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MULTIPLE SOLUTIONS FOR NONLINEAR BOUNDARY-VALUE PROBLEMS OF ODE*

КРАТНІ РОЗВ'ЯЗКИ НЕЛІНІЙНИХ ГРАНИЧНИХ ЗАДАЧ ДЛЯ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

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We consider a Hamilton system related to the Trott curve in Harnack's theorem. This theorem says that the maximal number of ovals for the fourth order curve is four. We treat the related Hamilton system which has more ovals that is prescribed by Harnack's theorem. We give explanation and consider the Dirichlet boundary-value problem for the system. Precise estimation is given for the number of solutions to the Dirichlet problem.

Розглянуто гамільтонову систему, пов'язану з кривою Тротта в теоремі Харнака, яка стверджує, що максимальна кількість овалів кривої четвертого порядку дорівнює 4. Розглянуто гамільтонову систему, що має більшу кількість овалів, ніж стверджується в теоремі Харнака. Наведено пояснення цього факту та розглянуто граничну задачу Діріхле для відповідної системи. Отримано точні оцінки кількості розв'язків задачі Діріхле.

1. Introduction. In this article we would like to consider two problems: the Harnack's theory about algebraic curves and the number of period annuli; two-dimensional Hamilton system and the number of solutions of this system with the Dirichlet boundary-value problem on a finite interval.

The paper has the following structure. In Section 2 we describe the Hamilton system, the Harnack's theorem and give a definition. In Section 3 we consider the Hamilton system of differential equations related to the Trott curve and analyze how many periodic solutions contain this system. In Section 4 we formulate the theorem and lemmas about the number of solutions for a system of the Trott curve with the Dirichlet boundary-value condition. In final Section 5 we summarize the results and make conclusions.

2. Preliminary results and definitions. Consider the two-dimensional nonlinear system

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y). \end{aligned} \tag{2.1}$$

Critical points of such system are to be determined from the relationships

$$f(x,y) = g(x,y) = 0.$$

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Suppose the critical points of system (2.1) are at $(x_i; y_i)$.

By linearization of the system the types of critical points can be determined and description can be given in the phase plane.

A critical point of system (2.1) is a center if it has a punctured neighborhood covered with non trivial cycles.

Definition 2.1 [2]. *A central region is the largest connected region covered with cycles around a center type critical point.*

Definition 2.2 [2]. *A period annulus is every connected region covered with nontrivial concentric cycles.*

Definition 2.3 [3]. *We will call a period annulus associated with a central region a trivial period annulus. Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center.*

Definition 2.4 [3]. *Respectively a period annulus enclosing several (more than one) critical points will be called a nontrivial period annulus.*

When studying periodic solutions of planar systems it is reasonable to consider the Hamilton systems which can be integrated [1].

A Hamiltonian function $H(x, y)$ is the first integral of the Hamilton's system

$$x' = \frac{\partial H(x,y)}{\partial y},$$

$$y' = -\frac{\partial H(x,y)}{\partial x}$$

because

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{dy}\frac{dy}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0.$$

The Hamilton function can be interpreted as the total energy of the system being described. For a closed system, it is the sum of the kinetic and potential energy in the system. There is a set of differential equations known as the Hamilton's equations which give the time evolution of the system and $H(x, y) = C$. Function $H(x, y)$ is often called Hamiltonian.

The Harnack's theorem about algebraic curves can be used in studying period annuli. It allows to construct fairly simple examples of existence of multiple period annuli for polynomials of lowest degree. We construct two-dimensional planar systems called Hamilton systems which have many period annuli as defined in the Harnack's theorem.

Theorem 2.1 [4]. *For any algebraic curve of degree n in the real projective plane, the number of components c is bounded by*

$$\frac{1-(-1)^n}{2} \leq c \leq \frac{(n-1)(n-2)}{2} + 1.$$

Any number of components in this range of possible values can be attained.

Definition 2.5. *A curve which attains the maximum number of real components is an M-curve.*

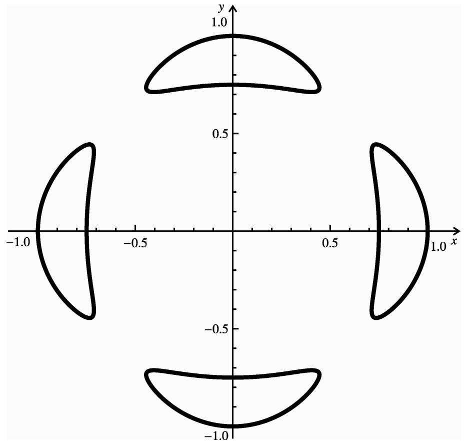

Fig. 3.1. The Trott curve.

There are the Hamilton systems with a number of period annuli greater than the number of ovals prescribed by the Harnack's theorem for related curves.

3. The system related to the Trott curve and periodic solutions. As an example, we consider a curve of degree four, which according to the Harnack's theorem has the maximal number of components four. It is a fourth degree curve, namely the Trott curve, which has the maximal number of components, exactly four.

Definition 3.1. *The Trott curve is an algebraic curve which satisfies the equation*

$$144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81 = 0.$$

The Trott curve (Fig. 3.1) has four separated ovals, the maximal number for a curve of degree four, and hence it is an M-curve.

Let us consider a function $H_1(x, y)$, a part of the polynomial Trott curve, $H_1(x, y) = 144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2$, and the Hamilton system

$$\begin{aligned} x' &= \frac{\partial H_1}{\partial y} = 576y^3 - 450y + 700x^2y, \\ y' &= -\frac{\partial H_1}{\partial x} = -576x^3 + 450x - 700xy^2. \end{aligned} \tag{3.1}$$

System (3.1) has 9 critical points, 5 of them are the points of type "center" and 4 are points of type "saddle":

$$\left(\frac{15}{\sqrt{638}}, \frac{15}{\sqrt{638}}\right), \quad \left(-\frac{15}{\sqrt{638}}, -\frac{15}{\sqrt{638}}\right), \quad \left(-\frac{15}{\sqrt{638}}, \frac{15}{\sqrt{638}}\right), \quad \left(\frac{15}{\sqrt{638}}, -\frac{15}{\sqrt{638}}\right).$$

The Hamilton system (3.1) contains five trivial period annuli around points of type "center":

$$(0,0), \quad \left(\frac{-5}{4\sqrt{2}}, 0\right), \quad \left(\frac{5}{4\sqrt{2}}, 0\right), \quad \left(0, \frac{5}{4\sqrt{2}}\right), \quad \left(0, \frac{-5}{4\sqrt{2}}\right)$$

and one nontrivial periodic annulus-around all critical points, totally six period annuli (Fig. 3.2).

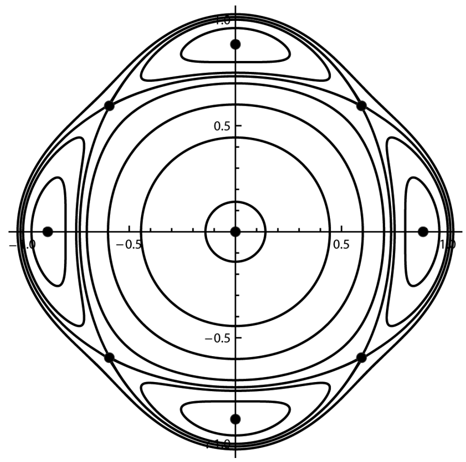

Fig. 3.2. The phase portrait of system (3.1).

The Harnack's theorem says about 4 ovals for the Trott curve. For the respective system (3.1) we have more, six period annuli. This is because we consider curves given by the relation $H_1(x, y) = C$ where C is arbitrary, and a specific Trott curve we obtain for $C = -81$.

4. A system related to the Trott curve and the Dirichlet boundary-value problem. We are interested in the question of how many solutions of the system satisfy the given boundary conditions.

Consider system (3.1) with the Dirichlet boundary conditions

$$x(0) = 0, \quad x(T) = 0. \tag{4.1}$$

Denote parts of the phase portrait of system (3.1): Region 1 in Fig. 4.1 (a); Region 2 in Fig. 4.1 (b); Region 3 in Fig. 4.1 (c); Region 4 in Fig. 4.1 (d):

Make the linearization of system (3.1).

$$\begin{aligned} f'_x &= 1400xy, \\ f'_y &= 1728y^2 - 450 + 700x^2, \\ g'_x &= -1728x^2 + 450 - 700y^2, \\ g'_y &= -1400xy. \end{aligned}$$

The linearized system for system (3.1) at a critical point (x^*, y^*) is

$$\begin{aligned} u' &= (1400x^*y^*)u + (1728y^{*2} - 450 + 700x^{*2})v, \\ v' &= (-1728x^{*2} + 450 - 700y^{*2})u + (-1400x^*y^*)v. \end{aligned}$$

Consider Region 1 and closed trajectories which are in a close proximity of the boundary of Region 1.

This boundary consists of the critical points $\left(-\dfrac{15}{\sqrt{638}}, \dfrac{15}{\sqrt{638}}\right)$, $\left(\dfrac{15}{\sqrt{638}}, \dfrac{15}{\sqrt{638}}\right)$ which are

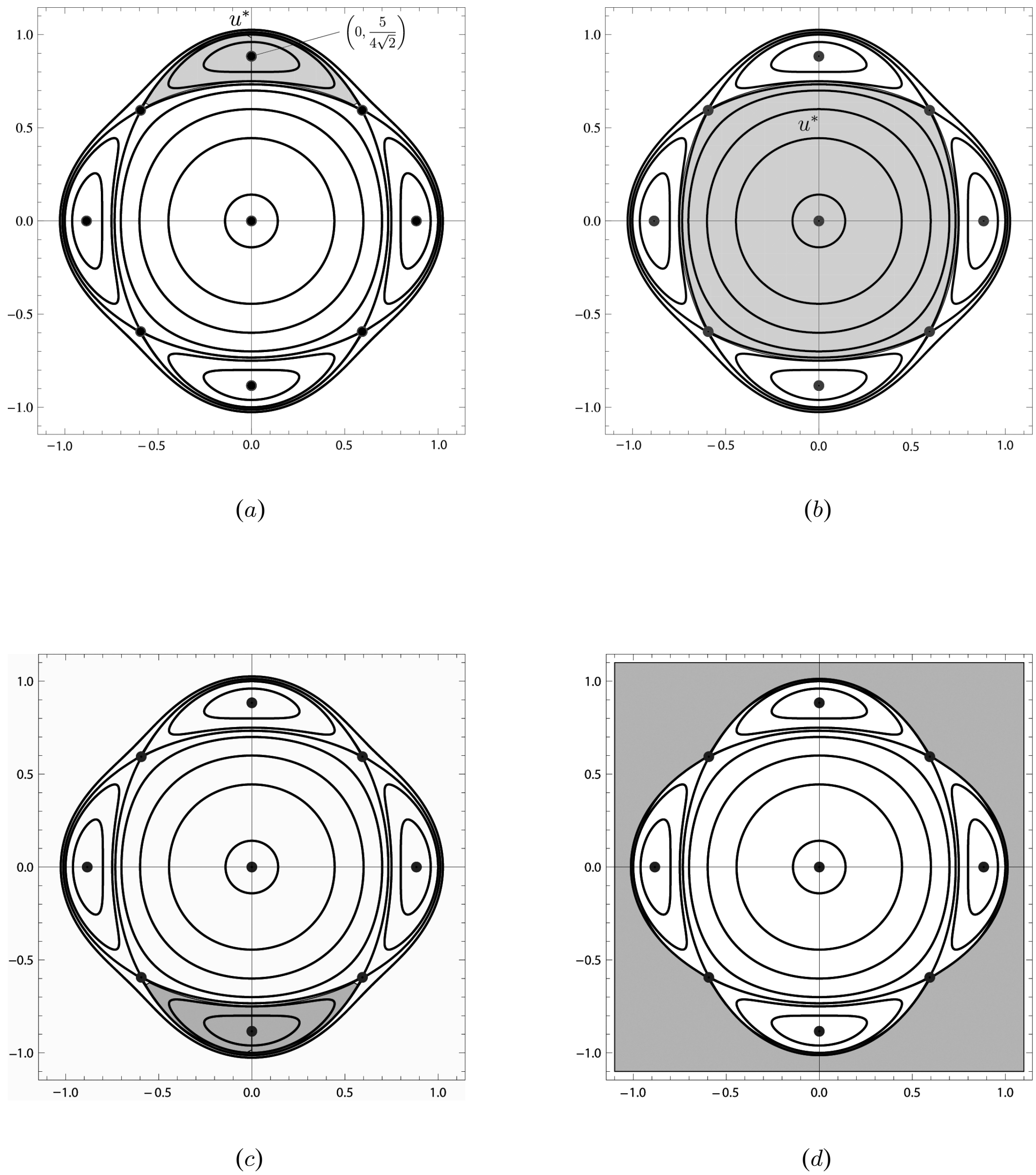

Fig. 4.1. The phase portrait of system (3.1).

of the “saddle” type, and of two heteroclinic solutions connecting them. Motion along heteroclinic solutions is very “slow” in the meaning that the time needed to pass from one critical point to another is infinity.

Denote a point of intersection of the “upper” heteroclinic solution with y axis $(0, u^*)$. The calculation gives $u^* \approx 1.01$.

For periodic solutions of system (3.1), which satisfy the initial conditions $(0, u_0)$, where $u_0 \sim \sim u^*$, $u_0 < u^*$, we have that the time $\tau(u_0)$ needed to reach the next intersection point with the y axis is arbitrarily large in the meaning that $\tau(u_0) \to +\infty$ as $u_0 \to u^*$.

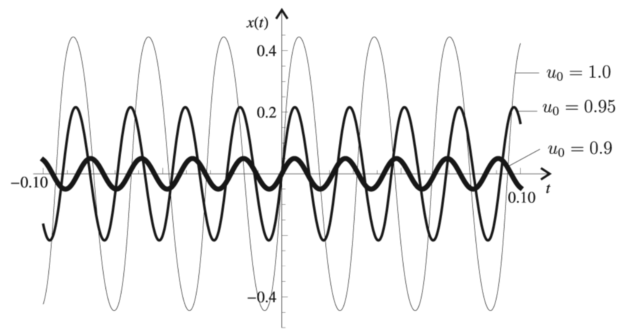

Fig. 4.2. A solution of system (3.1) in Region 1,
$x(t) \sim u(t)$, $u_0 = 0.9$, $u_0 = 0.95$, $u_0 = 1.0$.

On the other hand, trajectories in the interior of Region 1 surrounded the critical point $\left(0, \dfrac{5}{4\sqrt{2}}\right)$ are relatively "fast" (Fig. 4.2). We can compute the time $\tau(u_0)$, needed to pass the from y axis to another point on the y axis, in fact the half-period for $u_0 \to \dfrac{5}{4\sqrt{2}} \approx 0.884$.

For this, consider the linearized system of system (3.1) at the critical point $\left(0, \dfrac{5}{4\sqrt{2}}\right)$

$$\begin{aligned} u' &= 900v, \\ v' &= -96.875u \end{aligned} \tag{4.2}$$

(the value -96.875 is precise).

The eigenvalues of the linearized system (4.2) are $\lambda_{1,2} = \pm 295.27529i$ and the point $\left(0, \dfrac{5}{4\sqrt{2}}\right)$ is a point of type "center".

System (4.2) can be rewritten in the form

$$u'' = 900v' = -96.875 \cdot 900u = -87187.5u \tag{4.3}$$

and the solution of equation (4.3),

$$u(t) = \sin\sqrt{87187.5}t, \tag{4.4}$$

generates an approximation $(u(t), u'(t))$ to a solution of Cauchy problem (3.1), $x(0) = 0$, $y(0) = \dfrac{5}{4\sqrt{2}} \pm \epsilon$ where ϵ is a small value.

Therefore, in the Dirichlet problem (4.1), (4.4), $T = \dfrac{\tau}{2} = \dfrac{\pi}{\sqrt{87187.5}}$, where τ is a period of solution.

Lemma 4.1. *Let n be the largest integer such that $\dfrac{n\pi}{\sqrt{87187.5}} \leq T$. The Dirichlet problem (3.1), (4.1) in Region 1 has at least $2n$ nontrivial solutions.*

Proof. By considering the initial value problems (3.1),

$$(x(0), y(0)) = (0, u_0),$$

where $u_0 \in \left(\frac{5}{4\sqrt{2}}, u^*\right)$, we get n solutions and, for $u_0 < \frac{5}{4\sqrt{2}}$, the symmetric solutions which gives n more solutions.

Trajectories in Region 2 (Fig. 4.1 (b)), which are in a close proximity of the external boundary of Region 2 are very “slow”. This is because the boundary contains four critical points of the type “saddle”:

$$\left(\frac{15}{\sqrt{638}}, \frac{15}{\sqrt{638}}\right), \quad \left(-\frac{15}{\sqrt{638}}, -\frac{15}{\sqrt{638}}\right), \quad \left(-\frac{15}{\sqrt{638}}, \frac{15}{\sqrt{638}}\right), \quad \left(\frac{15}{\sqrt{638}}, -\frac{15}{\sqrt{638}}\right).$$

The linearized system of system (3.1) at the origin is

$$\begin{aligned} u' &= -450v, \\ v' &= 450u. \end{aligned} \tag{4.5}$$

The eigenvalues of linearized system (4.5) are $\lambda_{1,2} = \pm 450i$ and the origin is a point of type “center”.

System (4.5) can be rewritten in the form $u'' = -450v' = -450^2 u$ and $u(t) = \sin 450t$ generates an approximation $(u(t), u'(t))$ to a solution of Cauchy problem (3.1), $x(0) = 0$, $y(0) = \pm\epsilon$ where ϵ is a small value.

Therefore, in the Dirichlet condition (4.1), $T = \frac{\tau}{2} = \frac{\pi}{450}$, where τ is a period of solution (4.5).

Lemma 4.2. *Let m be the largest integer such that $\frac{m\pi}{450} \leq T$. The Dirichlet problem (3.1), (4.1) in Region 2 has at least $2m$ nontrivial solutions.*

Proof of Lemma 4.2 is similar to the proof of Lemma 4.1.

Consideration of Region 3 (Fig. 4.1 (c)) is similar to that for Region 1. The results can be formulated in the following lemma.

Lemma 4.3. *Let n be the largest integer such that $\frac{n\pi}{\sqrt{87187.5}} \leq T$. The Dirichlet problem (3.1), (4.1) in Region 3 has at least $2n$ nontrivial solutions.*

Proof of Lemma 4.3 is similar to the proof of Lemma 4.1.

The curves which are in Region 4 (Fig. 4.1 (d)) are closed. We want to evaluate the speed of rotation along these trajectories. For this, we consider the principal (cubic) part of system (3.1)

$$\begin{aligned} x' &= 576y^3 + 700x^2 y, \\ y' &= -576x^3 - 700xy^2. \end{aligned} \tag{4.6}$$

In the polar coordinates

$$\begin{aligned} x(t) &= r(t)\sin\Theta(t), \\ y(t) &= r(t)\cos\Theta(t), \end{aligned}$$

system (4.6) takes the form

$$\begin{aligned} r'(t) &= -31r^3(t)\sin 4\Theta(t), \\ \Theta'(t) &= r^2(t)(576 + 62\sin^2 2\Theta(t)). \end{aligned} \tag{4.7}$$

Lemma 4.4. *In Region 4 trajectories with large $u_0 = r_0$ move arbitrarily "fast".*

Proof. Consider Region 4 where $u > u_0$, $u_0 > 0$ may be arbitrarily large. It is true that $\Theta'(t) \geq 576\ \forall t \in [0, T]$.

The solution of system (3.1) with the initial condition $(x(0), y(0)) = (0, u_0)$ has a period $\tau(u_0)$ and $\tau(u_0) \to 0$ as $u_0 \to 0$. Since the expression in the parentheses in the right-hand side in the second equation (4.7) is never zero, we get that $\Theta'(t)$ is arbitrarily large as r^2 is large. Therefore in Region 4 trajectories with large u_0 move arbitrarily "fast".

Lemma 4.5. *Dirichlet problem (3.1), (4.1) for any T in Region 4 has infinitely many nontrivial solutions (a countable set).*

Therefore, we can formulate the following theorem which follows from Lemma 4.1 to Lemma 4.4.

Theorem 4.1. *The Dirichlet boundary-value problem (3.1), (4.1) in Regions 1, 2, 3 has at least $2(2n + m)$ nontrivial solutions. In external Region 4 there are infinitely many nontrivial solutions (a countable set).*

5. Conclusions. When analyzing the results it can be concluded that the number of period annuli of the Hamilton systems is significantly greater than the number of components of the respective curve in the Harnack's theorem. However this is not in contradiction with the Harnack's theorem, because in the latter equation the total number of component exceeds the number four which is the prescribed number of ovals in the Harnack's theory for a single curve given by $H_1 = C$.

For the system of differential equations (3.1) with the Dirichlet boundary conditions (4.1) there is a nontrivial solution and the number of solutions is infinite (a countable set).

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