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ON FINDING PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS IN A BANACH SPACE

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With the use of the numerical-analytic method of A.M. Samoilenko and a modification of Newton's method, we construct an approximation to the periodical solution of a difference equation in pertially ordered Banach spaces with an arbitrary given precision.

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There where a large number of works published for the last several years that deal with the research of problems of the reduction, existence of oscillating solutions, and construction of invariant manifolds of equations of different kinds in infinite dimension Banach spaces. Let's mention here some of them, namely [1 – 11]. In this article we use the numerical analytic method of A.M.Samoilenko and the modified method of Newton to construct an approximation of the periodical solution of the equation in partially ordered Banach spaces, which enables us to solve the problem with a given precision. It should be noticed that a similar problem for scalar equation of the second order is partly solved in [12] and, for the linear equation in Banach spaces, in [11].

Consider the equation

$$\Delta^2 x_n = f_n(x_n, x_{n+1}), \qquad n \in Z, \tag{1}$$

where $\Delta x_n = x_{n+1} - x_n$, the mapping $f_n(x,y)\colon W \times W \to W$, $x_n \in W$, W is a real Banach space with the norm $\|\cdot\|$, Z the set of the integers. Let's set $Z_0^+ = \{0,1,2,\ldots\}$, $Z^- = Z\backslash Z^+$, $Z^+ = Z_0^+\backslash\{0\}$.

It is easy to see that for any $\{x_0^*, x_1^*\} \subset W$, equation (1) has, on the set Z_0^+, the only solution $x_n = x_n(x_0^*, x_1^*)$ such that $x_0(x_0^*, x_1^*) = x_0^*, x_1(x_0^*, x_1^*) = x_1^*$. For $n \in \{2,3,4,\ldots\}$, it is given by the relation

$$x_n = -(n-1)x_0^* + nx_1^* + \sum_{i=1}^{n-1}(n-i)f_{i-1}(x_{i-1}, x_i). \tag{2}$$

Further we shall consider the function $f_n(x,y)$, N-periodic in n on the set Z.

Using the last equality, it's easy to see that the solution $x_n(x_0^*, x_1^*)$ of equations (1) is N-periodic on Z_0^+ if and only if its initial conditions, x_0^* and x_1^*, satisfy the system of equations

$$\begin{aligned} &\sum_{k=0}^{N-1} f_k(x_k(x_0^*, x_1^*), x_{k+1}(x_0^*, x_1^*)) = 0, \\ &x_1^* = x_0^* - \frac{1}{N}\sum_{i=1}^{N-1}(N-i)f_{i-1}(x_{i-1}, x_i). \end{aligned} \tag{3}$$

If such a solution exists, it is easy to continue it to the set Z with the help of periodicity.

Define a linear operator L that acts on the sequence of points $\{a_0, a_1, a_2, \dots, a_n, \dots\} \subset W$ by

$$La_0 = 0 \in W, \quad La_n = \sum_{i=0}^{n-1}(a_i - \overline{a_\nu}), \qquad \overline{a_\nu} = \frac{1}{N}\sum_{\nu=0}^{N-1} a_\nu, \quad n \in Z^+.$$

Then

$$L^2a_0 = 0 \in W, \quad L^2a_1 = -\frac{1}{N}\sum_{i=1}^{N-1}\sum_{s=0}^{i-1}(a_s - \overline{a_\nu}),$$

$$L^2a_n = \sum_{i=1}^{n-1}\sum_{k=0}^{i-1}(a_k - \overline{a_\nu}) - \frac{n}{N}\sum_{\nu=1}^{N-1}\sum_{k=0}^{\nu-1}(a_k - \overline{a_\nu}), \quad n \in Z^+\backslash\{1\}.$$

Set $L^2 f_n(x, y) = q_n(x, y)$ and write the equations

$$\Delta^2 x_n = f_n(x_n, x_{n+1}), \ \ n \in Z_0^+, \tag{4}$$

$$x_n = x_0 + g_n(x_n, x_{n+1}), \ \ n \in Z_0^+, \tag{5}$$

$$\Delta^2 x_n = f_n(x_n, x_{n+1}) - \mu, \ \ n \in Z_0^+, \tag{6}$$

where $\mu \in W$.

Using the relation (2), (3), it is easy to prove the following statements:

1) any N-periodic on Z_0^+ solution $x_n(x_0, x_1)$ of equation (4) is an N-periodic solution of equation (5) on Z_0^+;

2) if $x_n(x_0, x_1)$ is an N-periodic on Z_0^+ solution of equation (5), then it is an N-periodic solution on Z_0^+ of equation (6), where $\mu = \overline{f_\nu(x_\nu, x_{\nu+1})}$;

3) any N-periodic solution $x_n(x_0, x_1)$ of equation (6) on Z_0^+ is an N-periodic solution of equation (5) on this set.

The next conditions are referred to as conditions (A):

i)$f_n(x, y)$ is a function, N-periodic in $n \in Z$, continuous in x, y and defined in the region $D_0 = Z \times D \times D, \qquad D = \{x \in W \big| \|x\| \le R\}$, where

$$\|f_n(x, y)\| \le M,$$

$$\left\|f_n(x, y) - f_n(x', y')\right\| \le K_1 \left\|x - x'\right\| + K_2 \left\|y - y'\right\|,$$

and M, R, K_1, K_2 are positive constants;

ii) the set $D^f = \left\{ x \in D \Big| \|x\| \le R - \frac{N^2 M}{4} \right\} \subset D$ is nonempty, and $\gamma = \frac{N^2}{4}(K_1 + K_2) < 1$.

Define a sequence $\left\{ x_n^{(m)}(x_0) \right\}_{m=0}^{\infty}$, by the recurence relations,

$$x_n^{(0)} \equiv x_0, \quad x_0^{(m)}(x_0) \equiv x_0,$$
$$x_n^{(m)}(x_0) = x_0 + g_n(x_n^{(m-1)}(x_0), x_{n+1}^{(m-1)}(x_0)), m \in Z^+. \tag{7}$$

Under the conditions (A) the following statements take place:

4) equation (5) can't have more than one N-periodic solution x_n with the inital value $x_0 \in D^f$;

5) the function $\widetilde{x}_n(x_0) = \lim_{m\to\infty} x_n^{(m)}(x_0)$ is an N-periodic solution of equation (5) in Z_0^+ and also of equation (6), where $\mu = \mu^* = \overline{f_\nu(\widetilde{x}_\nu, \widetilde{x}_{\nu+1})}$, and

$$\left\| \widetilde{x}_n(x_0) - x_n^{(m)}(x_0) \right\| \le \sigma^*(m) = \frac{\left(\frac{N^2}{4}\right)^{m+1} M(K_1+K_2)^m}{1 - \frac{N^2}{4}(K_1+K_2)} \to 0$$

as $m \to \infty$;

6) for a function $\widetilde{x}_n(x_0)$ to be the only N-periodic on Z_0^+ solution of equation (4) with the initial value $x_0 \in D^f$, it is necessary and sufficient that $\mu^* = 0$;

7) if $x_n = x_n(x_0, x_1)$ is an N-periodic solution of equation (6) and $y_n = y_n(x_0, y_1)$ is an N-periodic solution of the equation $\Delta^2 y_n = f_n(y_n, y_{n+1}) - \mu_1$ on Z_0^+, then those solutions coincide on this set and $\mu_1 = \mu_2$.

The proof of the statements 1) – 7) doesn't essentially depend on the dimension of the space W and is partly stated in [10].

Thus to find an N-periodic solution of equation (1), it is necessary to find $x_0 \in D^f$ that satisfies the equation $\mu^* = 0$. Having continued the function $\widetilde{x}(x_0)$ by periodicity to Z^-, we get an N-periodic solution of equation (1).

Let's denote by $\frac{d_\phi Z(x)}{dx}$ the Frecher of the function $Z(x) : W \to W$, by $Z(x)\big|_{x_1}^{x_2}$ the difference $Z(x_2) - Z(x_1)$, set $\Phi = \{0, 1, 2, \ldots, N-1\}$ and prove first some auxiliary statements.

Lemma 1. *Let conditions (A) be satisfied and, moreover, the function $f_n(x, y)$ be defined for all $n \in \Phi$ in the area $Z \times D_1 \times D_1$, where $D_1 = \left\{ x \in W \Big| \|x\| < R + \rho \right\}$, ρ is a constant positive, Frechet differentiable in the area $\Omega \in D_1 \times D_1$ with $\left\| \frac{d_\Phi f_n(x,y)}{d(x,y)} \right\| \le P$, P a positive constant independent on $n \in Z, (x, y) \in \Omega$.*

If $\eta = \frac{N^2 P}{4} < 1$, then the functions $x_n^{(m)}(x)$ defined by relations (7) *are Frechet differentiable in the region $D_0^f = \left\{ x \in D^f \Big| \|x\| < R - \frac{N^2 M}{4} \right\}$ and*

$$\left\| \frac{d_\Phi x_n^{(m)}(x)}{dx} \right\| < \frac{1}{1-\eta} \tag{8}$$

for all $n \in \Phi, \quad m \in Z_0^+$.

Proof. Write relation (7) as

$$x_n^{(0)}(x) = x, \qquad x_0^{(m)}(x) = x, \qquad n \in \Phi,\ m \in Z_0^+, \tag{9}$$

$$x_1^{(m)}(x) = x - \frac{1}{N} \sum_{i=1}^{N-1} \sum_{s=0}^{i-1} (f_s(x_s^{(m-1)}(x), x_{s+1}^{(m-1)}(x))$$

$$- \overline{f_n(x_n^{(m-1)}(x), x_{n+1}^{(m-1)}(x))}), \qquad m \in Z^+, \tag{10}$$

$$x_n^{(m)}(x) = x + \sum_{i=1}^{n-1} \sum_{s=0}^{i-1} (f_s(x_s^{(m-1)}(x), x_{s+1}^{(m-1)}(x)) - \overline{f_n(x_n^{(m-1)}(x), x_{n+1}^{(m-1)}(x))})$$

$$- \frac{n}{N} \sum_{i=1}^{N-1} \sum_{s=0}^{i-1} (f_s(x_s^{m-1}(x), x_{s+1}^{(m-1)}(x)) - \overline{f_n(x_n^{(m-1)}(x), x_{n+1}^{(m-1)}(x))}), \tag{11}$$

$$n \in \Phi \backslash \{0, 1\}, m \in Z^+.$$

By the norm $\|(x, y)\|, \quad (x, y) \in \Omega$, we understand $\max\{\|x\|, \|y\|\}$, where $\|x\|, \|y\|$ is the norm in the region D_1. It is obvious that the set Ω is open in $W \times W$.

For $m = 0$ it follows from (9) that for all $n \in \Phi$,

$$\frac{d_\Phi x_n^{(0)}(x)}{dx} = E, \quad \left\| \frac{d_\Phi x_n^{(0)}(x)}{dx} \right\| = 1 < \frac{1}{1-\eta},$$

where E is the identy operator.

For $m \in Z^+$, we prove inequality (8) by induction.

Let $m = 1$.

Using (10) we write the equalities

$$\frac{d_\Phi x_1^{(1)}(x)}{dx} = E - \frac{1}{N} \sum_{i=1}^{N-1} \sum_{s=0}^{i-1} \left(\frac{d_\Phi f_s(x, x)}{dx} - \overline{\frac{d_\Phi f_n(x, x)}{dx}} \right)$$

$$= E - \frac{1}{N} \sum_{i=1}^{N-1} \sum_{s=0}^{i-1} \left(\frac{d_\Phi f_s(x, x)}{d(x, x)} \frac{d_\Phi g(x)}{dx} - \overline{\frac{d_\Phi f_s(x, x)}{d(x, x)} \frac{d_\Phi g(x)}{dx}} \right),$$

where $g(x)$ is a mapping of the open set D_0^f to the set Ω with the components, $g^1(x)$ and $g^2(x)$, being identity operators, because they map D_0^f in D_1.

Then

$$\left\|\frac{d_\Phi g(x)}{dx}\right\| = \sup_{\|h\|=1}\left\|\frac{d_\Phi g(x)}{dx}h\right\| = \sup_{\|h\|=1}\max\left\{\left\|\frac{d_\Phi g^1(x)}{dx}h\right\|,\right.$$

$$\left.\left\|\frac{d_\Phi g^2(x)}{dx}h\right\|\right\} \leq \sup_{\|h\|=1}\max\left\{\left\|\frac{d_\Phi g^1(x)}{dx}\right\|\|h\|, \left\|\frac{d_\Phi g^2(x)}{dx}\right\|\|h\|\right\}$$

$$= \max\left\{\left\|\frac{d_\Phi g^1(x)}{dx}\right\|, \left\|\frac{d_\Phi g^2(x)}{dx}\right\|\right\} = 1,\ \ h \in W.$$

Using proposition 1 from [11], where the space W is replaced by the space of linear continuous operators, $\pounds(W, W)$, with an appropriate norm, we come to the inequalities

$$\left\|\frac{d_\Phi x_1^{(1)}(x)}{dx}\right\| \leq 1 + \frac{N-1}{N}\,\frac{N}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x,x)}{d(x,x)}\right\|\ \left\|\frac{d_\Phi g(x)}{dx}\right\|\right\}$$

$$\leq 1 + \frac{P(N-1)}{2} \leq 1 + \frac{PN^2}{4} < \frac{1}{1-\eta},$$

since $N - 1 < \dfrac{N^2}{2}$ for all $N \in Z$.

Now let's estimate the norm of the derivative $\dfrac{d_\Phi x_n^{(1)}(x)}{dx}$ for $n \in \Phi\backslash\{0, 1\}$. Using (11) we have

$$\left\|\frac{d_\Phi x_n^{(1)}(x)}{dx}\right\| \leq 1 + \frac{N^2}{4}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x,x)}{dx}\right\|\right\} \leq 1 + \frac{N^2P}{4} < \frac{1}{1-\eta}.$$

So, estimation (8), for $m = 1$, takes place for all $n \in \Phi$.

Assume that it holds for all $1 < m \leq k$ uniformly in $n \in \Phi$ and show that it holds for $m = k + 1$. If $n = 0$, estimate (8) is obvious. For $n = 1$ we have

$$\left\|\frac{d_\Phi x_1^{(k+1)}(x)}{dx}\right\| \leq 1 + \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(k)}(x), x_{s+1}^{(k)}(x))}{dx}\right\|\right\}$$

$$\leq 1 + \frac{N-1}{2}P\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi x_s^{(k)}(x)}{dx}\right\|, \left\|\frac{d_\Phi x_{s+1}^{(k)}(x)}{dx}\right\|\right\}$$

$$\leq 1 + \frac{\eta}{1-\eta} = \frac{1}{1-\eta}.$$

At $n \in \Phi\backslash\{0,1\}$

$$\left\|\frac{d_\Phi x_n^{(k+1)}(x)}{dx}\right\| \leq 1+\frac{PN^2}{4}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi x_s^{(k)}(x)}{dx}\right\|,\left\|\frac{d_\Phi x_{s+1}^{(k)}(x)}{dx}\right\|\right\} \leq \frac{1}{1-\eta},$$

and this finishes the proof of Lemma 1.

Lemma 2. *Let, with conditions of Lemma 1, the following inequality hold for all* $\{x_1, x_2, y_1, y_2\} \subset D_1$ *uniformly in* $n \in \Phi$:

$$\left\|\frac{d_\Phi f_n(x,y)}{d(x,y)}\bigg|_{(x_2,y_2)}^{(x_1,y_1)}\right\| \leq L_0 \max\{\|x_1-x_2\|, \|y_1-y_2\|\},$$

where L_0 *is a positive constant. Then, uniformly in* $n \in \Phi, m \in Z_0^+$, *we have*

$$\left\|\frac{d_\Phi x_n^{(m)}(x)}{dx}\bigg|_{(x_2)}^{(x_1)}\right\| \leq \frac{L_0\eta}{P(1-\eta)^2(1-\gamma)}\|x_1-x_2\|. \tag{12}$$

Proof. For $m = 0$, the statement of Lemma 2 is obvious. For $m \in Z^+$, we'll prove inequality (12) by induction. Let $m = 1$. If $m = 1, n = 0$, inequality (12) is obvious.

Let's estimate $\left.\dfrac{d_\Phi x_1^{(1)}(x)}{dx}\right|_{x_2}^{x_1}$ in the norm. We have

$$\begin{aligned}
\left\|\frac{d_\Phi x_1^{(1)}(x)}{dx}\bigg|_{x_2}^{x_1}\right\| &\leq \frac{1}{N}\sum_{i=1}^{N-1}\left\|\sum_{s=0}^{i-1}\left(\frac{d_\Phi f_s(x,x)}{dx}\bigg|_{x_2}^{x_1} - \overline{\frac{d_\Phi f_s(x,x)}{dx}\bigg|_{x_2}^{x_1}}\right)\right\| \\
&\leq \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x,x)}{dx}\bigg|_{x_2}^{x_1}\right\|\right\} \\
&\leq \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x,x)}{d(x,x)}\right\|\left\|\frac{d_\Phi g(x)}{dx}\bigg|_{x_2}^{x_1}\right\| + \left\|\frac{d_\Phi f_s(x,x)}{d(x,x)}\bigg|_{x_2}^{x_1}\right\|\left\|\frac{d_\Phi g(x_2)}{dx}\right\|\right\} \\
&\leq \frac{N-1}{2}\max_{s\in\Phi}\{P\cdot 0 + L_0\|x_1-x_2\|\} = \\
&= \frac{N-1}{2}L_0\|x_1-x_2\| \leq \frac{\eta L_0}{P}\|x_1-x_2\|,
\end{aligned}$$

that is, inequality (12) takes place.

Now we shall prove this inequality for $m = 1, n \in \Phi\backslash\{0,1\}$,

$$\left\|\frac{d_\Phi x_n^{(1)}(x)}{dx}\bigg|_{x_2}^{x_1}\right\| \leq \frac{N^2}{4}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x,x)}{dx}\bigg|_{x_2}^{x_1}\right\|\right\} \leq \frac{\eta L_0}{P}\|x_1-x_2\|.$$

Thus, estimate (12) holds for $m = 1, n \in \Phi$.

Suppose that the inequality hold for all $1 < m \leq k$ and $n \in \Phi$,

$$\left\| \frac{d_\Phi x_n^{(m)}(x)}{dx} \Big|_{x_2}^{x_1} \right\| \leq \frac{L_0 \eta}{P(1-\eta)(1-\gamma)} \sum_{i=0}^{m-1} \eta^i \|x_1 - x_2\|, \tag{13}$$

from which, certainly, estimate (12) follows. Note that inequality (13) as it was shown above, for $m = 1$, takes place for $n \in \Phi$. Let's prove it for $m = k+1$.

For $x_0^{(k+1)}(x)$, inequality (13) is obvious. Taking into account that the set D_0^f is convex and denoting by $g_s^{(k)}(x)$ the mapping $D_0^f \to \Omega$ with components $g_s^{(k)_1}(x) = x_s^{(k)}(x)$, $g_s^{(k)_2}(x) = x_{s+1}^{(k)}(x)$, we write the following inequations:

$$\begin{aligned}
\left\| \frac{d_\Phi x_1^{(k+1)}(x)}{dx} \Big|_{x_2}^{x_1} \right\| &\leq \frac{N-1}{2} \max_{s\in\Phi} \left\{ \left\| \frac{d_\Phi f_s(x_s^{(k)}(x_1), x_{s+1}^{(k)}(x_1))}{d(x_s^{(k)}, x_{s+1}^{(k)}} \right\| \left\| \frac{d_\Phi g_s^{(k)}(x)}{dx} \Big|_{x_2}^{x_1} \right\| \right. \\
&\quad \left. + \left\| \frac{d_\Phi f_s(x_s^{(k)}(x), x_{s+1}^{(k)}(x))}{d(x_s^{(k)}, x_{s+1}^{(k)})} \Big|_{x_2}^{x_1} \right\| \left\| \frac{d_\Phi g_s^{(k)}(x_2)}{dx} \right\| \right\} \\
&\leq \frac{N-1}{2} \max_{s\in\Phi} \left\{ P \left\| \frac{d_\Phi g_s^{(k)}(x)}{dx} \Big|_{x_2}^{x_1} \right\| \right. \\
&\quad + L_0 \max \left\{ \left\| x_s^{(k)}(x) \Big|_{x_2}^{x_1} \right\|, \left\| x_{s+1}^{(k)}(x) \Big|_{x_2}^{x_1} \right\| \right\} \\
&\quad \left. \times \max \left\{ \left\| \frac{d_\Phi x_s^{(k)}(x_2)}{dx} \right\|, \left\| \frac{d_\Phi x_{s+1}^{(k)}(x_2)}{dx} \right\| \right\} \right\} \\
&\leq \frac{N-1}{2} \max_{s\in\Phi} \left\{ P \max \left\{ \left\| \frac{d_\Phi x_s^{(k)}(x)}{dx} \Big|_{x_2}^{x_1} \right\|, \left\| \frac{d_\Phi x_{s+1}^{(k)}(x)}{dx} \Big|_{x_2}^{x_1} \right\| \right\} \right. \\
&\quad \left. + \frac{L_0}{(1-\eta)(1-\gamma)} \|x_1 - x_2\| \right\} \\
&\leq \frac{\eta}{P} \left\{ P \frac{L_0 \eta}{P(1-\eta)(1-\gamma)} \sum_{i=0}^{k-1} \eta^i + \frac{L_0}{(1-\eta)(1-\gamma)} \right\} \|x_1 - x_2\| \\
&= \frac{L_0 \eta}{P(1-\eta)(1-\gamma)} \sum_{i=0}^{k} \eta^i \|x_1 - x_2\|.
\end{aligned}$$

It remains to estimate $\frac{d_\Phi x_n^{(k+1)}(x)}{dx} \Big|_{x_2}^{x_1}$ in the norm for $n \in \Phi \backslash \{0, 1\}$.

Using (11) we get similarly to the above that

$$\left\| \frac{d_\Phi x_n^{(k+1)}(x)}{dx} \Bigg|_{x_2}^{x_1} \right\| \le \frac{N^2}{4} \max_{s\in\Phi} \left\{ \left\| \frac{d_\Phi f_s(x_s^{(k)}(x), x_{s+1}^{(k)}(x))}{dx} \right\| \right\}$$

$$\le \frac{L_0 \eta}{P(1-\eta)(1-\gamma)} \sum_{i=0}^{k} \eta^i \|x_1 - x_2\|.$$

Lemma 2 is proved.

Lemma 3. *With the conditions of Lemma 2, the sequence* $\left\{ \frac{d_\Phi x_n^{(m)}(x)}{dx} \right\}_{m=0}^{\infty}$ *converges, as* $m \to \infty$, *uniformly in* $x \in D_0^f$ *for* $n \in \Phi$.

Proof. This sequence belongs to the space $\pounds(W, W)$ which is complete. Therefore, it is enough to prove that it is fundamental. For $n = 0$, the claim of the lemma is obvious. We'll prove that for all $m \in Z^+ \backslash \{1\}$ and $n \in \Phi \backslash \{0\}$, the following inequality holds:

$$\left\| \frac{d_\Phi x_n^{(m)}(x)}{dx} - \frac{d_\Phi x_n^{(m-1)}(x)}{dx} \right\| \le \frac{\eta}{P} \left(\sum_{i=0}^{m-2} \frac{\beta}{1-\eta} \gamma^{m-2-i} \eta^i + P\eta^{m-1} \right), \tag{14}$$

where $\beta = \dfrac{L_0 N^2 M}{4}$.

Let's prove estimate (14) for $m = 2$. Using equalities (4), (5) we have

$$\left\| \frac{d_\Phi x_1^{(1)}(x)}{dx} - \frac{d_\Phi x_1^{(0)}(x)}{dx} \right\| \le \frac{N-1}{2} \max_{s\in\Phi} \left\| \frac{d_\Phi f_s(x,x)}{dx} \right\| \le \frac{N-1}{2} P \le \eta,$$

$$\left\| \frac{d_\Phi x_n^{(1)}(x)}{dx} - \frac{d_\Phi x_n^{(0)}(x)}{dx} \right\| \le \frac{N^2}{4} \max_{s\in\Phi} \left\| \frac{d_\Phi f_s(x,x)}{dx} \right\| \le \frac{N^2}{4} P = \eta,$$

$$n \in \Phi \backslash \{0, 1\}.$$

Now we have the chain of inequalities

$$\left\|\frac{d_\Phi x_1^{(2)}(x)}{dx} - \frac{d_\Phi x_1^{(1)}(x)}{dx}\right\| \le \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(1)}(x), x_{s+1}^{(1)}(x))}{dx} - \frac{d_\Phi f_s(x,x)}{dx}\right\|\right\}$$

$$\le \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(1)}(x), x_{s+1}^{(1)}(x))}{d(x_s^{(1)}, x_{s+1}^{(1)})}\frac{d_\Phi g_s^{(1)}(x)}{dx}\right.\right.$$

$$\left.\left. - \frac{d_\Phi f_s(x,x)}{d(x,x)}\frac{d_\Phi g(x)}{dx}\right\|\right\} \le \frac{N-1}{2}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(1)}(x), x_{s+1}^{(1)}(x))}{d(x_s^{(1)}, x_{s+1}^{(1)})}\right.\right.$$

$$\left.\left. - \frac{d_\Phi f_s(x,x)}{d(x,x)}\right\| \left\|\frac{d_\Phi g_s^{(1)}(x)}{dx}\right\| + \left\|\frac{d_\Phi f_s(x,x)}{d(x,x)}\right\| \left\|\frac{d_\Phi g_s^{(1)}(x)}{dx} - \frac{d_\Phi g(x)}{dx}\right\|\right\}$$

$$\le \frac{N-1}{2}\max_{s\in\Phi}\left\{L_0 \max\left\{\left\|x_s^{(1)}(x) - x_s^{(0)}(x)\right\|,\ \left\|x_{s+1}^{(1)}(x) - x_{s+1}^{(0)}(x)\right\|\right\}\right.$$

$$\times \max\left\{\left\|\frac{d_\Phi x_s^{(1)}(x)}{dx}\right\|, \left\|\frac{d_\Phi x_{s+1}^{(1)}(x)}{dx}\right\|\right\}$$

$$\left. + P\max\left\{\left\|\frac{d_\Phi x_s^{(1)}(x)}{dx} - \frac{d_\Phi x_s^{(0)}(x)}{dx}\right\|,\ \left\|\frac{d_\Phi x_{s+1}^{(1)}(x)}{dx} - \frac{d_\Phi x_{s+1}^{(0)}(x)}{dx}\right\|\right\}\right\}$$

$$\le \frac{N-1}{2}\left(\frac{\beta}{1-\eta} + P\eta\right) \le \frac{\eta}{P}\left(\frac{\beta}{1-\eta} + P\eta\right).$$

Taking into account the last inequalities for $n \in \Phi\backslash\{0,1\}$ we have

$$\left\|\frac{d_\Phi x_n^{(2)}(x)}{dx} - \frac{d_\Phi x_n^{(1)}(x)}{dx}\right\| \le \frac{N^2}{4}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(1)}(x), x_{s+1}^{(1)}(x))}{dx} - \right.\right.$$

$$\left.\left. - \frac{d_\Phi f_s(x,x)}{dx}\right\|\right\} \le \frac{\eta}{P}\left(\frac{\beta}{1-\eta} + P\eta\right),$$

that is, estimate (14) holds for $m = 2,\ n \in \Phi\backslash\{0\}$.

Let's assume that it holds for all $2 < m \leq k$ and prove its validity for $m = k + 1$.
For all $n \in \Phi\backslash\{0\}$, we have

$$\left\|\frac{d_\Phi x_n^{(k+1)}(x)}{dx} - \frac{d_\Phi x_n^{(k)}(x)}{dx}\right\| \leq \frac{\eta}{P}\max_{s\in\Phi}\left\{\left\|\frac{d_\Phi f_s(x_s^{(k)}(x), x_{s+1}^{(k)}(x))}{dx}\right.\right.$$

$$\left.\left. - \frac{d_\Phi f_s(x_s^{(k-1)}(x), x_{s+1}^{(k-1)}(x))}{dx}\right\|\right\} \leq \frac{\eta}{P}\left\{\frac{\beta}{1-\eta}\gamma^{k-1}\right.$$

$$+ P\max\left\{\left\|\frac{d_\Phi x_s^{(k)}(x)}{dx} - \frac{d_\Phi x_s^{(k-1)}(x)}{dx}\right\|, \left\|\frac{d_\Phi x_{s+1}^{(k)}(x)}{dx}\right.\right.$$

$$\left.\left.\left. - \frac{d_\Phi x_{s+1}^{(k-1)}(x)}{dx}\right\|\right\}\right\} \leq \frac{\eta}{P}\left\{\frac{\beta}{1-\eta}\gamma^{k-1}\right.$$

$$\left. + \eta\left(\sum_{i=0}^{k-2}\frac{\beta}{1-\eta}\gamma^{k-2-i}\eta^i + P\eta^{k-1}\right)\right\},$$

which finishes the proof of estimate (14).

If $\eta \neq \gamma$ right-hand side of estimate (14) is equal to the expression

$$\frac{\eta\beta}{P(1-\eta)(\eta-\gamma)}(\eta^{m-1} - \gamma^{m-1}) + \eta^m.$$

Since $\eta < 1$ and $\gamma < 1$, the statement of Lemma 3 is proved.

If $\eta = \gamma$, then the right-hand side of estimate (14) is equal to the expression

$$\frac{\beta}{P(1-\gamma)}\gamma^{m-1}(m-1) + \gamma^m \leq \left(\frac{\beta}{P(1-\gamma)} + 1\right)\gamma^{m-1}(m-1), \ \ m \geq 2,$$

and it also shows that the sequence $\left\{\frac{d_\Phi x_n^{(m)}(x)}{dx}\right\}_{m=0}^{\infty}$, is fundamental since the series $\sum\limits_{m=2}^{\infty}\gamma^{m-1}(m-1)$ converges. Lemma 3 is proved.

Corollary 1. *With the conditions of Lemma 2, the function $\widetilde{x}_n(x)$ determined in claim 5), is Frechet differentiable on D_0^f for all $n \in \Phi$.*

The proof of this statement follows at once from Theorem 111 [13, p.780].

Lemma 4. *With the conditions of Lemma 2, the mapping $\Delta(x) = \overline{f_n(\widetilde{x}_n(x), \widetilde{x}_{n+1}(x))}$ is Frechet differentiable on the set D_0^f and, moveover, for all $\{x_1, x_2\} \subset D_0^f$ the following inequality holds:*

$$\left\|\frac{d_\Phi\Delta(x)}{dx}\Big|_{x_2}^{x_1}\right\| \leq L\|x_1 - x_2\|,$$

where $L = \mathrm{const} > 0$.

Proof. Taking into account Lemmas 1 – 3, it is not difficult to see that for all $n \in \Phi$ and $\{x_1, x_2\} \subset D_0^f$, the following relations take place:

$$\left\|\frac{d_\Phi \widetilde{x}_n(x)}{dx}\right\| \leq \frac{1}{1-\eta},$$

$$\left\|\frac{d_\Phi \widetilde{x}_n(x)}{dx}\bigg|_{x_2}^{x_1}\right\| \leq \frac{L_0\eta}{P(1-\eta)^2(1-\gamma)}\|x_1 - x_2\|,$$

$$\frac{d_\Phi \Delta(x)}{dx} = \frac{1}{N}\sum_{s=0}^{N-1} \frac{d_\Phi f_s(\widetilde{x}_s(x), \widetilde{x}_{s+1}(x))}{d(\widetilde{x}_s, \widetilde{x}_{s+1})}\frac{d\widetilde{g}_s(x)}{dx},$$

where $\widetilde{g}_s(x)$ is a mapping of the set D_0^f to the set Ω with components $\widetilde{g}_s^1(x) = \widetilde{x}_s(x),\ \widetilde{g}_s^2(x) = \widetilde{x}_{s+1}(x)$.

Taking into account the estimate for $\|\omega_n\|_0$ from Theorem 2 of [10], we have

$$\left\|\frac{d_\Phi \Delta(x)}{dx}\bigg|_{x_2}^{x_1}\right\| \leq \frac{1}{N}\sum_{s=0}^{N-1}\Bigg\{\left\|\frac{d_\Phi f_s(\widetilde{x}_s(x), \widetilde{x}_{s+1}(x))}{d(\widetilde{x}_s, \widetilde{x}_{s+1})}\bigg|_{x_2}^{x_1}\right\| \left\|\frac{d_\Phi \widetilde{g}_s(x_1)}{dx}\right\|$$

$$+\left\|\frac{d_\Phi f_s(\widetilde{x}_s(x_2), \widetilde{x}_{s+1}(x_2))}{d(\widetilde{x}_s, \widetilde{x}_{s+1})}\right\| \left\|\frac{d_\Phi \widetilde{g}_s(x)}{dx}\bigg|_{x_2}^{x_1}\right\|\Bigg\}$$

$$\leq \frac{1}{N}\sum_{s=0}^{N-1}\Bigg\{L_0 \max\Big\{\Big\|\widetilde{x}_s(x)|_{x_2}^{x_1}\Big\|, \Big\|\widetilde{x}_{s+1}(x)|_{x_2}^{x_1}\Big\|\Big\}$$

$$\times \max\left\{\left\|\frac{d_\Phi \widetilde{x}_s(x_1)}{dx}\right\|, \left\|\frac{d_\Phi \widetilde{x}_{s+1}(x_1)}{dx}\right\|\right\}$$

$$+ P\max\left\{\left\|\frac{d_\Phi \widetilde{x}_s(x)}{dx}\bigg|_{x_2}^{x_1}\right\|, \left\|\frac{d_\Phi \widetilde{x}_{s+1}(x)}{dx}\bigg|_{x_2}^{x_1}\right\|\right\}\Bigg\}$$

$$\leq \frac{L_0}{(1-\eta)(1-\gamma)}\left(1 + \frac{\eta}{1-\eta}\right)\|x_1 - x_2\|.$$

Denoting the constant term $\frac{L_0}{(1-\eta)(1-\gamma)}\left(1 + \frac{\eta}{1-\eta}\right)$ by L, we finish the proof of Lemma 4.

Lemma 5. *Let the conditions of Lemma 2 be satisfied, and there exist a point $x^0 \in D_0^f$ and a sequence of indexes $p_1 < p_2 < p_3 < \ldots < p_k < \ldots$ such that, for $s \in Z^+$,*

$$\left\|\frac{d_\Phi \Delta_{p_s}(x^0)}{dx^0} - E\right\| \leq l_0 < 1.$$

Then the mapping $\dfrac{d_\Phi \Delta(x^0)}{dx^0}$ *is circulating invertible and, moreover,*

$$\left\| \left[\frac{d_\Phi \Delta(x^0)}{dx^0} \right]^{-1} \right\| \le N_0^* = \frac{1}{1 - l_0}.$$

Proof. It is obvious that the statement of Lemma 5 follows at once if we pass to the limit,

$$\left\| \frac{d_\Phi \Delta(x)}{dx} - \frac{d_\Phi \Delta_m(x)}{dx} \right\| \to 0 \text{ as } m \to \infty. \tag{15}$$

First, let's estimate the difference $I_n^m(x) = \dfrac{d_\Phi \widetilde{x}_n(x)}{dx} - \dfrac{d_\Phi x_n^{(m)}(x)}{dx}$,

$$\begin{aligned} \|I_n^m(x)\| &= \lim_{p \to \infty} \left\| \frac{d_\Phi x_n^{m+p}(x)}{dx} - \frac{d_\Phi x_n^{(m)}(x)}{dx} \right\| \\ &\le \lim_{p \to +\infty} \sum_{i=1}^{p} \left\| \frac{d_\Phi x_n^{(m+i)}(x)}{dx} - \frac{d_\Phi x_n^{(m+i-1)}(x)}{dx} \right\|. \end{aligned}$$

Using Lemma 3 with $m \ge 2,\ n \in \Phi \backslash \{0\}$, and $\eta \ne \gamma$ (for example, $\eta > \gamma$), we come to the inequalities

$$\begin{aligned} \left\| I_n^{(m)}(x) \right\| &\le \sum_{i=1}^{\infty} \left(\frac{\eta\beta}{P(1-\eta)(\eta-\gamma)} + 1 \right) \eta^{m+i-1} \\ &\le \frac{\eta^m}{1-\eta} \left(\frac{\beta}{P(1-\eta)(\eta-\gamma)} + 1 \right), \end{aligned}$$

from which it follows that $\|I_n^m(x)\| \to 0$ as $m \to \infty$, since $\eta < 1$.

If $\gamma = \eta$, then

$$\begin{aligned} \left\| I_n^{(m)}(x) \right\| &\le \sum_{i=1}^{\infty} \left(\frac{\beta}{P(1-\gamma)} + 1 \right) \gamma^{m+i-1}(m+i-1) \\ &= \gamma^{m-1} \left(\frac{\beta}{1-\gamma} + 1 \right) \left(\sum_{i=1}^{\infty} i\gamma^i + \frac{(m-1)\gamma}{1-\gamma} \right) \to 0 \end{aligned}$$

as $m \to \infty$, since the series $\sum_{i=1}^{\infty} i\gamma^i$ converges to a number l^*.

Now we have

$$\left\|\frac{d_\Phi\Delta(x)}{dx} - \frac{d_\Phi\Delta_m(x)}{dx}\right\| \le \frac{1}{N}\sum_{s=0}^{N-1}\left\{\left\|\frac{d_\Phi f_s(\widetilde{x}_s(x),\widetilde{x}_{s+1}(x))}{d(\widetilde{x}_s,\widetilde{x}_{s+1})} - \frac{d_\Phi f_s(x_s^{(m)}(x),x_{s+1}^{(m)}(x))}{d(x_s^{(m)},x_{s+1}^{(m)})}\right\|\left\|\frac{d\widetilde{g}_s(x)}{dx}\right\| + \left\|\frac{d_\Phi f_s(x_s^{(m)}(x),x_{s+1}^{(m)}(x))}{d(x_s^{(m)},x_{s+1}^{(m)})}\right\|\left\|\frac{d_\Phi\widetilde{g}_s(x)}{dx} - \frac{dg_s^{(m)}(x)}{dx}\right\|\right\}$$

$$\le \frac{1}{N}\sum_{s=0}^{N-1}\left\{L_0\max\left\{\left\|\widetilde{x}_s(x)-x_s^{(m)}(x)\right\|,\left\|\widetilde{x}_{s+1}(x)-x_{s+1}^{(m)}(x)\right\|\right\}\right.$$
$$\times\max\left\{\left\|\frac{d_\Phi\widetilde{x}_s(x)}{dx}\right\|,\left\|\frac{d_\Phi\widetilde{x}_{s+1}(x)}{dx}\right\|\right\}$$
$$\left.+P\max\left\{\left\|I_s^m(x)\right\|,\left\|I_{s+1}^m(x)\right\|\right\}\right\} \le L_0\sigma^*(m)\frac{1}{1-\eta} + P\sigma_*(m) \to 0$$

as $m\to\infty$, because $\sigma^*(m) = \dfrac{MN^2\gamma^m}{4(1-\gamma)}$ defined in the statement 5) and $\sigma_*(m)$ given for $m \ge 2$ by the relation

$$\sigma_*(m) = \begin{cases} \dfrac{\eta^m}{1-\eta}\left(\dfrac{\beta}{P(1-\eta)(\eta-\gamma)}+1\right) & \text{if } \eta>\gamma;\\ \gamma^{m-1}\left(\dfrac{\beta}{P(1-\gamma)}+1\right)\left(l^*+\dfrac{(m-1)\gamma}{1-\gamma}\right) & \text{if } \eta=\gamma;\\ \dfrac{\gamma^m}{1-\gamma}\left(\dfrac{\beta}{P(1-\eta)(\gamma-\eta)}+1\right) & \text{if } \eta<\gamma, \end{cases}$$

approach zero as $m\to\infty$. This ends the proof of estimate (15) and Lemma 5.

Let's introduce the notations

$$N_0 = \left\|\left[\frac{d_\Phi\Delta(x^0)}{dx^0}\right]^{-1}\right\|,\ k = \left\|\left[\frac{d_\Phi\Delta(x^0)}{dx^0}\right]^{-1}\cdot\Delta(x^0)\right\|,\ h = N_0kL,$$

t_0 is the smallest root of the equation $ht^2 - t + 1 = 0$.

The lemmas proved above allow us to formulate the following statement which gives sufficient conditions for existence of an N-periodic solution of equation (1).

Theorem 1. *Let the conditions of Lemmas 4 and 5 hold. Suppose that the constants defined there, L and N_0^*, are such that $h^* = LN_0^{*2}M < 0{,}25$ and the closed ball $B^*(x^0, k^*t^*) \subset D_0^f$, where $k^* = N_0^*M, t^*$ is the smallest root of the equation $h^*t^2 - t + 1 = 0$.*

Then there is only one point x^* in the closed ball $B(x^0, kt_0) \subset B^*(x^0, k^*t^*)$ generating an N-periodic solution $\widetilde{x}_n(x^*), \widetilde{x}_0(x^*) = x^*$ of equation (1).In such a case we have

$$\left\|\widetilde{x}_n(x^*) - x_n^{(p)}(x_p)\right\| \le \frac{M}{1-\gamma}\left(\frac{N_0^* q^{*^p}}{1-q^*} + \frac{N^2\gamma^p}{4}\right), \tag{16}$$

where $q^* = \dfrac{1-\sqrt{1-4h^*}}{2} < 0,5$, *there function* $x_n^{(p)}(x_p)$ *are defined by relations (7), and* $\{x_p\}$ *is a sequence defined by the recurrence relation*

$$x_0 = x^0, \quad x_{n+1} = x_n - \left[\frac{d_\Phi \Delta(x^0)}{dx^0}\right]^{-1} \cdot \Delta(x_n), \quad n \in Z^+.$$

Proof. Since $k \le k^*, h \le h^*$, we have $t_0 \le t^*$ and the ball $B(x^0, kt_0)$ is embedded into the ball $B^*(x_0, k^*t^*)$ and, hence, in the set D_0^f. This allows to use Theorem 1 from [14, p. 430] and to write the estimate

$$\|x^* - x_p\| \le \frac{q^{*p}}{1-q^*} N_0^* M,$$

which leads to the inequalities

$$\begin{aligned}\left\|\widetilde{x}_n(x^*) - x_n^{(p)}(x_p)\right\| &\le \left\|\widetilde{x}_n(x^*) - \widetilde{x}_n(x_p)\right\| + \left\|\widetilde{x}_n(x_p) - x_n^{(p)}(x_p)\right\| \\ &\le \frac{\|x^* - x_p\|}{1-\gamma} + \sigma^*(p) \le \frac{q^{*^p}}{1-q^*} N_0^* M \frac{1}{1-\gamma} \\ &+ \frac{\left(\dfrac{N^2}{4}\right)^{p+1} M(K_1+K_2)^p}{1-\gamma}.\end{aligned} \tag{17}$$

The last estimate gives inequality (16), and this completes the demonstration of Theorem 1.
Now let's consider the equation

$$\Delta_{p_s}(x) = \overline{f_n\left(x_n^{(p_s)}(x), x_{n+1}^{(p_s)}(x)\right)} = 0. \tag{18}$$

In the same way as in the proof of Lemma 4, it is not difficult to see that for all $\{x_1, x_2\} \subset D_0^f$, the following inequality holds:

$$\left\|\left.\frac{d_\Phi \Delta_{p_s}(x)}{dx}\right|_{x_2}^{x_1}\right\| \le L\|x_1 - x_2\|, \quad s \in Z^+.$$

Besides, it is obvious that $\left\|\left[\dfrac{d_\Phi \Delta_{p_s}(x^0)}{dx^0}\right]^{-1}\right\| \le N_0^*$. Therefore, if the conditions of Theorem 1 holds, equation (18), for aech $s \in Z^+$ in the closed ball $B^*(x_0, k^*t^*)$, has a solution x_{p_s} which is a limit, as $k \to \infty$, of the sequence $\{x_k^{(p_s)}\}, x_0^{(p_s)} = x^0$, constructed using the recurrence formula

$$x_{k+1}^{(p_s)} = x_k^{(p_s)} - \left[\frac{d_\Phi \Delta_{p_s}(x^0)}{dx^0}\right]^{-1} \cdot \Delta_{p_s}(x_k^{(p_s)}).$$

For all $s \in Z^+, k \in Z_0^+$, the points $x_k^{(p_s)}$, together with the point x^*, belong to the closed ball $B^*(x^0, t^* k^*)$.

Let's introduce the notations

$$l_* = \sum_{i=1}^{\infty} i l_0^{i-1}, \qquad G = 1 + \frac{N_0^*(K_1 + K_2)}{1-\gamma},$$

$$\varepsilon(p_s) = l_* M \left\{ L_0 \sigma^*(p_s) \frac{1}{1-\eta} + P\sigma_*(p_s) \right\} + N_0^*(K_1 + K_2)\sigma^*(p_s),$$

$$\delta(p_s) = N_0^*(K_1 + K_2)\sigma^*(p_s) + M \left\{ L_0 \sigma^*(p_s) \frac{1}{1-\eta} + P\sigma_*(p_s) \right\}.$$

Since $\sigma^*(p_s)$ and $\sigma_*(p_s)$ approach zero as $s \to \infty$, both $\varepsilon(p_s)$ and $\delta(p_s)$ have the same property.

Let's formulate a statement which allows to approximate the function $\widetilde{x}_n(x^*)$ by the function $x_n^{(m)}(x_k^{(p_s)})$ with an arbitrary precision.

Theorem 2. *In the conditions of Theorem 1, the following relations hold:*

$$\lim_{k\to\infty} \lim_{s\to\infty} x_k^{(p_s)} = x^*, \tag{19}$$

moreover, if p_s and m are greater than two, we have

$$\left\| \widetilde{x}_n(x^*) - x_n^{(m)}(x_k^{(p_s)}) \right\| \leq \frac{\left\| x^* - x_k^{(p_s)} \right\|}{1-\gamma} + \sigma^*(m), \tag{20}$$

$$\left\| x^* - x_k^{(p_s)} \right\| \leq \frac{q^{*k}}{1-q^*} N_0^* M + \varepsilon(p_s) G^{k-1} + \delta(p_s) \sum_{i=0}^{k-2} G^i. \tag{21}$$

Proof. Estimate (20) is obtained in the same way as estimate (17). We'll show that, for $k \in Z_0^+$,

$$\lim_{s\to\infty} x_k^{(p_s)} = x_k, \tag{22}$$

where $\{x_k\}$ is a sequence of point obtained by using the modified method of Newton which consists in applying the recurrence formula from Theorem 1 with $x_0 = x^0$.

The inequality

$$\left\| \left[\frac{d_\Phi \Delta(x^0)}{dx^0} \right]^{-1} - \left[\frac{d_\Phi \Delta_{p_s}(x^0)}{dx^0} \right]^{-1} \right\| \le \left\| \frac{d_\Phi \Delta(x^0)}{dx^0} - \frac{d_\Phi \Delta_{p_s}(x^0)}{dx^0} \right\| l_*$$

leads to the estimate

$$\left\| x_1 - x_1^{(p_s)} \right\| \le \left\| \frac{d_\Phi \Delta(x^0)}{dx^0} - \frac{d_\Phi \Delta_{p_s}(x^0)}{dx^0} \right\| l_* M + N_0^*(K_1 + K_2)\sigma^*(p_s) \le \varepsilon(p_s).$$

Now write the following recurrence relation:

$$\begin{aligned} \left\| x_k - x_k^{(p_s)} \right\| &\le \left\| x_{k-1} - x_{k-1}^{(p_s)} \right\| + \left\{ L_0 \sigma^*(p_s) \frac{1}{1-\eta} + P\sigma_*(p_s) \right\} M \\ &\quad + N_0^* \left\| \Delta(x_{k-1}) - \Delta p_s(x_{k-1}^{(p_s)}) \right\| \\ &\le \left\| x_{k-1} - x_{k-1}^{(p_s)} \right\| \left\{ 1 + \frac{N_0^*(K_1+K_2)}{1-\gamma} \right\} + N_0^*(K_1+K_2)\sigma^*(p_s) \\ &\quad + M \left\{ L_0 \sigma^*(p_s) \frac{1}{1-\eta} + P\sigma_*(p_s) \right\} \\ &\le \left\| x_{k-1} - x_{k-1}^{(p_s)} \right\| G + \delta(p_s), \end{aligned}$$

from which we get:

$$\left\| x_k - x_k^{(p_s)} \right\| \le \varepsilon(p_s) G^{k-1} + \delta(p_s) \sum_{i=0}^{k-2} G^i$$

obtained by induction and from which relation (22) follows, where passing to the limit is not uniform in $k \in Z_0^+$.

From the last inequality it is easy to obtain inequality (21) which guarantees the validity of relation (19).

We must remark that nothing can be said about the validity of interchanging the limits (19). Theorem 2 is proved.

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