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NONLINEAR SIMULATION OF THE HYDROGEN ATOM BASED ON THE MODEL OF THE ELLIPTIC OSCILLATOR

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It is shown that the application of the model of the hydrogen atom, which is based on the theory of the elliptic oscillator, makes it possible to describe the structure of electron orbits. The investigation is based on asymptotics methods of nonlinear mechanics. It was established that the stable orbits of the electron correspond to certain resonant states. A numerical simulation of the problem was carried out.

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1. Introduction

The theories by Bohr, Sommerfeld, Heisenberg and Schrödinger [1 – 5] follow directly or indirectly from the Planck's postulate $E = h\nu n$. It is possible to show that, still using this postulate as a starting point, one can obtain other alternate quantum theories, whose reliability is the same as the previous ones. If we do not use the Planck's postulate and consider the nonlinearity of the electric field, that solves even the black body problem from the point of view of the classical continuity, it is possible to show that the equation $\ddot{x}+\gamma\dot{x}+\omega^2 x+\xi x^2 = \delta\cos(\Omega t+\varphi)$, where γ is the classical radiation pressure and $\varphi = 2/(3\cdot 137)$, describes correctly the hydrogen stationary states, and further forecasts new lines forbidden by the previous approaches. Finally, the stationary states of matter are interpreted in the clear classical environment as discrete resonance states between the electron bounded to the atom and even fully interacting with the surrounding environment.

2. The Nonlinearity of Coulomb's Law and Its Implications

Usually investigations on electron orbits were based on the model of a circular orbit, which results in a model of linear two-dimensional oscillator or the Heisenberg model, which includes postulation of the nonlinear term

$$\frac{d^2x}{dt^2} + \omega^2 x + \lambda x^2 = 0.$$

Based on these models it is possible to construct energy levels by means of the Plank's hypothesis ($E = h\nu n$).

In order to study the hydrogen atom without making recourse to Planck's postulate, it is necessary to highlight a fundamental feature of the electric field, i.e. its strong nonlinearity, and all its important consequences. It is easy to show that in most deductive reasonings of theoretical physics the nonlinearity of Coulomb's law has been always neglected.

Let us consider now the ellipse represented by the equation ($\varepsilon < 0$)

$$r = \frac{p}{1 + \varepsilon \cos \alpha}. \tag{1}$$

From (1) by setting $\cos \alpha = x/r$, we deduce that $r = p(1 - \varepsilon x/p)$. This implies that the projection of the force along the abscissae axis is given by

$$F_x = -\frac{e^2}{r^3} x = -\frac{e^2}{p^3 (1 - \varepsilon \frac{x}{p})^3} x.$$

Therefore, denoting

$$\omega^2 = \frac{e^2}{mp^3},$$

we finally obtain the equation along the x and y axes,

$$\frac{d^2 x}{dt^2} + \frac{\omega^2 x}{\left(1 - \varepsilon \frac{x}{p}\right)^3} = 0, \qquad \frac{d^2 y}{dt^2} + \frac{\omega^2 y}{\left(1 - \varepsilon \frac{x}{p}\right)^3} = 0.$$

Contrary to the classical approach, this procedure makes it possible to find the force-deformation law of the elliptic oscillator equation.

All these considerations imply that, to replicate the Rutherford's model with major accuracy, under the hypothesis that the external perturbation would not modify abruptly the original orbit, it should be necessary to consider, for example in the Planck's case, the equation

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \frac{\omega^2 x}{\left(1 - \varepsilon \frac{x}{p}\right)^3} = \frac{1}{e\sqrt{2\pi}} \int_{-\infty}^{+\infty} E(\Omega) \exp(i\Omega t) d\Omega, \tag{2}$$

that, up to small nonlinearity, coincides with the equation of Heisenberg. Indeed we get

$$\omega^2 x \left(1 - \varepsilon \frac{x}{p}\right)^{-3} \simeq \omega^2 x \left(1 + 3\varepsilon \frac{x}{p}\right) = \omega^2 x + 3\frac{\varepsilon}{p}\omega^2 x^2 = \omega^2 x + \lambda x^2.$$

The fact that, in the classical study of wave-particle interaction phenomena, the atom has always been represented by a linear model has clear and strong negative consequences. Actually, the transfer of energy from the wave to the matter, given the model adopted by Planck, occurs exclusively by means of the *fundamental* resonance phenomenon. When the nonlinearity is taken into account, it is well-known [6, 7] and easy to verify, that the *multiresonance* phenomenon occurs. Actually the resonance phenomenon occurs only when the external pulsation Ω tends to the natural pulsation ω of the system considered, but occurs even when $\Omega/\omega = n$ or $\omega/\Omega = n$ (this single fact would justify physically and would extend Planck's postulate). Therefore, when the nonlinearity of the electric field is neglected, we have a very limited forecast about the reality, and this is the main problem of a number of failures generally attributed to inner defects of classical mechanics (namely of its part relative to formulas deduced by the study of single central fields only). It is enough to consider that the harmonic oscillator can be obtained from the anharmonic oscillator by taking into account only the first term of an infinite series development. In this last case, the analytic forecast of the existence of infinite resonances can be justified intuitively by noting that when the distance from the central charge increases we have a series of linear oscillators with varying values of $\omega^2 = e^2/(md^3)$, so that a specific and distinct resonance frequency can be associated to each of these values.

In the following we will study the equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x + 3\frac{\varepsilon}{p}\omega^2 x^2 = \delta\exp(-i\Omega t), \tag{3}$$

but will also consider the more complete equation,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{\omega^2 x}{\left(1-\varepsilon\frac{x}{p}\right)^3} = \delta\exp(-i\Omega t). \tag{4}$$

We will see that the results of determining the electron orbits obtained by each of these two equations are completely analogous. Actually, as we can see by observing equation (4), we must study the possible stability and instability conditions of an electron that, in the absence of radiation emission and of interaction with the surrounding environment (represented by the external force), would describe, according to classical mechanics, an elliptic orbit.

3. Existence of a Linear Stationary Solution

When an electromagnetic wave hits the above-mentioned nonlinear oscillator, we can find a variety of solutions. It is known that they depend exclusively on initial conditions, which can give rise to such a great number of solutions that they could seem to be chaotic. As an example of the possible solutions that generally could occur, we can cite the case in which the peripheral particle, instead of orbiting around the central charge, oscillates harmonically along a small segment orthogonal to one of the equipotential surfaces generated by the central charge, with an energy loss due to the radiation emitted exactly equal to the energy absorbed from the external environment (stationary solution). This case would not be physically understandable if we still thought of a completely isolated atom.

With the hypothesis that $|\varepsilon| < 1$ we can write relation (4) in the following form:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x + 3\omega^2\frac{\varepsilon}{p}x^2 + 6\omega^2\frac{\varepsilon^2}{p^2}x^3 + 10\omega^2\frac{\varepsilon^3}{p^3}x^4 + \ldots = \delta\exp(-i\Omega t). \tag{5}$$

Now let us assume that there exists a solution of the type

$$x(t) = \sum_{n=1}^{\infty} A_n \exp(-in\Omega t). \tag{6}$$

Substituting (6) into (5) and assuming that $A_i >> A_{i+1}$, we find the following values for various coefficients of the series (6):

$$A_1 = \frac{\delta}{\Omega^2 - \omega^2 + i\gamma\Omega}, \quad A_2 = 3\frac{\omega^2 \varepsilon A_1^2}{p(4\Omega^2 - \omega^2 + 2i\gamma\Omega)}, \quad A_3 = 6\frac{\omega^2 \varepsilon A_1 (A_2 p + \varepsilon A_1^2)}{p^2(9\Omega^2 - \omega^2 + 3i\gamma\Omega)},$$

$$A_4 = \frac{\varepsilon\omega^2(6A_1A_3p^2 + 3A_2^2p^2 + 18\varepsilon A_1^2A_2p + 10\varepsilon^2A_1^4)}{p^3(16\Omega^2 - \omega^2 + 4i\gamma\Omega)}, \quad \dots, \quad A_n = \frac{\varepsilon\omega^2(F(A_i, p, \varepsilon))}{(n^2\Omega^2 - \omega^2 + ni\gamma\Omega)}. \tag{7}$$

We can observe that the expression of the generic term A_n (excluding the ε variable and the resonant term $1/(n^2\Omega^2 - \omega^2 + ni\gamma\Omega)$) is practically coincident with the corresponding terms which are obtained for the Heisenberg's equation for relevant x_i.

When we substitute A_1 into the subsequent terms we get

$$\Psi_1 = -\frac{\delta}{\Omega^2 - \omega^2 + i\gamma\Omega} \exp(-i\Omega t),$$

$$\Psi_2 = 3\frac{\omega^2\delta^2\varepsilon}{p(\Omega^2 - \omega^2 + i\gamma\Omega)^2(4\Omega^2 - \omega^2 + 2i\gamma\Omega)} \exp(-2i\Omega t),$$

$$\Psi_3 = -12\frac{\omega^2\delta^3\varepsilon^2(\omega^2 + 2\Omega^2 + i\gamma\Omega)}{p^2(\Omega^2 - \omega^2 + i\gamma\Omega)^3(4\Omega^2 - \omega^2 + 2i\gamma\Omega)(9\Omega^2 - \omega^2 + 3i\gamma\Omega)} \exp(-3i\Omega t), \tag{8}$$

$$\Psi_n = \dots$$

Even in this case we observe that the generic term in the previous series is resonant both at its own pulsation and at all previous critical pulsations. Furthermore we can see that this solution, expressed in terms of harmonics of the type $(n\Omega)$, shows a resonance phenomenon only if the condition

$$\omega = \Omega n \tag{9}$$

is satisfied. Going into the field of real numbers, we have

$$\Psi_1 = -\frac{\delta\left(\Omega^2 - \omega^2\right)}{\left(\Omega^2 - \omega^2\right)^2 + \gamma^2\Omega^2} \cos(\Omega t) + \frac{\delta\gamma\Omega}{\left(\Omega^2 - \omega^2\right)^2 + \gamma^2\Omega^2} \sin(\Omega t),$$

$$\Psi_2 = 3\left[\frac{\omega^2\varepsilon\delta^2\left(\left(\omega^2-\Omega^2\right)^2-\gamma^2\Omega^2\right)\left(4\Omega^2-\omega^2\right)-2\omega^2\varepsilon\delta^2\left(2\Omega^3\gamma-2\omega^2\gamma\Omega\right)^2\gamma\Omega}{\left(\left((\omega^2-\Omega^2)^2-\gamma^2\Omega^2\right)^2+(2\Omega^3\gamma-2\omega^2\gamma\Omega)^2\right)p\left((4\Omega^2-\omega^2)^2+4\gamma^2\Omega^2\right)}\right]\cos(2\Omega t)$$

$$+3\left[\frac{-\omega^2\varepsilon\delta^2\left(2\Omega^3\gamma-2\omega^2\gamma\Omega\right)^2\left(4\Omega^2-\omega^2\right)-2\omega^2\varepsilon\delta^2\left(\left(\omega^2-\Omega^2\right)^2-\gamma^2\Omega^2\right)\gamma\Omega}{\left(\left((\omega^2-\Omega^2)^2-\gamma^2\Omega^2\right)^2+(2\Omega^3\gamma-2\omega^2\gamma\Omega)^2\right)p\left((4\Omega^2-\omega^2)^2+4\gamma^2\Omega^2\right)}\right]\sin(2\Omega t), \tag{10}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\Psi_n = \frac{A_n}{\left(n^2\Omega^2-\omega^2\right)^2+n^2\gamma^2\Omega^2}\cos(n\Omega t)+\frac{B_n}{\left(n^2\Omega^2-\omega^2\right)^2+n^2\gamma^2\Omega^2}\sin(n\Omega t).$$

For obvious problems of space, in the last equation we have shown only the resonance $n\Omega$, but it is important to remember that each term of the series also contains resonances $(n-1)\Omega$, $(n-2)\Omega$, …. Therefore we can also have amplitudes $\Psi_{n,n-1}$, $\Psi_{n,n-2}, \ldots, \Psi_{n,1}$, besides $\Psi_{n,n}$. This concept can be expressed synthetically by a triangular matrix that embodies the features of this oscillator,

$$\begin{vmatrix} \Psi_{1,1} & & & \\ \Psi_{2,1} & \Psi_{2,2} & & \\ \Psi_{3,1} & \Psi_{3,2} & \Psi_{3,3} & \\ \ldots & & & \\ \Psi_{n,1} & \Psi_{n,2} & \Psi_{n,3} & \Psi_{n,n} \end{vmatrix}. \tag{11}$$

Actually, in the first row of this matrix, which expresses features of the first term of the series development, there is only the amplitude $\Psi_{1,1}$ due to the presence of the main resonance only; in the second row, which corresponds to the second term of the development, there are the first and second resonance, in the third row there are the first, second, and third resonances, and so on.

In the following, as usual, we will use the expression *elastic amplitude* to denote the coefficient of $\cos(n\Omega t)$ and the expression *absorption amplitude* to denote the coefficient of $\sin(n\Omega t)$, since this is the term that generates a speed in phase with the driving force.

Thus, a particular solution of equation (5) can be written in the form

$$x(t) = \sum_{n=1}^{\infty}\Psi_i. \tag{12}$$

The terms in series (12) are connected with the corresponding equations of system (10) has the same aspect of the stationary solution of the damped harmonic oscillator which is of the type

$$\Psi_1 = B_1\cos\Omega t + C_1\sin\Omega t, \quad \Psi_2 = (B_2\cos 2\Omega t + C_2\sin 2\Omega t)\varepsilon,$$

$$\ldots, \quad \Psi_n = (B_n\cos n\Omega t + C_n\sin n\Omega t)\varepsilon^n. \tag{13}$$

This implies that the solution of our equation can also be written in the following form

$$x = \sum_{n=1}^{\infty} \rho_n \cos(n\Omega t - \Phi_n), \tag{14}$$

where the following definitions have been used (note that each oscillator has a distinct phase that characterizes it further),

$$\rho_n = \sqrt{B_n^2 + C_n^2}, \quad \tan\Phi_n = \frac{C_n}{B_n}.$$

The considerations above lead us to the conclusion that, when secular terms are absent, relations (6) or their equivalent relation (14) represent a stationary solution of equation (5). It is reasonable to assume that it is also unique, due to the univocal determination of the terms A_i. Furthermore, we can observe that the solution we have found can be interpreted as a sum of the stationary solutions of an infinite set of damped harmonic oscillators driven by proper external forces and correlated with each other.

4. Evaluation of the Series Coefficients

Relations (7), in the hypothesis that ε is sufficiently smaller than 1, can be simplified by neglecting the terms depending on ε^2, and in such a case, they become

$$A_1 = \frac{\delta}{\Omega^2 - \omega^2 + i\gamma\Omega}, \quad A_2 = 3\frac{\omega^2\varepsilon A_1^2}{p(4\Omega^2 - \omega^2 + 2i\gamma\Omega)}, \quad A_3 = 3\frac{\omega^2\varepsilon 2A_1A_2}{p(9\Omega^2 - \omega^2 + 3i\gamma\Omega)}, \tag{15}$$

$$A_4 = 3\frac{\omega^2\varepsilon(2A_1A_3 + A_2^2)}{p(16\Omega^2 - \omega^2 + 4i\gamma\Omega)}, \quad A_5 = 3\frac{\omega^2\varepsilon(2A_1A_4 + 2A_2A_3)}{p(25\Omega^2 - \omega^2 + 5i\gamma\Omega)}, \quad \ldots .$$

We want to establish an algorithm that enables us to calculate any term of series (15). Let us consider, for example, the fourth term. If we define the two matrices

$$A = |A_1\ A_2\ A_3| \quad \text{and} \quad B = \begin{vmatrix} A_3 \\ A_2 \\ A_1 \end{vmatrix},$$

where B is the transposed and *reversed* matrix of A, we have

$$|A| \times |B| = 2A_1A_3 + A_2^2.$$

We can also verify that this rule, if we accept a less precise solution, is valid for any coefficient of the series. Therefore, we can write in short

$$x(t) = \sum_{n=1}^{\infty} \frac{3\omega^2\varepsilon}{p\left(n^2\Omega^2 - \omega^2 + ni\gamma\Omega\right)} \left(|A_1\ A_2\ A_3 ... A_{n-1}| \times \begin{vmatrix} A_{n-1} \\ \cdots \\ A_3 \\ A_2 \\ A_1 \end{vmatrix} \right) \exp(-ni\Omega t), \tag{16}$$

where the initial term is

$$A_1 = \frac{\delta}{\Omega^2 - \omega^2 + i\gamma\Omega},$$

and the symbol $\times$ has been used for the matrix product (that in this case does not commute).

5. Graphical Representations

We will limit ourselves to considerations concerning the first four terms of series (10), and will set the following values[1]:

$$(\varepsilon = 0.8;\ p = 30;\ \delta = 300;\ \omega = 16;\ \gamma = 0.1)\,.$$

With these choices we find the following 3D-graph for $t \in [0, 3]$ and $\Omega \in [3.5, 10]$. Since the critical pulsation of the oscillator is equal to 16, and since we have considered only the first four terms, we will have a resonance condition when

$$(\Omega = 16;\ \Omega = 8;\ \Omega = 5.333;\ \Omega = 4)\,.$$

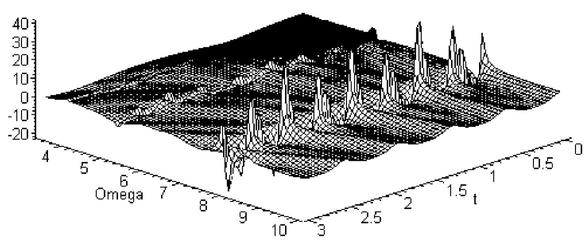


Fig. 1. Miscellaneus resonances of an elliptic oscillator.

In Fig. 1 above, we can see the peaks associated to the three resonances $\Omega = 4$, $\Omega = 5.33$, and $\Omega = 8$. The peak at $\Omega = 16$ is not represented, since it is outside the range chosen for Ω.

Thus, if the present model is valid, there will be an energy transfer from the driving force to the oscillator only when the above mentioned resonances occur. In any other case, the external wave goes through the system leaving it practically unperturbed.

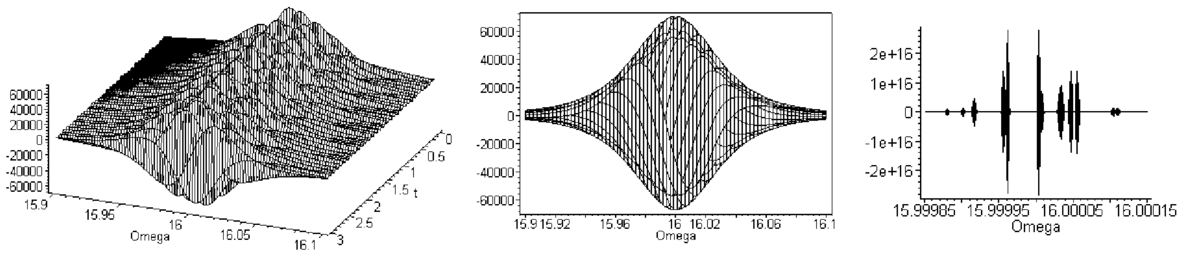
Moreover, it is important to observe the following. Each component of the series contains its proper resonance pulsation $n\Omega$, and also contains all the previous $(n-1)$ critical pulsations. Thus, the resonance $n\Omega$ also appears in all the subsequent terms, and, therefore, when the driving force stimulates this resonance, it *activates* all the infinite oscillators $n, n+1, n+2, \ldots$. Thus the energy, represented by the square of the amplitude, will distribute with a certain law

[1]These values are taken conventionally for qualitative demonstration of the approach potential and do not correspond to the real physical values.

among these oscillators. Noting that A_i is already much greater than A_{i+1}, if the nonlinearity parameter ε is small, the following inequality

$$(A_n \varepsilon^n)^2 >> \left(A_{n+1} \varepsilon^{n+1}\right)^2$$

will be certainly satisfied. This can be illustrated by Fig. 1 that was obtained for a moderately high eccentricity. The figure also shows that the second resonance that occurs for $\Omega = 8$ generates amplitudes much higher than those relative to the third resonance that arises for $\Omega = 5.3$. This resonance, in turn, is much higher than the fourth resonance, which corresponds to $\Omega = 4$.



a *b* *c*

Fig. 2. Envelopment amplitude.

This implies that, if the driving force stimulates the n-th term of the series and if ε is very small, it is possible to set, with good approximation,

$$(x(t))^2 \simeq (\Psi_n)^2 . \tag{17}$$

On the contrary, if the nonlinearity is higher ($\varepsilon \simeq 1$), the various resonance peaks sums and produce a composite profile, due to the subsequent terms that are no more negligible. This effect can be understood by observing the graph in Fig. 2*a*, that is relative to the fundamental resonance $\Omega = 16$ that appears in all infinite terms of the series.

In Fig. 2*b*, which corresponds to the high pressure, we can notice various crests due to infinitely many subsequent terms of the series that resonate when $\Omega = \omega$. These crests have a quite large mean width due to the fact that the value γ of the pressure is quite high.

On the contrary, if we lower the pressure γ to small values, the above mentioned crests transform into very subtle and very high lines, as can be seen in Fig. 2*c*.

The fact that a single vibrating particle can assume the aspect of a boundless line spectrum is singular.

6. Evaluation of Energy Levels

Of course, since now we don't start from Planck's hypothesis, we will find completely general theoretical relations that, for the description of the hydrogen atom, will require that some constants of the equation assume special values. On the other hand, it is easy to see that, without prejudice for the precision of our calculations, we can consider the much more simple equation,

$$\ddot{x} + \gamma \dot{x} + \omega^2 x + 3\omega^2 \frac{\varepsilon}{p} x^2 = \delta \cos(\Omega t + \varphi) , \tag{18}$$

which is analogous to the Heisenberg equation[2] and is suitable to be solved with simple calculations whose extension is in any case immediate and in agreement with the results already obtained.

In the following we will limit ourselves only to the determination of the stationary levels of Bohr and to an analytical outline of the emission process. To simplify the calculations as much as possible (calculations that can easily be extended to formulas containing more than one nonlinear term), we shall assume that there exists a solution of the type

$$x = A_1 \cos(\Omega t) + A_2 \cos(2\Omega t). \tag{19}$$

After substituting this form of the solution in equation (18) and by equating values of the same order of smallness we obtain the following solution:

$$A_1 = \frac{\delta}{\sqrt{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^2}}, \qquad A_2 = -\frac{3}{2}\frac{\omega^2 \varepsilon A_1^2}{p\,(\omega^2 - 4\Omega^2)}. \tag{20}$$

We want to evaluate the work done by the external force. Multiplying (18) by (dx) and integrating over the period $(2\pi/\Omega)$, (18) and (19) will give us

$$\int v dv + \omega^2 \int x dx + \int 3\omega^2 \frac{\varepsilon}{p} x^2 dx + \gamma\Omega^2 \int \left(A_1 \sin(\Omega t) + 2A_2 \sin(2\Omega t) + \ldots\right)^2 dt =$$

$$= -\Omega\delta \int \cos(\Omega t + \varphi)\left(A_1 \sin(\Omega t) + 2A_2 \sin(2\Omega t) + \ldots\right) dt. \tag{21}$$

The first three integrals of (21) are equal to zero owing to the periodicity, so we have

$$\gamma\Omega^2 \int \left(A_1 \sin(\Omega t) + 2A_2 \sin(2\Omega t) + \ldots\right)^2 dt$$

$$= -\Omega\delta \int \cos(\Omega t + \varphi)\left(A_1 \sin(\Omega t) + 2A_2 \sin(2\Omega t) + \ldots\right) dt,$$

from which it follows that

$$\frac{\gamma\pi}{\Omega}\left(A_1^2\,(1\Omega)^2 + A_2^2\,(2\Omega)^2 + A_3^2\,(3\Omega)^2 + \ldots\right) = \pi\delta A_1 \sin\varphi.$$

Therefore,

$$\frac{1}{2} m \left[A_1^2\,(1\Omega)^2 + A_2^2\,(2\Omega)^2 + A_3^2\,(3\Omega)^2 + \ldots\right] = \frac{1}{2} m \frac{\delta^2}{\gamma^2} \sin^2\varphi. \tag{22}$$

This equation sets relation between the amplitudes A_i and their corresponding frequencies.

[2]Even Heisenberg used, at the beginning, the damping term, but subsequently, to simplify the calculations, he eliminated it. In his treatments the external force is always absent because probably he realized that the use of Planck's postulate eliminated the orbit stability problem.

Let us consider another interesting property of nolinearity manifestation. If we assume that there is a driving force of the type

$$F = \sum_{j=1}^{n} \delta_j \exp(-in\Omega_j t),$$

we can say that the terms activated will be only the corresponding terms relative to the resonances Ω_j and their linear combinations. In fact the equation

$$\ddot{x} + \omega^2 x + \lambda x^2 = \delta_1 \cos(\Omega_1 t) + \delta_2 \cos(\Omega_2 t)$$

admits the stationary solution

$$x = -\frac{\lambda}{2\omega^2}(A_1^2 + A_2^2) + A_1 \cos(\Omega_1 t) + A_2 \cos(\Omega_2 t) + \frac{\lambda A_1^2 \cos(2\Omega_1 t)}{2(4\Omega_1^2 - \omega^2)}$$
$$+ \frac{\lambda A_2^2 \cos(2\Omega_2 t)}{2(4\Omega_2^2 - \omega^2)} + \frac{\lambda A_1 A_2 \cos(\Omega_1 - \Omega_2)\, t}{((\Omega_1 - \Omega_2)^2 - \omega^2)} + \frac{\lambda A_1 A_2 \cos(\Omega_1 + \Omega_2)\, t}{((\Omega_1 + \Omega_2)^2 - \omega^2)}$$

with

$$A_1 = \frac{\delta_1}{\omega^2 - \Omega_1^2}, \quad A_2 = \frac{\delta_2}{\omega^2 - \Omega_2^2}.$$

It is easy to see that, besides the occurrence of resonances at $\omega = \Omega_1, \omega = \Omega_2, \omega = 2\Omega_1$, and $\omega = 2\Omega_2$, we will also have resonances at $\omega = \Omega_1 + \Omega_2$ and $\omega = \Omega_1 - \Omega_2$. This phenomenon has a resemblance with the addition and subtraction principle discovered by W. Ritz in the hydrogen atom spectrum.

7. Stability and Instability

Let us consider, for the sake of simplicity, the following equation[3]

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x + 3\omega^2\frac{\varepsilon}{p}x^2 = \delta\cos(\Omega t + \varphi),$$

and let us assume that there exists a solution of the type

$$x(t) = A_1\cos(\Omega t) + A_2\cos(2\Omega t).$$

Substituting this solution in the first equation, collecting the terms containing sines and cosines, and requiring that they must be equal to zero, we find the following identities:

$$\frac{1}{2}\frac{-2A_1\Omega^2 p + 6\omega^2\varepsilon A_1 A_2 + 2\omega^2 A_1 p}{p} = \delta\cos\varphi,$$

[3]It is necessary to note that this equation contains only terms of the second order in comparison with equation (5). Neglecting the terms of the third order can essentially influence the inclination of the amplitude-frequency characteristics. On the other hand we focused our attention on studying the equation similar to the Heisenberg one for showing its basic regularities.

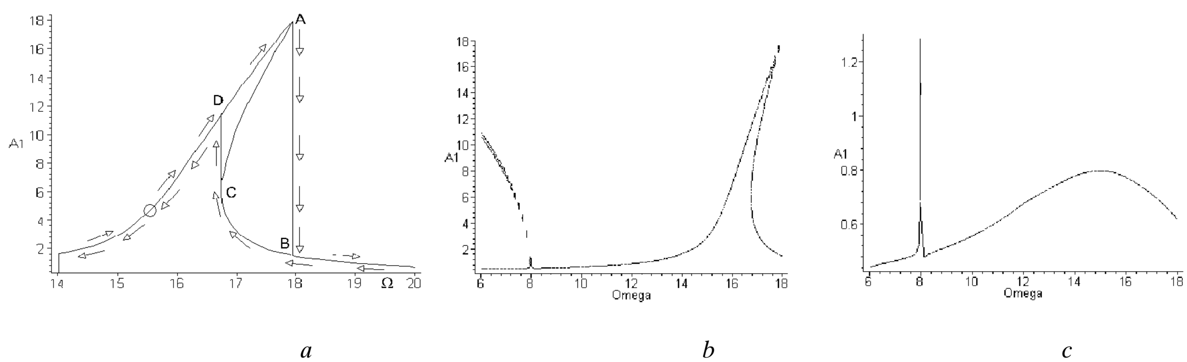


Fig. 3. Amplitude-frequency characteristics.

$$A_2 = \frac{3}{2}\frac{\omega^2 \varepsilon A_1^2}{p(4\Omega^2 - \omega^2)}, \quad \gamma A_1 \Omega = \delta \sin(\varphi).$$

Substituting the second equation into the first, squaring and summing the remaining equations, we find

$$\left[(\omega^2 - \Omega^2) + \frac{9}{2}\frac{\omega^4 \varepsilon^2 A_1^2}{p^2\,(4\Omega^2 - \omega^2)}\right]^2 + (\gamma\Omega)^2 = \left(\frac{\delta}{A_1}\right)^2. \tag{23}$$

We have studied this implicit function for the following values:

$$\omega = 16;\ \varepsilon = 0.8;\ p = 30;\ \gamma = 8;\ \delta = 100,$$

and have obtained a graph shown in Fig. 3*a*.

Let us assume that we are able to increase gradually the pulsation Ω starting from the value 14. The corresponding ordinate is forced to trace the ascending branch of the curve. Let us consider a small sphere that moves along this branch because of a hypothetical gravitational force. We can imagine that, after having moved it along the mentioned branch, the small sphere is let free just when it is very near to the abscissa $\Omega = 18$. Then the sphere, being forced to describe the curve because of the above mentioned gravitational force, will fall again towards the potential well that corresponds to the resonance $\Omega = 8$, as can be seen clearly in Fig. 3*b*.

On the contrary, if our small sphere is let free just at the highest point of the curve A, then it can both return towards the resonance $\Omega = 8$ or fall *vertically* onto the lower branch of the curve at the right side of the previously mentioned highest point. This vertical fall A-B is due to the fact that our external action forces a pulsation increase while the sphere, in order to describe the broken part of the curve, should cause on the contrary a decrease of the pulsation.

Conversely, starting from a pulsation equal to 20 and imposing a gradual decrease of its value, the particle will describe the above-mentioned curve until it reaches the pulsation $\Omega = 16.4$ shown in Fig. 3*b*, and subsequently, for the same reason as described above, will be forced to jump up to the higher branch.

It is evident that the particle, when the external variation of the pulsation disappears abruptly, will remain in a vicinity of another resonance condition, giving back, under the form of radiation, the energy gap between the two stationary states. Returning back to Fig. 3*a*, we can say that, starting from the pulsation $\Omega = 14$, the small sphere describes the upper curve following the arrows, until it reaches A. In this point it should return back, but the pulsation increase forbids this, so it falls vertically into B, and then continues its movement on the right side of the lower curve. On the other hand, going on with a continuous decrease of the pulsation, starting from $\Omega = 20$, the small sphere follows the path marked by the arrows below the curve, and thus, once reached the point C, it leaves the curve and moves vertically until it reaches the upper curve in the point D, continuing then towards the left side. These forecasts, besides describing the shift of the lines due to the radiation pressure, also show that in the vicinity of the resonance frequencies there are abrupt amplitude variations due to said instability phenomena.

Our curves are extremely sensitive to pressure variations. A pressure increase transforms the graph shown in Fig. 3*b* into the graph shown in Fig. 3*c*.

In this figure we see the line corresponding to the resonance $\Omega = 8$ that should reach an infinite value. Of course this value is due to the fact that we have studied a solution by truncating the series in an excessive way.

What we have said above, besides giving a justification to the shifts of spectrum lines, even implies that the position occupied by the particle in a point of the space should be given by a function indissolubly related to the value that the radiation pressure assumes in that point. Of course this is a microscopic description of the reality that can be adapted to the hydrogen atom with a procedure totally similar to the procedure used above. Therefore, in principle, the fact that the atom has just a specific size should be related to the value that the radiation pressure has at that point.

It could seem that there is a conflict between the former solution represented by relation (20), that we will denote explicit solution, and the present implicit (and more precise) solution. However, it is easy to see that, while (20) is an explicit solution of the differential equation that makes it possible to determine various coefficients of the series, the solution adopted in the present case does not have this property. The conflict disappears when we observe that (23) makes it possible to write explicitly the term A_1 only if we set $\varepsilon^2 = 0$. In such a case we get

$$A_1 = \frac{\delta}{\sqrt{(\omega^2 - \Omega^2) + \gamma^2\Omega^2}},$$

and this expression is the same as that given by the explicit solution (20).

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