Nonlinear Oscillations, Vol. 4, No. 3, 2001

THE CANONICAL REDUCTION METHOD FOR SYMPLECTIC STRUCTURES AND ITS APPLICATIONS

A. K. Prykarpats'ky

Institute of Applied Problems of Mechanics and Mathematics, NAS of Ukraine, Naukova St., 3b, L'viv, Ukraine; and AGH, Krakow, 30059, Poland
e-mail: prykanat@cybergal.com
prykanat@start.com.au

V. Hr. Samoylenko

Kyiv National Taras Shevchenko University
Volodymyr'ska St., 64, Kyiv, 01033, Ukraine
e-mail: vsam@imath.kiev.ua
vsam@mail.univ.kiev.ua

The canonical reduction method is analized in detail and applied to Maxwell and Yang–Mills equations considered as Hamiltonian systems on some fiber bundles with symplectic and connection structures. The minimum interaction principle is proved to have geometric origin within the reduction method devised.

AMS Subject Classification: 35 Q58, 58 F07

1. Preliminaries

We begin by reviewing the backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. The material is partly available in [1 – 3], so here we only sketch, but in the notation suitable for us, the necessary definitions and statements.

Let G denote a given Lie group with the unity element $e \in G$ and the corresponding Lie algebra $\mathcal{G} \simeq T_e(G)$. Consider a principal fiber bundle $p : (M, \varphi) \to N$ with the structure group G and a base manifold N on which the Lie group G acts by means of a mapping $\varphi : M \times G \to M$. Namely, for each $g \in G$ there is a group diffeomorphism $\varphi_g : M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u} : G \to M$, where $\hat{u}(g) = \varphi_g(u)$.

On the principal fiber bundle $p : (M, \varphi) \to N$ there is assigned a connection $\Gamma(\mathcal{A})$ by means of such a morphism $\mathcal{A} : (T(M), \varphi_{g*}) \to (\mathcal{G}, Ad_{g^{-1}})$ that for each $u \in M$, the mapping $\mathcal{A}(u) : T_u(M) \to \mathcal{G}$ is a left inverse of the mapping $\hat{u}_*(e) : \mathcal{G} \to T_u(M)$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1. \tag{1}$$

As usual, denote by $\varphi_g^* : T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g : M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G-invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$, given by formula

$$\omega^{(2)} := d\,\mathrm{pr}^*\alpha^{(1)},$$

generates the corresponding momentum mapping $l : T^*(M) \to \mathcal{G}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}_*(e)\alpha^{(1)}(u) \tag{2}$$

for all $u \in M$.

Remark here that the principal fiber bundle structure $p : (M, \varphi) \to N$ means, in part, the exactness of the following sequences of mappings:

$$0 \to \mathcal{G} \stackrel{\hat{u}_*(e)}{\to} T_u(M) \stackrel{p_*(u)}{\to} T_{p(u)}(N) = 0,$$

that is,

$$p_*(u)\hat{u}_*(e) = 0 = \hat{u}^*(e)p_*(u) \tag{3}$$

for all $u \in M$.

Combining (3) with (1) and (2), one obtains the following embedding for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$:

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in \operatorname{range} p^*(u). \tag{4}$$

The expression (4) means that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0$$

for all $u \in M$.

Taking now into account that the mapping $p^*(u) : T^*(N) \to T^*(M)$ for each $u \in M$ is injective, we conclude that it has a unique inverse mapping $(p^*(u))^{-1}$ on its image $p^*(u)T^*_{p(u)}(N) \subset T^*_u(M)$. Thereby, for each $u \in M$ one can define a morphism $p_{\mathcal{A}} : (T^*(M), \varphi^*_g) \to T^*(N)$ by

$$p_{\mathcal{A}}(u) : \alpha^{(1)}(u) \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{5}$$

Based on definition (5), one can easily check that the diagram

$$\begin{array}{ccc} T^*(M) & \stackrel{p_{\mathcal{A}}}{\to} & T^*(N) \\ \mathrm{pr}_M\downarrow & & \downarrow\mathrm{pr}_N \\ M & \stackrel{p}{\to} & N \end{array}$$

is commutative.

Let an element $\xi \in \mathcal{G}^*$ be G-invariant, that is, $Ad^*_{g^{-1}}\xi = \xi$ for all $g \in G$. Denote also by $p^{\xi}_{\mathcal{A}}$ the restriction of mapping (5) to the subset $l^{-1}(\xi) \in T^*(M)$, that is, $p^{\xi}_{\mathcal{A}} : l^{-1}(\xi) \to T^*(N)$, where, for all $u \in M$, we have

$$p^{\xi}_{\mathcal{A}}(u) : l^{-1}(\xi) \to (p^*(u))^{-1} [1 - \mathcal{A}^*(u)\hat{u}^*(e)]l^{-1}(\xi).$$

Now one can characterize the structure of the reduced phase space $l^{-1}(\xi)/G$ by means of the following lemma.

Lemma 1. *The mapping $p^{\xi}_{\mathcal{A}}(u) \,:\, l^{-1}(\xi) \;\to\; T^{*}(N)$ is a principal fiber G-bundle with the reduced space $l^{-1}(\xi)/G$ being diffeomorphic to $T^{*}(N)$.*

Denote by $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ the standard Ad-invariant nondegenerate scalar product on $\mathcal{G}^{*}\times\mathcal{G}$. Based on Lemma 1, one deduces the following characterization theorem.

Theorem 1. *Given a principal fiber G-bundle with a connection $\Gamma(\mathcal{A})$ and a G-invariant element $\xi\in\mathcal{G}^{*}$, then every such a connection $\Gamma(A)$ defines a symplectomorphism $\nu_{\xi} : l^{-1}(\xi)/G \to T^{*}(N)$ between the reduced phase space $l^{-1}(\xi)/G$ and the cotangent bundle $T^{*}(N)$, where $l : T^{*}(M) \;\to\; \mathcal{G}^{*}$ is the naturally associated momentum mapping for the group G-action on M.*

Moreover, the following equality:

$$(p^{\xi}_{\mathcal{A}})(d\,\mathrm{pr}^{*}\beta^{(1)}+\mathrm{pr}^{*}\,\Omega^{(2)}_{\xi}) \;=\; d\,\mathrm{pr}^{*}\alpha^{(1)}\Big|_{l^{-1}(\xi)} \tag{6}$$

holds for the canonical 1-forms $\beta^{(1)}\in\Lambda^{1}(N)$ and $\alpha^{(1)}\in\Lambda^{1}(M)$, where $\Omega^{(2)}_{\xi} := \langle\Omega^{(2)},\xi\rangle_{\mathcal{G}}$ is the ξ-component of the corresponding curvature form $\Omega^{(2)}\in\Lambda^{(2)}(N)\otimes\mathcal{G}$.

Proof. On $l^{-1}(\xi)\subset M$ due to (5), we have

$$p^{*}(u)p^{\xi}_{\mathcal{A}}(\alpha^{(1)}(u)) \,=\, p^{*}(u)\beta^{(1)}(\mathrm{pr}_{N}(u)) \,=\, \alpha^{(1)}(u)-\mathcal{A}^{*}(u)\hat{u}^{*}(e)\alpha^{(1)}(u)$$

for any $\beta^{(1)}\in T^{*}(N)$ and all $u\in M_{\xi} := p_{M}l^{-1}(\xi)\subset M$. Thus we easily get that

$$\alpha^{(1)}(u) \,=\, (p^{\xi}_{\mathcal{A}})^{-1}\beta^{(1)}(p_{N}(u)) \,=\, p^{*}(u)\beta^{(1)}\,(\mathrm{pr}_{N}(u))+\langle\mathcal{A}(u),\xi\rangle$$

for all $u\in M_{\xi}$. Recall now that in virtue of (6), one gets on M_{ξ}

$$p\,\mathrm{pr}_{M_{\xi}} \,=\, \mathrm{pr}_{N}\,p^{\xi}_{\mathcal{A}},\ \mathrm{pr}^{*}_{M_{\xi}}\,p^{*} \,=\, (p^{\xi}_{\mathcal{A}})^{*}\,\mathrm{pr}^{*}_{N}\,.$$

Therefore we can write now that

$$\begin{aligned}\mathrm{pr}^{*}_{M_{\xi}}\alpha^{(1)}(u) \,&=\, \mathrm{pr}^{*}_{M_{\xi}}\beta^{(1)}(p_{N}(u))+\mathrm{pr}^{*}_{M_{\xi}}\langle\mathcal{A}(u),\xi\rangle\\ &=\, (p^{\xi}_{\mathcal{A}})^{*}(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+\mathrm{pr}^{*}_{M_{\xi}}\langle\mathcal{A}(u),\xi\rangle.\end{aligned}$$

Whence taking the external differential, one arrives at the following equality:

$$\begin{aligned}d\,\mathrm{pr}^{*}_{M_{\xi}}\alpha^{(1)}(u) \,&=\, (p^{\xi}_{\mathcal{A}})^{*}d(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+\mathrm{pr}^{*}_{M_{\xi}}\langle d\,\mathcal{A}(u),\xi\rangle\\ &=\, (p^{\xi}_{\mathcal{A}})^{*}d(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+\mathrm{pr}^{*}_{M_{\xi}}\langle\Omega(p(u)),\xi\rangle\\ &=\, (p^{\xi}_{\mathcal{A}})^{*}d(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+\mathrm{pr}^{*}_{M_{\xi}}p^{*}\langle\Omega,\xi\rangle(u)\\ &=\, (p^{\xi}_{\mathcal{A}})^{*}d(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+(p^{\xi}_{\mathcal{A}})^{*}\mathrm{pr}^{*}_{N}\langle\Omega,\xi\rangle(u)\\ &=\, (p^{\xi}_{\mathcal{A}})^{*}[d(\mathrm{pr}^{*}_{N}\beta^{(1)})(u)+\mathrm{pr}^{*}_{N}\langle\Omega,\xi\rangle(u)].\end{aligned}$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form $\Omega \in \Lambda^2(M) \otimes \mathcal{G}$:

$$\begin{aligned}\langle d\mathcal{A}(u), \xi\rangle &= \langle d\mathcal{A}(u) + \mathcal{A}(u) \wedge \mathcal{A}(u), \xi\rangle - \langle \mathcal{A}(u) \wedge \mathcal{A}(u), \xi\rangle \\ &= \langle \Omega(p_N(u)), \xi\rangle = \langle p_N^* \Omega, \xi\rangle(u) \end{aligned} \tag{7}$$

at any $u \in M_\xi$, since for any $A, B \in \mathcal{G}$, $\langle [A, B], \xi\rangle = \langle B, (Ad\ A)^* \xi\rangle = 0$ in virtue of the invariance condition $Ad_G \xi = \xi$. Thereby the proof is finished.

Remark. Since the canonical 2-form $d\,\mathrm{pr}^* \alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is G-invariant on $T^*(M)$, due to the construction, it is evident that its restriction to the G-invariant submanifold $l^{-1}(\xi) \subset T^*(M)$ will be effectively defined only on the reduced space $l^{-1}(\xi)/G$, which ensures the validity of the equality sign in (6).

As a consequence of Theorem 1 one can formulate the following results useful enough for applications.

Theorem 2. *Let an element $\xi \in \mathcal{G}^*$ have the isotropy group G_ξ acting on the subset $l^{-1}(\xi) \subset T^*(M)$ freely and properly, so that the reduced phase space $(l^{-1}(\xi)/G_\xi\ , \sigma_\xi^{(2)})$ is symplectic, where by the definition,*

$$\sigma_\xi^{(2)} := d\,\mathrm{pr}^* \alpha^{(1)}\Big|_{l^{-1}(\xi)} .$$

If the principal fiber bundle $p : (M, \varphi) \to N$ has the structure group coinciding with G_ξ, then the reduced symplectic space $(l^{-1}(\xi)/G_\xi, \sigma_\xi^{(2)})$ is symplectomorphic to the cotangent symplectic space $(T^*(N), \omega_\xi^{(2)})$, where

$$\omega_\xi^{(2)} = d\,\mathrm{pr}^* \beta^{(1)} + \mathrm{pr}^* \Omega_\xi^{(2)},$$

and the corresponding symplectomorphism is given by a relation similar to (6).

Theorem 3. *In order that two symplectic spaces $(l^{-1}(\xi)/G, \sigma_\xi^{(2)})$ and $(T^*(N), d\,\mathrm{pr}^* \beta^{(1)})$ were symplectomorphic, it is necessary and sufficient that the element $\xi \in \ker h$, where for a G-invariant element $\xi \in \mathcal{G}^*$, the mapping $h : \xi \to [\Omega_\xi^{(2)}] \in H^2(N; \mathbf{Z})$, with $H^2(N; \mathbf{Z})$ being the cohomology class of 2-forms on the manifold N.*

In case where a Lie group G is given the tangent space $T(G)$ is also Lie group isomorphic to the semidirect product $\tilde{G} := G \otimes_{Ad} \mathcal{G}$ of the Lie group G and its Lie algebra $\mathcal{G}$ under the adjoint Ad-action of G on $\mathcal{G}$.

The Lie algebra $\widetilde{\mathcal{G}}$ of $\tilde{G}$ is, correspondingly, the semidirect product of $\mathcal{G}$ with itself, regarded as a trivial Abelian Lie algebra, under the adjoint ad-action and thus has the bracket defined by

$$[(a_1, m_1), (a_2, m_2)] := ([a_1, a_2], [a_1, m_2] + [a_2, m_1])$$

for all $(a_j, m_j) \in \mathcal{G} \otimes_{ad} \mathcal{G}$, $j = \overline{1, 2}$.

Take now any element $\xi \in \mathcal{G}^*$ and compute its isotropy group G_ξ under the coadjoint action Ad^* of G on $\mathcal{G}^*$, and denote by $\mathcal{G}_\xi$ its Lie algebra.

The cotangent bundle $T^*(G)$ is obviously diffeomorphic to $M := G \times \mathcal{G}^*$ on which the Lie group G_ξ acts freely and properly (due to the construction) by left translation on the first factor and Ad^*-action on the second one.

The corresponding momentum mapping $l : G \times \mathcal{G}^* \to \mathcal{G}_\xi^*$ is obtained as

$$l(h, \alpha) = Ad^*_{h^{-1}} \alpha \big|_{\mathcal{G}_\xi^*}$$

with no critical points.

Let now $\eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$. Therefore, the reduced space $(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \sigma_\xi^{(2)})$ has to be symplectic due to the well known Marsden – Weinstein reduction theorem [4, 5], where $G_\xi^{\eta(\xi)}$ is the isotropy subgroup of the G_ξ-coadjoint action on $\eta(\xi) \in \mathcal{G}_\xi^*$ and the symplectic form $\sigma_\xi^{(2)} := d\,\mathrm{p}r^* \alpha^{(1)}\big|_{l^{-1}(\eta(\xi))}$ is naturally induced from the canonical symplectic structure on $T^*(G)$.

Define now for $\eta(\xi) \in \mathcal{G}_\xi^*$ the one-form $\alpha^{(1)}_{\eta(\xi)} \in \Lambda^1(G)$ by

$$\alpha^{(1)}_{\eta(\xi)}(h) := R_h^* \eta(\xi), \tag{8}$$

where $R_h : G \to G$ is the right translation by an element $h \in G$.

It is easy to check that element (8) is right G-invariant and left $G_\xi^{\eta(\xi)}$-invariant, thus inducing a one-form on the quotient $N_\xi := G/G_\xi^{\eta(\xi)}$.

Denote by $\mathrm{pr}^* \alpha^{(1)}_{\eta(\xi)}$ its pull-back to $T^*(N_\xi)$ and form the symplectic manifold $(T^*(N_\xi), d\,\mathrm{pr}^* \beta^{(1)} + d\,\mathrm{pr}^* \alpha^{(1)}_{\eta(\xi)})$, where $d\,\mathrm{pr}^* \beta^{(1)} \in \Lambda^{(2)}(T^*(N_\xi))$ is the canonical symplectic form on $T^*(N_\xi)$.

The construction above can now be summarized as the following theorem.

Theorem 4. *Let $\xi, \eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$ be fixed. Then the reduced symplectic manifold $(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \sigma_\xi^{(2)})$ is a symplectic covering of the coadjoint orbit $Or(\xi, \eta(\xi); \tilde{G})$ and symplectically embeds onto a subbundle over $G/G_\xi^{\eta(\xi)}$ of $(T^*(G/G_\xi^{\eta(\xi)}), \omega_\xi^{(2)})$, with*

$$\omega_\xi^{(2)} := d\,\mathrm{pr}^* \beta^{(1)} + d\,\mathrm{pr}^* \alpha^{(1)}_{\eta(\xi)} \in \Lambda^2(T^*(G/G_\xi^{\eta(\xi)}).$$

The statement above fits into the conditions of Theorem 2 if one defines a connection 1-form $\mathcal{A}(g) : T_g(G) \to \mathcal{G}_\xi$ as follows:

$$\langle \mathcal{A}(g), \xi \rangle_{\mathcal{G}} := R_g^* \eta(\xi) \tag{9}$$

for any $g \in G$.

Expression (9) generates a completely horizontal 2-form $d\langle \mathcal{A}(g), \xi \rangle_{\mathcal{G}}$ on the Lie group G, which immediately gives rise to the symplectic structure $\omega_\xi^{(2)}$ on the reduced phase space $T^*(G/G_\xi^{\eta(\xi)})$.

2. The Maxwell Electromagnetic Equations

Under the Maxwell electromagnetic equations we understand the following relationships on the cotangent phase space $T^*(N)$ with $N \subset T(D; \mathbf{R}^3)$ being a manifold of vector fields on some almost everywhere smooth enough domain $D \subset \mathbf{R}^3$:

$$\partial E/\partial t = \operatorname{rot} B, \;\; \partial B/\partial t = -\operatorname{rot} E,$$
$$\operatorname{div} E = \rho, \quad \operatorname{div} B = 0, \tag{10}$$

where $(E, B) \in T^*(N)$ is a vector of electric and magnetic fields and $\rho \in C(D; \mathbf{R})$ is some fixed density function for a smeared out ambient charge.

Aiming to repesent equations (10) as those on reduced symplectic space, define as in [6] an appropriate configuration space, $M \subset \mathcal{T}(D; \mathbf{R}^3)$, with a vector potential field coordinate $A \in M$.

The cotangent space $T^*(M)$ may be identified with pairs $(A, Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D; \mathbf{R}^3)$ is a vector field density in D.

On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M)$, where $\omega^{(2)} := d\operatorname{pr}^*\alpha^{(1)}$, and

$$\alpha^{(1)}(A, Y) = \int_D d^3x \langle Y, dA\rangle := (Y, dA), \tag{11}$$

where by $\langle\cdot,\cdot\rangle$ we denote the standard scalar product in $\mathbf{R}^3$ endowed with the measure d^3x, and by $\operatorname{pr} : T^*(M) \to M$ we denote the usual basepoint projection upon the base space M.

Define now a Hamiltonian function $H \in \mathcal{D}(T^*(M))$ by

$$H(A, Y) = \frac{1}{2}((Y, Y) + (\operatorname{rot} A, \operatorname{rot} A)), \tag{12}$$

which is evidently invariant with respect to the following symmetry group G acting on the base manifold M and lifted to $T^*(M)$: for any $\psi \in \mathcal{G} \subset C^{(1)}(D; \mathbf{R})$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \quad \Phi_\psi(Y) = Y. \tag{13}$$

Under the transformation (13) the 1-form (11) is evidently invariant too, since

$$\varphi_\psi^* \alpha^{(1)}(A, Y) = (Y, dA + \nabla d\psi) = (Y, dA) - (\operatorname{div} Y, d\psi) = \alpha^{(1)}(A, Y),$$

where we made use of the condition that $d\psi \simeq 0$ in $\Lambda^1(M)$.

Thus the corresponding momentum mapping (2) is given as $l(A, Y) = -\operatorname{div} Y$ for all $(A, Y) \in T^*(M)$.

If $\rho \in \mathcal{G}^*$, one can define the reduced space $l^{-1}(\rho)/G$, since evidently the isotropy group $G_\rho = G$ due to its commutativity.

Consider now a principal fiber bundle $p : M \to N$ with the Abelian structure group G and a base manifold N taken to be $N := \{B \in \mathcal{T}(D; \mathbf{R}^3) : \operatorname{div} B = 0\}$, where, by definition, $p(A) := B = \operatorname{rot} A$.

Over this bundle, one can build a connection 1-form $\mathcal{A} : T(M) \to \mathcal{G}$, where for all $A \in M$,

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \;\; d\langle \mathcal{A}(A), \rho \rangle_{\mathcal{G}} = \Omega_\rho^{(2)}(B)$$

in virtue of commutativity of the Lie algebra $\mathcal{G}$.

Then, due to Theorem 2, the cotangent manifold $T^*(N)$ is symplectomorphic to the reduced phase space

$$l^{-1}(\rho)/G \cong \{(B,S) \in T^*(N) : \operatorname{div} E(S) = \rho, \;\; \operatorname{div} B = 0\}$$

with the canonical symplectic 2-form

$$\omega_\rho^{(2)}(B,S) = (dS, \wedge dB) + d\,\langle \mathcal{A}(A), \rho \rangle_{\mathcal{G}}, \tag{14}$$

where we put $\operatorname{rot} S - F = -E$ and, by definition, $\operatorname{div} F = \rho$ for some mapping $F \in C^1(D; \mathbf{R}^3)$.

The Hamiltonian (12) is correspondingly reduced to the following classical form:

$$H(B,E) = \frac{1}{2}((B,B) + (E,E)). \tag{15}$$

As a result, the Maxwell equations (9) become a Hamiltonian system on the reduced phase space $T^*(N)$ endowed with the quasicanonical symplectic structure (14) and the new Hamiltonian function (15).

It is well known that Maxwell equations (10) admit a one more canonical symplectic structure on $T^*(N)$, namely,

$$\bar{\omega}^{(2)} := (dB, \wedge dE), \tag{16}$$

with respect to which they are Hamiltonian too and whose "helicitylike" conservative Hamiltonian function is

$$\bar{H}(B,E) = \frac{1}{2}((\operatorname{rot} E, E) + (\operatorname{rot} B, B)), \tag{17}$$

where $(B,E) \in T^*(N)$.

It easy to see that (17) is also an invariant function with respect to the Maxwell equations (10). Subject to the Maxwell equations (10), a group theoretical interpretation of the symplectic structure (16) is still waiting for search.

Notice finally that both symplectic structure (16) and Hamiltonian (17) are invariant with respect to the following Abelian group $G^2 = G \times G$-action:

$$G^2 \ni (\psi, \chi) : (B,E) \to (B + \nabla\psi, E + \nabla\chi) \tag{18}$$

for all $(B,E) \in T^*(N)$.

Corresponding to (18), the momentum mapping $l : T^*(N) \to \mathcal{G}^* \times \mathcal{G}^*$ is calculated as

$$l(B, E) = (\operatorname{div} E, -\operatorname{div} B) \tag{19}$$

for any $(B, E) \in T^*(N)$.

Fixing a value in (19) as $l(B, E) = \xi := (\rho, 0)$, that is,

$$\operatorname{div} E = \rho, \;\; \operatorname{div} B = 0, \tag{20}$$

one obtains the reduced phase space $l^{-1}(\xi)/G^2$, since the isotropy subgroup G^2_ξ of the element $\xi \in \mathcal{G}^* \times \mathcal{G}^*$ coincides with entire group G^2.

Thus the reduced phase space, due to Theorem 2, is endowed with the canonical symplectic structure

$$\bar{\omega}^{(2)}(A, Y) = (dY, \wedge dA) + d\,\langle \mathcal{A}(A), \xi \rangle_{\mathcal{G}},$$

where $T^*(M) \ni (A, Y)$ are variables constituting the corresponding coordinates on the cotangent space over an associated fibre bundle $\bar{p} : N \to M$ with a curvature 1-form $\mathcal{A} : T(N) \to \mathcal{G} \times \mathcal{G}$.

In virtue of (20) one can define the projection map $\bar{p} : N \to M$ as follows: $\bar{p}(B) := \operatorname{rot}^{-1} B = A$ for any $A \in M \subset \mathcal{T}(D; \mathbf{R}^3)$.

It is evident that the second condition of (20) is satisfied automatically on the cotangent bundle $T^*(M)$.

Subject to the coadjoint variables $Y \in T^*_A(M)$ and $E \in T^*_B(N)$ for all $A \in M$ and $E \in N$, one can easily obtain from the equality $\bar{p}^* \beta^{(1)} = \alpha^{(1)}$ the expression

$$Y = -\operatorname{rot} E$$

satisfying the evident condition $\operatorname{div} Y = 0$. The Hamiltonians (15) and (17) take, correspondingly, on $T^*(M)$ the forms

$$\bar{\mathcal{H}}(A, Y) = \frac{1}{2} \left((\operatorname{rot}^3 A, A) + (\operatorname{rot}^{-1} Y, Y) \right)$$

and

$$\mathcal{H}(A, Y) = \frac{1}{2} \left((\operatorname{rot}^{-1} Y, \operatorname{rot}^{-1} Y) + (\operatorname{rot} A, \operatorname{rot} A) \right),$$

being obviously invariant too with respect to common evolutions on $T^*(M)$.

As was mentioned in [1], the invariant like (17) admits the following geometrical interpretation: its quantity is a helicity structure related with dynamical equations, that is a number of closed linkages of the vortex lines present in the ambient phase space.

If one to consider now a motion of a charged particle under a Maxwell field, it is convenient to introduce another fiber bundle structure $p : M \to N$, namely, the one such that $M = N \times G$, $N := D \subset \mathbf{R}^3$, and $G := \mathbf{R} \backslash \{0\}$ being the corresponding (Abelian) structure Lie group.

An analysis similar to the above gives rise to a symplectic structure reduced to the space $l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathcal{G}$,

$$\omega^{(2)}(q, p) = \langle dp, \wedge dq \rangle + d\langle \mathcal{A}(q, g), \xi \rangle_{\mathcal{G}},$$

where

$$\mathcal{A}(q,g) := \langle A(q), dq\rangle + g^{-1}dg$$

is a usual connection 1-form on M with $(q,p) \in T^*(N)$ and $g \in G$.

The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i, q^j\} = 0, \quad \{p_j, q^i\} = \delta^i_j, \quad \{p_i, p_j\} = F_{ji}(q)$$

for all $(q,p) \in T^*(N)$.

If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q,p)$, it is easy to verify that $\omega^{(2)}_\xi \to \tilde{\omega}^{(2)}_\xi := \langle d\tilde{p}, \wedge dq\rangle$, giving rise to the following Poisson brackets [7, 8]:

$$\{q^i, q^j\} = 0, \quad \{\tilde{p}_j, q^i\} = \delta^i_j, \quad \{\tilde{p}_i, \tilde{p}_j\} = 0,$$

where $i,j = \overline{1,3}$, iff for all $i,j,k = \overline{1,3}$ the standard Maxwell field equations are satisfied on N:

$$\frac{\partial F_{ij}}{\partial q_k} + \frac{\partial F_{jk}}{\partial q_i} + \frac{\partial F_{ki}}{\partial q_j} = 0$$

with the curvature tensor

$$F_{ij}(q) := \frac{\partial A_j}{\partial q^i} - \frac{\partial A_i}{\partial q^j},$$

where $i,j = \overline{1,3}$, $q \in N$.

Such a construction permits a natural generalization to the case of non-Abelian structure Lie group yielding a description of Yang – Mills field equations within the reduction approach.

3. A Charged Particle Phase Space Structure and Yang – Mills Field Equations

As before, we start by defining a phase space M of a particle under a Yang – Mills field in a region $D \subset \mathbf{R}^3$ as $M := D \times G$, where G is a Lie group (not in general semisimple) acting on M from the right.

Over the space M one can define quite naturally a connection $\Gamma(\mathcal{A})$ if to consider the following trivial principal fiber bundle $p : M \to N$, where $N := D$, with the structure group G. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q,g)$ can be written [1, 3, 9] as

$$\mathcal{A}(q;g) := g^{-1}\left(d + \sum_{i=1}^{n} a_i A^{(i)}(q)\right) g, \tag{21}$$

where $\{a_i \in \mathcal{G} : i = \overline{1,n}\}$ is a basis of the Lie algebra $\mathcal{G}$ of the Lie group G, and $A_i : D \to \Lambda^1(D)$, $i = \overline{1,n}$, are the Yang – Mills fields in the physical space $D \subset \mathbf{R}^3$.

Now one defines the natural left invariant Liouville form on M by

$$\alpha^{(1)}(q;g) := \langle p, dq\rangle + \langle y, g^{-1}dg\rangle_{\mathcal{G}}, \tag{22}$$

where $y \in T^*(G)$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ denotes as before the usual Ad-invariant nondegenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, as evidently $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$.

The main assumption we need to make for the sequel is that the connection 1-form is in accordance with the action of the Lie group G on M. The latter means that the condition

$$R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g)$$

is satisfied for all $(q, g) \in M$ and $h \in G$, where $R_h : G \to G$ means the right translation by an element $h \in G$ on the Lie group G.

Having stated all preliminary conditions needed for the reduction Theorem 2 to be applied to our model, suppose that the Lie group G canonical action on M is naturally lifted to that on the cotangent space $T^*(M)$ endowed, due to (22), with the following G-invariant canonical symplectic structure:

$$\begin{aligned} \omega^{(2)}(q, p; g, y) := d\,\mathrm{pr}^* \alpha^{(1)}(q, p; g, y) &= \langle dp, \wedge dq \rangle \\ &+ \langle dy, \wedge g^{-1} dg \rangle_{\mathcal{G}} + \langle y dg^{-1}, \wedge dg \rangle_{\mathcal{G}} \end{aligned} \tag{23}$$

for all $(q, p; g, y) \in T^*(M)$.

Take now an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is, $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ can only be the trivial one, $\xi = 0$ as it happens in the case of the Lie group $G = SL_2(\mathbf{R})$.

Then one can construct the reduced phase space $l^{-1}(\xi)/G$ symplectomorphic to $(T^*(N), \omega_\xi^{(2)})$, where due to (7) for any $(q, p) \in T^*(N)$ we have

$$\begin{aligned} \omega_\xi^{(2)}(q, p) &= \langle dp, \wedge dq \rangle + \langle \Omega^{(2)}(q), \xi \rangle_{\mathcal{G}} \\ &= \langle dp, \wedge dq \rangle + \sum_{s=1}^{n} \sum_{i,j=1}^{3} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j. \end{aligned} \tag{24}$$

In the above we have expanded the element

$$\mathcal{G}^* \ni \xi = \sum_{i=1}^{n} e_i a^i$$

with respect to the biorthogonal basis $\{a^i \in \mathcal{G}^* : \langle a^i, a_j \rangle_{\mathcal{G}} = \delta_j^i, i, j = \overline{1, n}\}$ with $e_i \in \mathbf{R}$, $i = \overline{1, 3}$, being some constants and we denoted by $F_{ij}^{(s)}(q)$, $i, j = \overline{1, 3}$, $s = \overline{1, n}$, the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ components, that is,

$$\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{i,j=1}^{3} a_s F_{ij}^{(s)}(q) dq^i \wedge dq^j$$

for any point $q \in N$. Summarizing the above calculations we can formulate the following result.

Theorem 5. *Suppose a Yang – Mills field (21) on the fiber bundle $p : M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G-action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad_G^* \xi = \xi$. Then for the naturally constructed momentum mapping $l : T^*(M) \to G^*$ (being equivariant) the reduced phase space $l^{-1}(\xi)/G \simeq T^*(N)$ is endowed with the canonical symplectic structure (24) having the following component-wise Poissoin brackets form:*

$$\{p_i, q^j\}_\xi = \delta_i^j, \ \ \{q^i, q^j\}_\xi = 0, \ \ \{p_i, p_j\}_\xi = \sum_{s=1}^n e_s F_{ji}^{(s)}(q)$$

for all $i, j = \overline{1,3}$ and $(q, p) \in T^*(N)$.

The correspondingly extended Poisson bracket on the whole cotangent space $T^*(M)$ amounts, due to (23), to the following set of Poisson relations:

$$\begin{gathered} \{y_s, y_k\} = \sum_{r=1}^n c_{sk}^r y_r, \ \ \{p_i, q^j\} = \delta_i^j\,, \\ \{y_s, p_j\} = 0 = \{q^i, q^j\}, \ \ \{p_i, p_j\} = \sum_{s=1}^n y_s F_{ji}^{(s)}(q), \end{gathered} \tag{25}$$

where $i, j = \overline{1,3}$, $c_{sk}^r \in \mathbf{R}$, $s, k, r = \overline{1,n}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion

$$A^{(s)}(q) = \sum_{j=1}^3 A_j^{(s)}(q) dq^j$$

as well made alternative values $e_i := y_i$, $i = \overline{1,n}$.

The result (25) can be seen easily if one rewrites expression (23) in an extended form, $\omega^{(2)} := \omega^{(2)}_{ext}$, where

$$\omega^{(2)}_{ext} := \omega^{(2)}\Big|_{\mathcal{A}_0 \to \mathcal{A}}, \ \ \mathcal{A}_0(g) := g^{-1} dg, \ \ g \in G.$$

Thereby one can obtain in virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$ that

$$\begin{aligned} \omega^{(2)}_{ext}(q, p; u, y) &= \langle dp, \wedge dq \rangle + d\langle y(g), Ad_{g^{-1}} \mathcal{A}(q; e)\rangle_{\mathcal{G}} \\ &= \langle dp, \wedge dq \rangle + \langle d\, Ad^*_{g^{-1}} y(g), \wedge \mathcal{A}(q; e)\rangle_{\mathcal{G}} \\ &= \langle dp, \wedge dq \rangle + \sum_{s=1}^n dy_s \wedge du^s + + \sum_{j=1}^3 \sum_{s=1}^n A_j^{(s)}(q) dy_s \wedge dq \\ &\quad - \langle Ad^*_{g^{-1}} y(g), \mathcal{A}(q, e) \wedge \mathcal{A}(q, e)\rangle_{\mathcal{G}} \\ &\quad + \sum_{k \geq s=1}^n \sum_{l=1}^n y_l\, c_{sk}^l\, du^k \wedge du^s + \sum_{s=1}^n \sum_{i \geq j=1}^3 y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \end{aligned} \tag{26}$$

where the coordinate points $(q, p; u, y) \in T^*(M)$ are defined as follows:

$$\mathcal{A}_0(e) := \sum_{s=1}^{n} du^i\, a_i, \quad Ad^*_{g^{-1}} y(g) = y(e) := \sum_{s=1}^{n} y_s\, a^s$$

for any element $g \in G$.

Whence one immediately gets the Poisson brackets (25) plus additional brackets connected with the conjugated sets of variables $\{u^s \in \mathbf{R} : s = \overline{1,n}\} \in \mathcal{G}^*$ and $\{y_s \in \mathbf{R} : s = \overline{1,n}\} \in \mathcal{G}$ of the following form:

$$\{y_s, u^k\} = \delta_s^k,\ \{u^k, q^j\} = 0,\ \{p_j, u^s\} = A_j^{(s)}(q),\ \{u^s, u^k\} = 0, \tag{27}$$

where $j = \overline{1,3},\ k, s = \overline{1,n},$ and $q \in N$.

Note here that the suggested above transition from the symplectic structure $\omega^{(2)}$ on $T^*(N)$ to its extension $\omega^{(2)}_{ext}$ on $T^*(M)$ just consists formally in adding to the symplectic structure $\omega^{(2)}$ an exact part which transforms it into equivalent one.

Looking now at expressions (26), one can immediately infer that the element

$$\xi := \sum_{s=1}^{n} e_s a^s \in \mathcal{G}^*$$

will be invariant with respect to the Ad^*-action of the Lie group G iff

$$\{y_s, y_k\}|_{y_s = e_s} = \sum_{r=1}^{n} c^r_{sk}\, e_r \equiv 0$$

identically for all $s, k = \overline{1,n},\ j = \overline{1,3}$ and $q \in N$.

In this and only this case, the reduction scheme elaborated above will go through.

Returning attention to the expression (27), one can easily write the following exact expression:

$$\omega^{(2)}_{ext}(q, p; u, y) = \omega^{(2)}(q, p + \sum_{s=1}^{n} y_s\, A^{(s)}(q); u, y), \tag{28}$$

on the phase space $T^*(M) \ni (q, p; u, y)$, where we denoted for brevity $\langle A^{(s)}(q), dq\rangle$ by

$$\sum_{j=1}^{3} A_j^{(s)}(q)\, dq^j.$$

The transformation like (28) was discussed within a somewhat different context in paper [7] containing also a good background for the infinite-dimensional generalization of symplectic structure techniques.

Having observed from (28) that the simple change of variable

$$\tilde{p} := p + \sum_{s=1}^{n} y_s\, A^{(s)}(q)$$

of the cotangent space $T^*(N)$ recasts our symplectic structure (26) into the old canonical form (23), one obtains the following new set of Poisson brackets on $T^*(M) \ni (q, \tilde{p}; u, y)$ of the form:

$$\{y_s, y_k\} = \sum_{r=1}^{n} c_{sk}^r \, y_r, \;\; \{\tilde{p}_i, \tilde{p}_j\} = 0, \;\; \{\tilde{p}_i, q^j\} = \delta_i^j,$$

$$\{y_s, q^j\} = 0 = \{q^i, q^j\}, \; \{u^s, u^k\} = 0, \;\; \{y_s, \tilde{p}_j\} = 0,$$

$$\{u^s, q^i\} = 0, \;\; \{y_s, u^k\} = \delta_s^k, \;\; \{u^s, \tilde{p}_j\} = 0,$$

where $k, s = \overline{1, n}$ and $i, j = \overline{1, 3}$, holds iff the Yang – Mills equations

$$\frac{\partial F_{ij}^{(s)}}{\partial q^l} + \frac{\partial F_{jl}^{(s)}}{\partial q^i} + \frac{\partial F_{li}^{(s)}}{\partial q^j} + \sum_{k,r=1}^{n} c_{kr}^s (F_{ij}^{(k)} A_l^{(r)} + F_{jl}^{(k)} A_i^{(r)} + F_{li}^{(k)} A_j^{(r)}) = 0$$

are fulfilled for all $s = \overline{1, n}$ and $i, j, l = \overline{1, 3}$ on the base manifold N.

This effect of complete reduction of Yang – Mills variables from the symplectic structure (26) is known in literature [1, 7] as the principle of minimal interaction and appeared to be useful enough for studying different interacting systems as in papers [6, 10]. In Part 2 of this paper we shall continue a study of reduced symplectic structures connected with infinite dimensional coupled dynamical systems like Yang – Mills – Vlasov, Yang – Mills – Bogoliubov and Yang – Mills – Josephson ones.

4. Acknowledgements

One of the authors (A.P.) is cordially indebted to Prof. B.A. Kupershmidt for sending a set of inspiring reprints of his papers some of which are cited through this work as well as to Prof. J. Zagrodzinski for sending his paper before publication.

Special thanks for nice hospitality and warm research atmosphere are due to the staff of Department of Physics at the EMU of N.Cyprus, especially to Profs. M. Halilsoy, E. Aydiroglu, Ufuk Taneri and Ping Zhang for valuable discussions of problems under study.

REFERENCES

1. *Gillemin V. and Sternberg S.* "On the equations of motion of a classical particle in a Yang – Mills field and the principle of general covariance," Hadronic J., **1**, 1 – 32 (1978).
2. *Kummer J.* "On the construction of the Reduced phase space of a Hamiltonian system with symmetry," Indiana Univ. Math. J., **30**, No. 2, 261 – 281 (1981).
3. *Prykarpatsky A. and Mykytiuk I.* Algebraic Integrability of Nonlinear Dynamical Systems on Manifolds: Classical and Quantum Aspects, Kluver, Dordrecht (1998).
4. *Abraham R. and Marsden J.* Foundations of Mechanics, Second Edition, Addison-Wesley, New York (1978).
5. *Ratiu T.* "Euler – Poisson equations on Lie algebras and the N-dimensional heavy rigid body," Proc. Nat. Acad. Sci., **78**, No. 3, 1327 – 1328 (1981).

6. *Marsden J. and Weinstein A.* "The Hamiltonian structure of the Maxwell – Vlasov equations," Physica D, **4**, 394 – 406 (1982).

7. *Kupershmidt B.A.* "Infinite-dimensional analogs of the minimal coupling principle and of the Poincare lemma for differential two-forms," Different. Geom. and Appl., **2**, 275 – 293 (1992).

8. *Holm D. and Kupershmidt B.* "Superfluid plasmas: multivelocity nonlinear hydrodynamics of superfluid solutions with charged condensates coupled electromagnetically," Phys. Rev., **36A**, No. 8, 3947 – 3956 (1987).

9. *Moor J.D.* "Lectures on Seiberg – Witten invariants," Lect. Notes Math., Springer, Berlin (1996).

10. *Prykarpatsky A. and Zagrodzinski J.* "Dynamical aspects of Josephson type media," Ann. Inst. H. Poincaré A, **70**, No. 5, 497 – 524 (1999).

Received 21.01.2001