{H}(\Omega, \mathbf{R})$, $\hat{W}(\Omega, \mathbf{R})$ and $\hat{V}'(\Omega, \mathbf{R})$ the spaces of the Fourier transforms with respect to time of the corresponding functions of $H(\Omega, \mathbf{R}^+)$, $W(\Omega, \mathbf{R}^+)$ and $V'(\Omega, \mathbf{R}^+)$. Between each pair of these spaces there exists an isomorphism by virtue of Plancherel's theorem, which allows us to define in a natural way the scalar products in $\hat{H}(\Omega, \mathbf{R})$, $\hat{W}(\Omega, \mathbf{R})$, $\hat{V}'(\Omega, \mathbf{R})$ and to transform our problem as follows.

By use of Plancherel's theorem and (5.9) with zero initial data, (5.7) yields

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{\Omega}\Big(\Big\{-i\omega[\varepsilon\hat{\mathbf{E}}(\mathbf{x},\omega)+\hat{\vartheta}(\mathbf{x},\omega)\mathbf{a}]+\nabla\times\hat{\mathbf{H}}(\mathbf{x},\omega)+\hat{\mathbf{J}}(\mathbf{x},\omega)\Big\}\cdot\hat{\mathbf{e}}^*(\mathbf{x},\omega)$$
$$+[i\omega\mu\hat{\mathbf{H}}(\mathbf{x},\omega)+\nabla\times\hat{\mathbf{E}}(\mathbf{x},\omega)]\cdot\hat{\mathbf{h}}^*(\mathbf{x},\omega)+[(1+i\omega\alpha)\hat{\mathbf{J}}(\mathbf{x},\omega)$$
$$-\sigma\hat{\mathbf{E}}(\mathbf{x},\omega)]\cdot\hat{\mathbf{u}}^*(\mathbf{x},\omega)+[(1+i\omega\tau)\hat{\mathbf{q}}(\mathbf{x},\omega)+k\nabla\hat{\vartheta}(\mathbf{x},\omega)]\cdot\hat{\mathbf{v}}^*(\mathbf{x},\omega)$$
$$+i\omega\Theta_0[(c/\Theta_0+\mathbf{a}^2/\varepsilon)\hat{\vartheta}(\mathbf{x},\omega)+\mathbf{a}\cdot\hat{\mathbf{E}}(\mathbf{x},\omega)]\hat{\beta}^*(\mathbf{x},\omega)-\hat{\mathbf{q}}(\mathbf{x},\omega)$$
$$\cdot\nabla\hat{\beta}^*(\mathbf{x},\omega)\Big)\,d\mathbf{x}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{\Omega}[\hat{\mathbf{F}}(\mathbf{x},\omega)\cdot\hat{\mathbf{e}}^*(\mathbf{x},\omega)+\hat{\mathbf{G}}(\mathbf{x},\omega)\cdot\hat{\mathbf{h}}^*(\mathbf{x},\omega)$$
$$+\hat{\mathbf{I}}(\mathbf{x},\omega)\cdot\hat{\mathbf{u}}^*(\mathbf{x},\omega)+\hat{\mathbf{Q}}(\mathbf{x},\omega)\cdot\hat{\mathbf{v}}^*(\mathbf{x},\omega)+\hat{R}(\mathbf{x},\omega)\hat{\beta}^*(\mathbf{x},\omega)]d\mathbf{x}d\omega \tag{5.10}$$

for any $(\hat{\mathbf{e}},\hat{\mathbf{h}},\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\beta})\in\hat{W}(\Omega,\mathbf{R})$.

Here * denotes the complex conjugate.

Remark 5.1. Let $(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}},\hat{\mathbf{q}},\hat{\vartheta})$ be the Fourier transform of a weak solution of P′, as a consequence of (5.10) it follows that it is a weak solution of the problem we can derive by the formal application of the Fourier transfom with respect to time to the problem P′, i.e.,

$$-i\omega[\varepsilon\hat{\mathbf{E}}(\mathbf{x},\omega)+\hat{\vartheta}(\mathbf{x},\omega)\mathbf{a}])+\nabla\times\hat{\mathbf{H}}(\mathbf{x},\omega)-\hat{\mathbf{J}}(\mathbf{x},\omega)=\hat{\mathbf{F}}(\mathbf{x},\omega), \tag{5.11}$$

$$i\omega\mu\hat{\mathbf{H}}(\mathbf{x},\omega)+\nabla\times\hat{\mathbf{E}}(\mathbf{x},\omega)=\hat{\mathbf{G}}(\mathbf{x},\omega), \tag{5.12}$$

$$i\omega\left[\Theta_0\mathbf{a}\cdot\hat{\mathbf{E}}(\mathbf{x},\omega)+\left(c+\Theta_0\mathbf{a}^2/\varepsilon\right)\hat{\vartheta}(\mathbf{x},\omega)\right]+\nabla\cdot\hat{\mathbf{q}}(\mathbf{x},\omega)=\hat{R}(\mathbf{x},\omega), \tag{5.13}$$

$$(1+i\omega\alpha)\hat{\mathbf{J}}(\mathbf{x},\omega)-\sigma\hat{\mathbf{E}}(\mathbf{x},\omega)=\hat{\mathbf{I}}(\mathbf{x},\omega), \tag{5.14}$$

$$(1+i\omega\tau)\hat{\mathbf{q}}(\mathbf{x},\omega)+k\nabla\hat{\vartheta}(\mathbf{x},\omega)=\hat{\mathbf{Q}}(\mathbf{x},\omega), \tag{5.15}$$

with

$$\hat{\mathbf{E}}(\mathbf{x},\omega)\times\mathbf{n}=\mathbf{0},\qquad \hat{\vartheta}(\mathbf{x},\omega)=0 \tag{5.16}$$

for all $\omega\in\mathbf{R}$.

We observe that these boundary conditions, the first of which yields $\mathbf{H}\cdot\mathbf{n}=0$ too, and the assumed hypothesis on the greater regularities of $\vartheta\in H^1_{\vartheta}(\Omega)$ and $\mathbf{H}\in H^1_H(\Omega)$ with Ω simply connected [8] yield the following inequalities:

$$\int_{\Omega}\left|\hat{\vartheta}\right|^2 d\mathbf{x}\le\beta_{\vartheta}(\Omega)\int_{\Omega}\left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x},\qquad \int_{\Omega}\left|\hat{\mathbf{H}}\right|^2 d\mathbf{x}\le\beta_H(\Omega)\int_{\Omega}\left|\nabla\times\hat{\mathbf{H}}\right|^2 d\mathbf{x}, \tag{5.17}$$

where $\beta_\vartheta(\Omega)$ and $\beta_H(\Omega)$ are constant and depend only on the domain Ω.

We denote by $\Lambda[(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta})]$ and by $\langle(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta})\rangle$ the first and the second integral of (5.10), respectively.

Theorem 5.1. *For any* $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}) \in \hat{V}'(\Omega, \mathbf{R})$ *there exists a unique solution* $(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}) \in \hat{H}(\Omega, \mathbf{R})$ *such that*

$$\Lambda[(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta})] = \langle(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}), (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta})\rangle \tag{5.18}$$

for any $(\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta}) \in \hat{W}(\Omega, \mathbf{R})$.

Proof. *Uniqueness*. To show the uniqueness of the solution, we prove that the homogeneous system obtained by (5.11) – (5.15), where the right-hand side of each equation is put equal to zero, has only the zero solution for all $\omega \in \mathbf{R}$.

Thus, keeping always in mind the system so modified, if we consider the inner product of the conjugate of (5.12) with $\hat{\mathbf{H}}$ and integrate over Ω we have

$$-i\omega\mu \int_\Omega \left|\hat{\mathbf{H}}\right|^2 d\mathbf{x} + \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}} d\mathbf{x} = 0, \tag{5.19}$$

whose real part gives

$$\operatorname{Re} \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}} d\mathbf{x} = 0. \tag{5.20}$$

From (5.11), the inner product with $\hat{\mathbf{E}}^*$, taking account of the identity $\nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}} - \nabla \times \hat{\mathbf{H}} \cdot \hat{\mathbf{E}}^* = \nabla \cdot (\hat{\mathbf{E}}^* \times \hat{\mathbf{H}})$ and $(5.16)_1$, yields

$$-i\omega\varepsilon \int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} - i\omega\mathbf{a} \cdot \int_\Omega \hat{\vartheta}\hat{\mathbf{E}}^* d\mathbf{x} + \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}} d\mathbf{x} - \int_\Omega \hat{\mathbf{J}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} = 0, \tag{5.21}$$

from which and (5.20) it follows that

$$\omega \operatorname{Im} \mathbf{a} \cdot \int_\Omega \hat{\vartheta}\hat{\mathbf{E}}^* d\mathbf{x} = \operatorname{Re} \int_\Omega \hat{\mathbf{J}} \cdot \hat{\mathbf{E}}^* d\mathbf{x}. \tag{5.22}$$

The real part of

$$\frac{1}{\sigma}(1 - i\omega\alpha) \int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} - \int_\Omega \hat{\mathbf{E}}^* \cdot \hat{\mathbf{J}} d\mathbf{x} = 0, \tag{5.23}$$

derived by the inner product of the conjugate of (5.14) with $\hat{\mathbf{J}}$, together with (5.22) give

$$\omega \operatorname{Im} \mathbf{a} \cdot \int_\Omega \hat{\vartheta}\hat{\mathbf{E}}^* d\mathbf{x} = \frac{1}{\sigma} \int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x}. \tag{5.24}$$

From (5.13), multiplying by $\hat{\vartheta}^*$, upon an integration by parts and on account of $(5.16)_2$, it follows that

$$i\omega\left[\Theta_0\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} + \int_\Omega \left(c+\Theta_0\mathbf{a}^2/\varepsilon\right)\left|\hat{\vartheta}\right|^2 d\mathbf{x}\right] - \int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} = 0, \tag{5.25}$$

whence

$$\omega\,\mathrm{Im}\,\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} = -\frac{1}{\Theta_0}\,\mathrm{Re}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x}. \tag{5.26}$$

The real part of

$$\frac{1}{k}(1-i\omega\tau)\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} + \int_\Omega \nabla\hat{\vartheta}^*\cdot\hat{\mathbf{q}} d\mathbf{x} = 0, \tag{5.27}$$

obtained by taking the inner product of the conjugate of (5.15) with $\hat{\mathbf{q}}$, allows us to rewrite (5.26) as

$$\omega\,\mathrm{Im}\,\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} = \frac{1}{k\Theta_0}\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}. \tag{5.28}$$

Adding (5.24) and (5.28) we get

$$\frac{1}{\sigma}\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} + \frac{1}{k\Theta_0}\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} = 0, \tag{5.29}$$

from which, taking account of $(3.9)_{3,4}$, we have

$$\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} = 0, \quad \int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} = 0. \tag{5.30}$$

Then, from (5.14) and (5.15) it follows that

$$\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} = 0, \quad \int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} = 0 \tag{5.31}$$

and from (5.17), or (5.13) and (5.12), analogous results hold for $\hat{\vartheta}$ and $\hat{\mathbf{H}}$.

Existence. To prove the existence we consider the following lemmas.

Lemma 5.1. *For all* $\omega \in \mathbf{R}$*, any weak solution to the problem (5.11) – (5.16) satisfies the following inequality*

$$\mathcal{G}(\omega) \leq \nu^2(\omega) \int_\Omega \left(\left|\hat{\mathbf{F}}\right|^2 + \left|\hat{\mathbf{G}}\right|^2 + \left|\hat{\mathbf{I}}\right|^2 + \left|\hat{\mathbf{Q}}\right|^2 + \left|\hat{R}\right|^2 \right) d\mathbf{x}, \tag{5.32}$$

where

$$\mathcal{G}(\omega) = \int_\Omega \Bigg(\left|\hat{\mathbf{E}}\right|^2 + \left|\hat{\mathbf{H}}\right|^2 + \left|\hat{\mathbf{J}}\right|^2 + \left|\hat{\mathbf{q}}\right|^2 + \left|\hat{\vartheta}\right|^2 + \left|\nabla \times \hat{\mathbf{E}}\right|^2 + \left|\nabla \times \hat{\mathbf{H}}\right|^2 + \left|\nabla \hat{\vartheta}\right|^2 \Bigg) d\mathbf{x} \tag{5.33}$$

and $\nu(\omega)$ *is a positive function of* ω, Ω *and the material constants.*

Proof. Let us integrate over Ω the relations derived by taking the inner products of (5.11) by $\hat{\mathbf{E}}^*$, $\mathbf{a}\hat{\vartheta}^*$ and $\nabla \times \hat{\mathbf{H}}^*$, of the conjugate of (5.12) by $\hat{\mathbf{H}}$ and $\nabla \times \hat{\mathbf{E}}$, of (5.14) by $\hat{\mathbf{E}}^*$ and of the conjugate of the same (5.14) by $\hat{\mathbf{J}}$, $\nabla \times \hat{\mathbf{H}}$ and $\mathbf{a}\hat{\vartheta}$, of (5.15) by $\nabla\hat{\vartheta}^*$, of the conjugate of the same (5.15) by $\hat{\mathbf{q}}$ and multiplying (5.13) by $\hat{\vartheta}^*$.

Thus, from (5.11), with two integrations similar to the one made to derive (5.21), we get

$$- i\omega\varepsilon \int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} - i\omega\mathbf{a} \cdot \int_\Omega \hat{\vartheta}\hat{\mathbf{E}}^* d\mathbf{x} + \int_\Omega \hat{\mathbf{H}} \cdot \nabla \times \hat{\mathbf{E}}^* d\mathbf{x} - \int_\Omega \hat{\mathbf{J}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} = \int_\Omega \hat{\mathbf{F}} \cdot \hat{\mathbf{E}}^* d\mathbf{x}, \tag{5.34}$$

$$- i\omega\varepsilon\mathbf{a} \cdot \int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} - i\omega\mathbf{a}^2 \int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x} + \mathbf{a} \cdot \int_\Omega \nabla \times \hat{\mathbf{H}}\hat{\vartheta}^* d\mathbf{x} - \mathbf{a} \cdot \int_\Omega \hat{\mathbf{J}}\vartheta^* d\mathbf{x} = \int_\Omega \hat{\mathbf{F}} \cdot \mathbf{a}\hat{\vartheta}^* d\mathbf{x}, \tag{5.35}$$

$$- i\omega\varepsilon \int_\Omega \nabla \times \hat{\mathbf{E}} \cdot \hat{\mathbf{H}}^* d\mathbf{x} - i\omega\mathbf{a} \cdot \int_\Omega \hat{\vartheta}\nabla \times \hat{\mathbf{H}}^* d\mathbf{x} + \int_\Omega \left|\nabla \times \hat{\mathbf{H}}\right|^2 d\mathbf{x} - \int_\Omega \hat{\mathbf{J}} \cdot \nabla \times \hat{\mathbf{H}}^* d\mathbf{x} = \int_\Omega \hat{\mathbf{F}} \cdot \nabla \times \hat{\mathbf{H}}^* d\mathbf{x}; \tag{5.36}$$

from (5.12) we have

$$-i\omega\mu\int_\Omega \left|\hat{\mathbf{H}}\right|^2 d\mathbf{x}+\int_\Omega \nabla\times\hat{\mathbf{E}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}=\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}, \tag{5.37}$$

$$-i\omega\mu\int_\Omega \hat{\mathbf{H}}^*\cdot\nabla\times\hat{\mathbf{E}}d\mathbf{x}+\int_\Omega \left|\nabla\times\hat{\mathbf{E}}\right|^2 d\mathbf{x}=\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}d\mathbf{x}; \tag{5.38}$$

from (5.13), with an integration by parts taking account of $(5.16)_2$, we obtain

$$i\omega\Theta_0\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^*d\mathbf{x}+i\omega\left(c+\Theta_0\mathbf{a}^2/\varepsilon\right)\int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x}-\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}=\int_\Omega \hat{R}\hat{\vartheta}^*d\mathbf{x}. \tag{5.39}$$

Moreover, (5.14) yields

$$(1-i\omega\alpha)\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x}-\sigma\int_\Omega \hat{\mathbf{E}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}=\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}, \tag{5.40}$$

$$(1+i\omega\alpha)\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}-\sigma\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x}=\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}, \tag{5.41}$$

$$(1-i\omega\alpha)\int_\Omega \hat{\mathbf{J}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}-\sigma\int_\Omega \nabla\times\hat{\mathbf{E}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}=\int_\Omega \hat{\mathbf{I}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}, \tag{5.42}$$

$$(1-i\omega\alpha)\mathbf{a}\cdot\int_\Omega \hat{\mathbf{J}}^*\hat{\vartheta}d\mathbf{x}-\sigma\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}^*\vartheta d\mathbf{x}=\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta}d\mathbf{x}, \tag{5.43}$$

in the last relation by one we have integrated as for (5.21).

Finally, (5.15) gives

$$(1-i\omega\tau)\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}+k\int_\Omega \nabla\hat{\vartheta}^*\cdot\hat{\mathbf{q}}d\mathbf{x}=\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}, \tag{5.44}$$

$$(1+i\omega\tau)\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}+k\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x}=\int_\Omega \hat{\mathbf{Q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}. \tag{5.45}$$

We first consider that the real and the imaginary parts of (5.44)

$$\mathrm{Re}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}=\frac{1}{k}\left(\mathrm{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}-\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}\right), \tag{5.46}$$

$$\mathrm{Im}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}=\frac{1}{k}\left(\mathrm{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}+\omega\tau\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}\right), \tag{5.47}$$

from which, subtracting (5.47) multiplied by $\omega\tau$ from (5.46) and taking account of

$$\operatorname{Re}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} - \omega\tau \operatorname{Im}\int_\Omega \hat{\mathbf{q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} = \operatorname{Re}\int_\Omega \hat{\mathbf{Q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} - k\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x}, \tag{5.48}$$

the real part of (5.45), we obtain

$$k\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} - \frac{1}{k}(1+\tau^2\omega^2)\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} = \operatorname{Re}\int_\Omega \hat{\mathbf{Q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\left(\omega\tau \operatorname{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x} - \operatorname{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\right). \tag{5.49}$$

Analogously, from the real and the imaginary parts of (5.40),

$$\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} - \sigma \operatorname{Re}\int_\Omega \hat{\mathbf{E}}^*\cdot\hat{\mathbf{J}}d\mathbf{x} = \operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}, \tag{5.50}$$

$$-\omega\alpha\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} - \sigma \operatorname{Im}\int_\Omega \hat{\mathbf{E}}^*\cdot\hat{\mathbf{J}}d\mathbf{x} = \operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}, \tag{5.51}$$

and the real part of (5.41),

$$-\operatorname{Re}\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} + \omega\alpha \operatorname{Im}\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} = -\sigma\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} - \operatorname{Re}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x}, \tag{5.52}$$

we have

$$\sigma\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} - \frac{1}{\sigma}(1+\alpha^2\omega^2)\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x} = -\operatorname{Re}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} - \frac{1}{\sigma}\left(\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x} - \omega\alpha \operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}\right). \tag{5.53}$$

A useful relation follows at once from the real part of (5.37),

$$\operatorname{Re}\int_\Omega \nabla\times\hat{\mathbf{E}}^*\cdot\hat{\mathbf{H}}d\mathbf{x} = \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}. \tag{5.54}$$

If we consider (5.41) again, adding its real part (5.52) to the imaginary one, multiplied by $\omega\alpha$, we obtain the relation

$$\operatorname{Re}\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} = \frac{1}{1+\alpha^2\omega^2}\left(\sigma\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x} + \operatorname{Re}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} + \omega\alpha \operatorname{Im}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x}\right), \tag{5.55}$$

which, together with (5.54), allows us to write the real part of (5.34) in as:

$$\omega \operatorname{Im} \mathbf{a} \cdot \int_\Omega \hat{\vartheta} \hat{\mathbf{E}}^* d\mathbf{x} = \operatorname{Re} \int_\Omega \hat{\mathbf{F}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} - \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}} d\mathbf{x}$$
$$+ \frac{1}{1+\alpha^2\omega^2} \left(\sigma \int_\Omega \left| \hat{\mathbf{E}} \right|^2 d\mathbf{x} + \operatorname{Re} \int_\Omega \hat{\mathbf{I}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} + \omega\alpha \operatorname{Im} \int_\Omega \hat{\mathbf{I}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} \right). \tag{5.56}$$

A different expression for the left-hand side of (5.56) can be derived from the real part of (5.39), which, on account of (5.46), can be written as follows:

$$\omega \operatorname{Im} \mathbf{a} \cdot \int_\Omega \hat{\vartheta} \hat{\mathbf{E}}^* d\mathbf{x} = \frac{1}{\Theta_0} \left[\operatorname{Re} \int_\Omega \hat{R} \hat{\vartheta}^* d\mathbf{x} + \frac{1}{k} \left(\operatorname{Re} \int_\Omega \hat{\mathbf{Q}}^* \hat{\mathbf{q}} d\mathbf{x} - \int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} \right) \right]. \tag{5.57}$$

Finally, subtracting (5.57) from (5.56), we get

$$\sigma \int_\Omega \left| \hat{\mathbf{E}} \right|^2 d\mathbf{x} + \frac{1}{k\Theta_0} (1+\alpha^2\omega^2) \int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}$$
$$= (1+\alpha^2\omega^2) \left[\frac{1}{k\Theta_0} \left(\operatorname{Re} \int_\Omega \hat{\mathbf{Q}}^* \cdot \hat{\mathbf{q}} d\mathbf{x} + k \operatorname{Re} \int_\Omega \hat{R} \hat{\vartheta}^* d\mathbf{x} \right) - \operatorname{Re} \int_\Omega \hat{\mathbf{F}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} \right.$$
$$\left. + \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}} d\mathbf{x} \right] - \operatorname{Re} \int_\Omega \hat{\mathbf{I}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} - \omega\alpha \operatorname{Im} \int_\Omega \hat{\mathbf{I}} \cdot \hat{\mathbf{E}}^* d\mathbf{x}. \tag{5.58}$$

This relation, if we subtract (5.53), yields

$$\frac{1}{\sigma} \int_\Omega \left| \hat{\mathbf{J}} \right|^2 d\mathbf{x} + \frac{1}{k\Theta_0} \int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} = \frac{1}{k\Theta_0} \left(\operatorname{Re} \int_\Omega \hat{\mathbf{Q}}^* \cdot \hat{\mathbf{q}} d\mathbf{x} + k \operatorname{Re} \int_\Omega \hat{R} \hat{\vartheta}^* d\mathbf{x} \right)$$
$$+ \frac{1}{1+\alpha^2\omega^2} \left[\frac{1}{\sigma} \operatorname{Re} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}} d\mathbf{x} - \omega\alpha \left(\operatorname{Im} \int_\Omega \hat{\mathbf{I}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} + \frac{1}{\sigma} \operatorname{Im} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}} d\mathbf{x} \right) \right]$$
$$- \operatorname{Re} \int_\Omega \hat{\mathbf{F}} \cdot \hat{\mathbf{E}}^* d\mathbf{x} + \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}} d\mathbf{x} \tag{5.59}$$

and, moreover, gives an increase for $\frac{1}{k\Theta_0} \int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x}$, which allows us to derive from (5.49) the inequality

$$\frac{k}{\Theta_0}\int_\Omega \left|\nabla\hat{\vartheta}\right|^2 d\mathbf{x} \le \frac{1}{\Theta_0}\left[\operatorname{Re}\int_\Omega \hat{\mathbf{Q}}\cdot\nabla\hat{\vartheta}^* d\mathbf{x} + \omega\frac{\tau}{k}\left(\operatorname{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\right.\right.$$

$$\left.\left. +\omega\tau\operatorname{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\right)\right] + (1+\tau^2\omega^2)\left(\frac{1}{\Theta_0}\operatorname{Re}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x} + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}\right.$$

$$\left. -\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^* d\mathbf{x}\right) - \frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}\left(\operatorname{Re}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} + \omega\alpha\operatorname{Im}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^* d\mathbf{x}\right). \tag{5.60}$$

It remains to consider the real parts of (5.38) and (5.36), that is

$$\frac{1}{\mu}\int_\Omega \left|\nabla\times\hat{\mathbf{E}}\right|^2 d\mathbf{x} = \frac{1}{\mu}\operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}d\mathbf{x} - \omega\operatorname{Im}\int_\Omega \nabla\times\hat{\mathbf{E}}\cdot\hat{\mathbf{H}}^* d\mathbf{x}, \tag{5.61}$$

$$\frac{1}{\varepsilon}\int_\Omega \left|\nabla\times\hat{\mathbf{H}}\right|^2 d\mathbf{x} = \frac{1}{\varepsilon}\left(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^* d\mathbf{x} + \operatorname{Re}\int_\Omega \hat{\mathbf{J}}\cdot\nabla\times\hat{\mathbf{H}}^* d\mathbf{x}\right.$$

$$\left. -\omega\operatorname{Im}\mathbf{a}\cdot\int_\Omega \hat{\vartheta}\nabla\times\hat{\mathbf{H}}^* d\mathbf{x}\right) - \omega\operatorname{Im}\int_\Omega \nabla\times\hat{\mathbf{E}}\cdot\hat{\mathbf{H}}^* d\mathbf{x}, \tag{5.62}$$

where only the last three terms must be evaluated.

To do this we first consider the imaginary part of (5.39), which, on account of (5.47), assumes the following form:

$$\omega\operatorname{Re}\mathbf{a}\cdot\int_\Omega \hat{\mathbf{E}}\hat{\vartheta}^* d\mathbf{x} = \frac{1}{\Theta_0}\left(\operatorname{Im}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\operatorname{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\right)$$

$$+\omega\left[\frac{\tau}{k\Theta_0}\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} - \left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)\int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x}\right]. \tag{5.63}$$

The imaginary part of (5.34), together with (5.51) and (5.63), yields the required quantity,

$$-\omega\,\mathrm{Im}\int_\Omega \nabla\times\hat{\mathbf{E}}\cdot\hat{\mathbf{H}}^* d\mathbf{x} = \omega\left[\mathrm{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^* d\mathbf{x} - \frac{1}{\sigma}\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}} d\mathbf{x} \right.$$
$$\left. + \frac{1}{\Theta_0}\left(\mathrm{Im}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\mathrm{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}} d\mathbf{x}\right)\right] + \omega^2\left[\varepsilon\int_\Omega \left|\hat{\mathbf{E}}\right|^2 d\mathbf{x}\right.$$
$$\left. + \frac{\tau}{k\Theta_0}\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} - \left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)\int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x} - \frac{\alpha}{\sigma}\int_\Omega \left|\hat{\mathbf{J}}\right|^2 d\mathbf{x}\right]. \tag{5.64}$$

Then, from (5.43), adding its imaginary part to the real one multiplied by $\omega\alpha$, we obtain a relation, which, after introducing (5.57) and (5.63), gives

$$\frac{\omega}{\varepsilon}\mathrm{Im}\,\mathbf{a}\cdot\int_\Omega \hat{\mathbf{J}}\hat{\vartheta}^* d\mathbf{x} = \frac{1}{1+\alpha^2\omega^2}\frac{\sigma}{\varepsilon}\left\{\omega^2\alpha\left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)\int_\Omega \left|\hat{\vartheta}\right|^2 d\mathbf{x}\right.$$
$$+ (1-\omega^2\alpha\tau)\frac{1}{k\Theta_0}\int_\Omega |\hat{\mathbf{q}}|^2 d\mathbf{x} - \omega\frac{\alpha}{\Theta_0}\left(\mathrm{Im}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\mathrm{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}} d\mathbf{x}\right)$$
$$- \omega^2\frac{\alpha}{\sigma}\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta} d\mathbf{x} - \frac{1}{\Theta_0}\left(\mathrm{Re}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x} + \frac{1}{k}\mathrm{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}} d\mathbf{x}\right)$$
$$\left. - \frac{\omega}{\sigma}\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta} d\mathbf{x}\right\}. \tag{5.65}$$

This expression, together with (5.63), allows us to evaluate, from the imaginary part of (5.35), the other quantity

$$-\frac{\omega}{\varepsilon}\mathrm{Im}\,\mathbf{a}\cdot\int_\Omega \hat{\vartheta}\nabla\times\hat{\mathbf{H}}^* d\mathbf{x} = \frac{\omega}{\varepsilon}\mathrm{Im}\int_\Omega \hat{\mathbf{F}}\cdot\mathbf{a}\hat{\vartheta}^* d\mathbf{x} - \frac{1}{1+\alpha^2\omega^2}\frac{1}{\varepsilon}\left[\frac{\sigma}{\Theta_0}\left(\mathrm{Re}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x}\right.\right.$$
$$\left.\left. + \frac{1}{k}\mathrm{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}} d\mathbf{x}\right) + \omega\left(\omega\alpha\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta} d\mathbf{x} + \mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta} d\mathbf{x}\right)\right]$$
$$+ \left(1-\frac{\alpha\sigma}{\varepsilon}\frac{1}{1+\alpha^2\omega^2}\right)\frac{\omega}{k\Theta_0}\left(\mathrm{Im}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}} d\mathbf{x} + k\,\mathrm{Im}\int_\Omega \hat{R}\hat{\vartheta}^* d\mathbf{x}\right)$$

$$+\omega^2\left[\frac{1}{1+\alpha^2\omega^2}\frac{\alpha\sigma}{\varepsilon}\left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)-\frac{c}{\Theta_0}\right]\int_\Omega\left|\hat{\vartheta}\right|^2 d\mathbf{x}$$

$$+\left[\tau\omega^2\left(1-\frac{1}{1+\alpha^2\omega^2}\frac{\alpha\sigma}{\varepsilon}\right)+\frac{1}{1+\alpha^2\omega^2}\frac{\sigma}{\varepsilon}\right]\frac{1}{k\Theta_0}\int_\Omega\left|\hat{\mathbf{q}}\right|^2 d\mathbf{x}. \qquad (5.66)$$

Finally, for the third quantity we must derive, consider (5.42). Subtracting from its real part the imaginary one multiplied by $\omega\alpha$ and taking into account (5.54) and (5.64), we get

$$\frac{1}{\varepsilon}\operatorname{Re}\int_\Omega\hat{\mathbf{J}}\cdot\nabla\times\hat{\mathbf{H}}^*d\mathbf{x}=\frac{1}{1+\alpha^2\omega^2}\frac{1}{\varepsilon}\left\{\sigma\operatorname{Re}\int_\Omega\hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}+\operatorname{Re}\int_\Omega\hat{\mathbf{I}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}\right.$$

$$-\omega\alpha\operatorname{Im}\int_\Omega\hat{\mathbf{I}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}-\omega\alpha\sigma\left[\operatorname{Im}\int_\Omega\hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}-\frac{1}{\sigma}\operatorname{Im}\int_\Omega\hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}+\right.$$

$$+\frac{1}{k\Theta_0}\left(\operatorname{Im}\int_\Omega\hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}+k\operatorname{Im}\int_\Omega\hat{R}\hat{\vartheta}^*d\mathbf{x}\right)\bigg]+\omega^2\alpha\sigma\left(\left[\frac{\alpha}{\sigma}\int_\Omega\left|\hat{\mathbf{J}}\right|^2 d\mathbf{x}\right.\right.$$

$$\left.\left.\left.+\left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)\int_\Omega\left|\hat{\vartheta}\right|^2 d\mathbf{x}\right]-\left(\varepsilon\int_\Omega\left|\hat{\mathbf{E}}\right|^2 d\mathbf{x}+\frac{\tau}{k\Theta_0}\int_\Omega\left|\hat{\mathbf{q}}\right|^2 d\mathbf{x}\right)\right)\right\}. \qquad (5.67)$$

Thus, we have an estimate for all the quantities of (5.33), which, taking account of (5.17), can be increased as follows:

$$\xi\mathcal{G}(\omega)\le\int_\Omega\left\{\left[\sigma\left|\hat{\mathbf{E}}\right|^2+\frac{1}{k\Theta_0}(1+\alpha^2\omega^2)\left|\hat{\mathbf{q}}\right|^2\right]\left(\frac{1}{\sigma}\left|\hat{\mathbf{J}}\right|^2+\frac{1}{k\Theta_0}\left|\hat{\mathbf{q}}\right|^2\right)+\frac{1}{\mu}\left|\nabla\times\hat{\mathbf{E}}\right|^2\right.$$

$$\left.+\left[1+\varepsilon\beta_H(\Omega)\right]\frac{1}{\varepsilon}\left|\nabla\times\hat{\mathbf{H}}\right|^2 d\mathbf{x}+\left[1+\frac{\Theta_0}{k}\beta_\vartheta(\Omega)\right]\frac{k}{\Theta_0}\left|\nabla\hat{\vartheta}\right|^2\right\}, \qquad (5.68)$$

where $\xi=\min\{\sigma,1,1/\sigma,1/k\Theta_0,1/\mu,1/\varepsilon,k/\Theta_0\}$.

We consider the relations (5.58), (5.59) and (5.60) together with (5.61) and (5.62), which must be substituted into this inequality. We observe that in (5.64), (5.66) and (5.67), to be considered for (5.61) and (5.62), there are some negative terms which can be neglected, but there are the following positive terms

$$\left\{[2+\varepsilon\beta_H(\Omega)]\tau\omega^2+[1+\varepsilon\beta_H(\Omega)]\left(\tau\omega^2+\frac{\sigma}{\varepsilon}\frac{1}{1+\alpha^2\omega^2}\right)\right\}\left(\frac{1}{k\Theta_0}\int_\Omega|\hat{\mathbf{q}}|^2d\mathbf{x}\right)$$

$$+[2+\varepsilon\beta_H(\Omega)]\frac{\varepsilon}{\sigma}\omega^2\left(\sigma\int_\Omega\left|\hat{\mathbf{E}}\right|^2d\mathbf{x}\right)+\frac{\sigma\alpha^2}{\varepsilon}\left[1+\varepsilon\beta_H(\Omega)\right]\frac{\omega^2}{1+\alpha^2\omega^2}\left(\frac{1}{\sigma}\int_\Omega\left|\hat{\mathbf{J}}\right|^2d\mathbf{x}\right)$$

$$+2\frac{\Theta_0}{k}[1+\varepsilon\beta_H(\Omega)]\frac{\alpha\sigma}{\varepsilon}\left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)\frac{\omega^2}{1+\alpha^2\omega^2}\left(\frac{k}{\Theta_0}\int_\Omega\left|\hat{\vartheta}\right|^2d\mathbf{x}\right),$$

whose four expressions, in parenthesis and containing the four integrals, can be increased by the right-hand sides of (5.59), (5.58), (5.59), $(5.17)_1$ and (5.60), respectively.

To simplify the result, it is useful to put

$$\beta_1=[1+\varepsilon\beta_H(\Omega)]\,\sigma/\varepsilon,\quad \beta_2=[2+\varepsilon\beta_H(\Omega)]\,\varepsilon/\sigma,\quad \beta_3=[3+2\varepsilon\beta_H(\Omega)]\,\tau,$$
$$\beta_4=1+\Theta_0\beta_\vartheta(\Omega)/k,\quad \beta_5=2\left(c/\Theta_0+\mathbf{a}^2/\varepsilon\right)\alpha\Theta_0\beta_\vartheta(\Omega)/k,$$

with which we define

$$c_1=2+\beta_1,\quad c_2=\alpha^2+\beta_2+\beta_3,\quad c_3=\beta_2\alpha^2,\quad c_4=\beta_1\beta_5,$$
$$c_5=\beta_2\sigma/\varepsilon-\beta_1\alpha,\quad c_6=\beta_1/\sigma,\quad c_7=1+\beta_1$$

and finally we introduce

$$d_1=c_1+\beta_4,\quad d_2=c_2+\beta_4\tau,\quad d_3=kc_1+\beta_4/\Theta_0,\quad d_4=kc_2+\beta_4\tau^2/\Theta_0,$$
$$d_5=c_5-c_6\alpha\sigma,\quad d_6=c_2+\beta_4\tau^2,\quad d_7=\beta_4\tau+c_6\varepsilon.$$

Thus, (5.68) becomes

$$\xi\mathcal{G}(\omega)\leq\Bigg(-\left[d_1+\left(d_2+c_4\frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}\right)\omega^2+c_3\omega^4\right]\operatorname{Re}\int_\Omega\hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}$$

$$+c_5\frac{\omega}{1+\alpha^2\omega^2}\operatorname{Im}\int_\Omega\hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}\Bigg)+\left(c_6\operatorname{Re}\int_\Omega\hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*d\mathbf{x}\right)+\left(c_6\omega\operatorname{Im}\int_\Omega\hat{\mathbf{F}}\cdot\mathbf{a}\hat{\vartheta}^*d\mathbf{x}\right)$$

$$+\left(\left[d_1+\left(d_2+c_4\frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}\right)\omega^2+c_3\omega^4+\beta_1\frac{1}{1+\alpha^2\omega^2}\right]\operatorname{Re}\int_\Omega\hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}d\mathbf{x}\right)$$

$$
\begin{aligned}
&+\left(\frac{1}{\mu}\operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}d\mathbf{x}\right)+\Bigg(\Bigg[d_3+\left(d_4+\frac{c_4}{\Theta_0}\frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}\right)\omega^2+kc_3\omega^4\\
&-\frac{\beta_1}{\Theta_0}\frac{1}{1+\alpha^2\omega^2}\Bigg]\operatorname{Re}\int_\Omega \hat{R}\hat{\vartheta}^*d\mathbf{x}+\frac{1}{\Theta_0}\left(c_6\varepsilon+d_5\frac{1}{1+\alpha^2\omega^2}\right)\omega\operatorname{Im}\int_\Omega \hat{R}\hat{\vartheta}^*d\mathbf{x}\Bigg)\\
&-\Bigg(\Bigg[1+\beta_2\omega^2+\left(\beta_4+c_4\frac{\omega^2}{1+\alpha^2\omega^2}\right)\frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}\Bigg]\operatorname{Re}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}+\Big[1+\beta_2\omega^2\\
&+\left(\beta_4+c_4\frac{\omega^2}{1+\alpha^2\omega^2}\right)\frac{1+\tau^2\omega^2}{1+\alpha^2\omega^2}+(c_7+\beta_3\omega^2)\frac{1}{1+\alpha^2\omega^2}\Big]\alpha\omega\operatorname{Im}\int_\Omega \hat{\mathbf{I}}\cdot\hat{\mathbf{E}}^*d\mathbf{x}\Bigg)\\
&+\left(\frac{1}{\sigma}\frac{1}{1+\alpha^2\omega^2}\left\{(c_7+\beta_3\omega^2)\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}-[c_5+(c_7+\beta_3\omega^2)\alpha]\omega\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}d\mathbf{x}\right\}\right)\\
&+\left(c_6\frac{1}{1+\alpha^2\omega^2}\left[\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}-\alpha\omega\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\nabla\times\hat{\mathbf{H}}d\mathbf{x}\right]\right)\\
&-\left(c_6\frac{\omega}{1+\alpha^2\omega^2}\left[\alpha\omega\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta}d\mathbf{x}+\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\mathbf{a}\hat{\vartheta}d\mathbf{x}\right]\right)\\
&+\Bigg(\frac{1}{k\Theta_0}\Bigg\{\left[c_1+d_6\omega^2+\left(c_3+c_4\tau^2\frac{1}{1+\alpha^2\omega^2}\right)\omega^4-\beta_1\frac{1}{1+\alpha^2\omega^2}\right]\operatorname{Re}\int_\Omega \hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\\
&+\left[d_7+(d_5+c_4\tau\omega^2)\frac{1}{1+\alpha^2\omega^2}\right]\omega\operatorname{Im}\hat{\mathbf{Q}}^*\cdot\hat{\mathbf{q}}d\mathbf{x}\Bigg\}\Bigg)\\
&+\left(\frac{1}{\Theta_0}\left(\beta_4+c_4\frac{\omega^2}{1+\alpha^2\omega^2}\right)\operatorname{Re}\int_\Omega \hat{\mathbf{Q}}\cdot\nabla\hat{\vartheta}^*d\mathbf{x}\right).
\end{aligned}
\tag{5.69}
$$

If we denote by the $\lambda_{i,j}(\omega),\ i=1,2,...,12;\ j=1,2$, the coefficients of the real $(j=1)$ and the imaginary $(j=2)$ parts of the twelve different integrals in (5.69) and consider

$$
\lambda(\omega)=\frac{1}{\xi}\max\left\{|\lambda_{i,1}(\omega)|+|\lambda_{i,2}(\omega)|,\quad i=1,2,\dots,12\right\},
\tag{5.70}
$$

from (5.69) with applications of Schwarz's inequality it follows that

$$
\mathcal{G}(\omega)\le 12\lambda(\omega)\max\{1,|\mathbf{a}|\}\left[\int_\Omega\left(\left|\hat{\mathbf{F}}\right|^2+\left|\hat{\mathbf{G}}\right|^2+\left|\hat{\mathbf{I}}\right|^2+\left|\hat{\mathbf{Q}}\right|^2+\left|\hat{R}\right|^2\right)d\mathbf{x}\right]^{1/2}\mathcal{G}^{1/2}(\omega)
\tag{5.71}
$$

and hence (5.32).

Lemma 5.2. *If the sources* $(\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \in V'(\Omega, \mathbf{R}^+)$*, the hypotheses there stated assure that the inverse Fourier transforms of* $(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}) \in \hat{H}(\Omega, \mathbf{R})$ *exist and are* L^2*-functions with zero initial data.*

Proof. Since the 5-tuple $(\mathbf{F}, \mathbf{G}, \mathbf{I}, \mathbf{Q}, R) \in V'(\Omega, \mathbf{R}^+)$, the hypotheses assumed on the sources together with the form of $\nu(\omega)$, which is a continuous function of $\omega \in \mathbf{R}$ and approaches infinity as ω^4, allow us to integrate over $\mathbf{R}$ the right-hand side of (5.32), i.e.,

$$\int_{-\infty}^{+\infty} \int_{\Omega} \nu^2(\omega) \left(\left|\hat{\mathbf{F}}\right|^2 + \left|\hat{\mathbf{G}}\right|^2 + \left|\hat{\mathbf{I}}\right|^2 + \left|\hat{\mathbf{Q}}\right|^2 + \left|\hat{R}\right|^2 \right) d\mathbf{x} d\omega < +\infty. \tag{5.72}$$

Thus, from the inequality (5.32) it follows that

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathcal{G}(\omega) d\omega &= 2\pi \|(\hat{\mathbf{E}}(\mathbf{x},\omega), \hat{\mathbf{H}}(\mathbf{x},\omega), \hat{\mathbf{J}}(\mathbf{x},\omega), \hat{\mathbf{q}}(\mathbf{x},\omega), \hat{\vartheta}(\mathbf{x},\omega))\|^2_{\hat{H}} \\ &\leq \int_{-\infty}^{+\infty} \int_{\Omega} \nu^2(\omega) \left(\left|\hat{\mathbf{F}}\right|^2 + \left|\hat{\mathbf{G}}\right|^2 + \left|\hat{\mathbf{I}}\right|^2 + \left|\hat{\mathbf{Q}}\right|^2 + \left|\hat{R}\right|^2 \right) d\mathbf{x} d\omega, \end{aligned} \tag{5.73}$$

whence, by virtue of Plancherel's theorem, the inverse Fourier transforms of $(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta})$ exist.

From the linearity of system (5.11) – (5.15) and inequality (5.32) of Lemma 5.1 we have this corollary.

Corollary 5.1. *Let* $(\hat{\mathbf{F}}^{(i)}, \hat{\mathbf{G}}^{(i)}, \hat{\mathbf{I}}^{(i)}, \hat{\mathbf{Q}}^{(i)}, \hat{R}^{(i)})$*,* $i = 1, 2$*, be two source fields. The corresponding solutions of our problem* $(\hat{\mathbf{E}}^{(i)}, \hat{\mathbf{H}}^{(i)}, \hat{\mathbf{J}}^{(i)}, \hat{\mathbf{q}}^{(i)}, \hat{\vartheta}^{(i)})$ *satisfy*

$$\begin{aligned} &2\pi \left\| (\hat{\mathbf{E}}^{(1)} - \hat{\mathbf{E}}^{(2)}, \hat{\mathbf{H}}^{(1)} - \hat{\mathbf{H}}^{(2)}, \hat{\mathbf{J}}^{(1)} - \hat{\mathbf{J}}^{(2)}, \hat{\mathbf{q}}^{(1)} - \hat{\mathbf{q}}^{(2)}, \hat{\vartheta}^{(1)} - \hat{\vartheta}^{(2)}) \right\|^2_{\hat{H}} \\ &\quad \leq \int_{-\infty}^{+\infty} \int_{\Omega} \nu^2(\omega) \left(\left|\hat{\mathbf{F}}^{(1)} - \hat{\mathbf{F}}^{(2)}\right|^2 + \left|\hat{\mathbf{G}}^{(1)} - \hat{\mathbf{G}}^{(2)}\right|^2 + \left|\hat{\mathbf{I}}^{(1)} - \hat{\mathbf{I}}^{(2)}\right|^2 \right. \\ &\qquad \left. + \left|\hat{\mathbf{Q}}^{(1)} - \hat{\mathbf{Q}}^{(2)}\right|^2 + \left|\hat{R}^{(1)} - \hat{R}^{(2)}\right|^2 \right) d\mathbf{x} d\omega. \end{aligned} \tag{5.74}$$

Lemma 5.3. *The subset*

$$\begin{aligned} S_a = \Big\{ &(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}) \in \hat{V}'(\Omega, \mathbf{R}) : \exists\, (\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}) \in \hat{H}(\Omega, \mathbf{R}) \\ &\text{which satisfies } (5.10)\ \forall (\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\beta}) \in \hat{W}(\Omega, \mathbf{R}) \Big\} \end{aligned} \tag{5.75}$$

is dense and closed in $\hat{V}'(\Omega, \mathbf{R})$.

Proof. To prove that S_a is dense we suppose that there is a nonzero element $(\hat{\mathbf{F}}^0, \hat{\mathbf{G}}^0, \hat{\mathbf{I}}^0, \hat{\mathbf{Q}}^0, \hat{R}^0) \in \hat{V}'(\Omega, \mathbf{R}) \backslash \bar{S}_a$, $\bar{S}_a$ being the closure of S_a in $\hat{V}'(\Omega, \mathbf{R})$. We use the Hahn–Banach theorem, which states that there exists $(\hat{\mathbf{e}}^0, \hat{\mathbf{h}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0, \hat{\beta}^0) \in \hat{W}(\Omega, \mathbf{R})$ which satisfies these relations

$$\langle (\hat{\mathbf{F}}^0, \hat{\mathbf{G}}^0, \hat{\mathbf{I}}^0, \hat{\mathbf{Q}}^0, \hat{R}^0), (\hat{\mathbf{e}}^0, \hat{\mathbf{h}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0, \hat{\beta}^0) \rangle \neq 0,$$

$$\langle (\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}), (\hat{\mathbf{e}}^0, \hat{\mathbf{h}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0, \hat{\beta}^0) \rangle = 0 \quad \forall (\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}) \in S_a. \tag{5.76}$$

The second condition is equivalent to

$$\Lambda[(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}), (\hat{\mathbf{e}}^0, \hat{\mathbf{h}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0, \hat{\beta}^0)] = 0 \quad \forall (\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}) \in \hat{H}(\Omega, \mathbf{R}) \tag{5.77}$$

because of (5.18), whence it follows that

$$(\hat{\mathbf{e}}^0, \hat{\mathbf{h}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0, \hat{\beta}^0) = 0, \tag{5.78}$$

which does not satisfies (5.76). Hence, S_a is dense in $\hat{V}'(\Omega, \mathbf{R})$.

To show the closure of S_a in $\hat{V}'(\Omega, \mathbf{R})$ we consider a sequence of sources, which is denoted by $\left\{ (\hat{\mathbf{F}}^{(n)}, \hat{\mathbf{G}}^{(n)}, \hat{\mathbf{I}}^{(n)}, \hat{\mathbf{Q}}^{(n)}, \hat{R}^{(n)}) \in S_a, \ n = 1, 2, ... \right\}$ and assumed convergent to $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}) \in \hat{V}'(\Omega, \mathbf{R})$.

Denoting by $(\hat{\mathbf{E}}^{(n)}, \hat{\mathbf{H}}^{(n)}, \hat{\mathbf{J}}^{(n)}, \hat{\mathbf{q}}^{(n)}, \hat{\vartheta}^{(n)}) \in \hat{H}(\Omega, \mathbf{R})$ the corresponding solutions, we consider (5.74) of Corollary 5.1, which gives

$$\begin{aligned} &\left\| (\hat{\mathbf{E}}^{(n)} - \hat{\mathbf{E}}^{(m)}, \hat{\mathbf{H}}^{(n)} - \hat{\mathbf{H}}^{(m)}, \hat{\mathbf{J}}^{(n)} - \hat{\mathbf{J}}^{(m)}, \hat{\mathbf{q}}^{(n)} - \hat{\mathbf{q}}^{(m)}, \hat{\vartheta}^{(n)} - \hat{\vartheta}^{(m)}) \right\|_{\hat{H}}^2 \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\Omega} \nu^2(\omega) \left(\left| \hat{\mathbf{F}}^{(n)} - \hat{\mathbf{F}}^{(m)} \right|^2 + \left| \hat{\mathbf{G}}^{(n)} - \hat{\mathbf{G}}^{(m)} \right|^2 + \left| \hat{\mathbf{I}}^{(n)} - \hat{\mathbf{I}}^{(m)} \right|^2 \right. \\ &\quad \left. + \left| \hat{\mathbf{Q}}^{(n)} - \hat{\mathbf{Q}}^{(m)} \right|^2 + \left| \hat{R}^{(n)} - \hat{R}^{(m)} \right|^2 \right) d\mathbf{x} d\omega, \end{aligned} \tag{5.79}$$

whence it follows that $\left\{ (\hat{\mathbf{E}}^{(n)}, \hat{\mathbf{H}}^{(n)}, \hat{\mathbf{J}}^{(n)}, \hat{\mathbf{q}}^{(n)}, \hat{\vartheta}^{(n)}), \ n = 1, 2, ... \right\}$ is a Cauchy sequence and the completeness of the space gives

$$\lim_{n \to +\infty} (\hat{\mathbf{E}}^{(n)}, \hat{\mathbf{H}}^{(n)}, \hat{\mathbf{J}}^{(n)}, \hat{\mathbf{q}}^{(n)}, \hat{\vartheta}^{(n)}) = (\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta}) \in \hat{H}(\Omega, \mathbf{R}). \tag{5.80}$$

Thus, it is enough to consider the sequence of identities obtained by substituting the solutions $(\hat{\mathbf{E}}^{(n)}, \hat{\mathbf{H}}^{(n)}, \hat{\mathbf{J}}^{(n)}, \hat{\mathbf{q}}^{(n)}, \hat{\vartheta}^{(n)})$ and the sources $(\hat{\mathbf{F}}^{(n)}, \hat{\mathbf{G}}^{(n)}, \hat{\mathbf{I}}^{(n)}, \hat{\mathbf{Q}}^{(n)}, \hat{R}^{(n)})$ into (5.18); the limit of these identities as $n \to +\infty$, consists of a similar identity in terms of the limits $(\hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{J}}, \hat{\mathbf{q}}, \hat{\vartheta})$ and $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R})$; therefore, we conclude that $(\hat{\mathbf{F}}, \hat{\mathbf{G}}, \hat{\mathbf{I}}, \hat{\mathbf{Q}}, \hat{R}) \in S_a$.

The application of the Plancherel theorem yields the existence of $(\mathbf{E}, \mathbf{H}, \mathbf{J}, \mathbf{q}, \vartheta) \in H(\Omega, \mathbf{R}^+)$, which is the solution of our problem.

Thus, the proof of Theorem 5.1 is complete.

REFERENCES

1. *Coleman B.D. and Dill E.H.* "Thermodynamic restrictions on the constitutive equations of electromagnetic theory," ZAMP, **22**, 691 – 702 (1971).
2. *Coleman B.D. and Dill E.H.* "On the Thermodynamics of electromagnetic fields in materials with memory," Arch. Ration. Mech. and Anal., **41**, 132 – 162 (1971).
3. *Amendola G.* "Linear stability for a thermoelectromagnetic material with memory," Quart. Appl. Math., **1**, 67 – 84 (2001).
4. *Noll W.* "A new mathematical theory of simple materials," Arch. Ration. Mech. and Anal., **48**, 1 – 50 (1972).
5. *Coleman B.D. and Owen D.R.* "A mathematical foundation of Thermodynamics," Arch. Ration. Mech. and Anal., **54,** 1 – 104 (1974).
6. *Cattaneo C.* "Sulla conduzione del calore," Atti Sem. Mat. Fis. Univ. Modena, **3**, 83 – 101(1948).
7. *Fabrizio M., Lazzari B., and Muñoz Rivera J.E.* "Asymptotic behavior in linear thermoelasticity," J. Math. Anal. and Appl., **232**, 138 – 165 (1999).
8. *Bykhousekis E.B. and Smirnov N.V.* "On the orthogonal decomposition of the space of vector functions square summable in a given domain," Tr. Mat. Inst. Steklov, **59**, 6 – 36 (1960).

Received 17.10.2001