Nonlinear Oscillations, Vol. 4, No. 3, 2001

INTEGRAL REPRESENTATION OF HYPERPARABOLIC EQUATION

I. V. Samoilenko

Institute of Mathematics, NAS of Ukraine
Tereshchenkivs'ka St., 3, Kyiv, 01601, Ukraine

In this work, for functions that satisfy a Cauchy problem for hyperparabolic equations, we write integral equations and solve them using the method of successive approximations.

AMS Subject Classification: 60J30, 35Q40, 45L10

Introduction

The Riemman method is a well-known method for solving a Cauchy problem for the telegraph equation [1]. But this method can't be used for an analogue of the telegraph equation in R^n (hyperparabolic equations), and also nonlinear hyperparabolic equations that are defined later.

1. Cauchy Problem for Telegraph Equation

The telegraph process and the corresponding random evolution is a model for the motion of a physical particle on a line, when the particle changes its direction of motion to the opposite in random moments of time. Besides, the telegraph process defines a "rectangular wave" as an oscillating process in problems of radioengineering [2].

It's well-known [3] that a special form of a functional of the Markov random evolution satisfies the Cauchy problem

$$\frac{\partial^2}{\partial t^2}u(x,t) = -2\lambda\frac{\partial}{\partial t}u(x,t) + V^2\frac{\partial^2}{\partial x^2}u(x,t),$$

$$u(x,0) = f(x), \frac{\partial}{\partial t}u(x,t)\Big|_{t=0} = V\frac{d}{dx}f(x). \tag{1}$$

But, for the functional $u(x,t)$, it is possible to write an integral equation from the first jump of the process,

$$u(x,t) = e^{-\lambda t}f(x+Vt) + \lambda e^{-\lambda t}\int_0^t f(x - V(t+2s))ds$$

$$+ \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}u(x+V(s-l), t-(l+s))dlds. \tag{2}$$

The problem (1) can be obtained from (2). To do this, let us change the variables in (2),

$$u(x,t) = e^{-\lambda t} f(x+Vt) + \lambda e^{-\lambda t} \int_0^t f(x-V(t+2s))ds$$

$$- \lambda^2 e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f} u(x+V(2s+f-t),f)dsdf\,.$$

Differentiating the last expression with respect to x and t we have

$$\frac{\partial^2}{\partial x^2} u(x,t) = e^{-\lambda t} \frac{d^2}{dx^2} f(x+Vt) + \lambda e^{-\lambda t} \int_0^t \frac{d^2}{dx^2} f(x-V(t+2s))ds$$

$$- \lambda^2 e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f} \frac{\partial^2}{\partial x^2} u(x+V(2s+f-t),f)dsdf,$$

$$\frac{\partial}{\partial t} u(x,t) = - \lambda \Bigg[e^{-\lambda t} f(x+Vt) + \lambda e^{-\lambda t} \int_0^t f(x-V(t-2s))ds$$

$$- \lambda^2 e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f} u(x+V(2s+f-t),f)dsdf \Bigg] + \Bigg[V e^{-\lambda t} \frac{d}{dx} f(x+Vt)$$

$$- \lambda V e^{-\lambda t} \int_0^t \frac{d}{dx} f(x-V(t-2s))ds + \lambda^2 V e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f}$$

$$\times \frac{\partial}{\partial x} u(x+V(2s+f-t),f)dsdf \Bigg] + \lambda e^{-\lambda t} f(x+Vt) - \lambda^2 e^{-\lambda t} \int_0^t e^{\lambda f}$$

$$\times u(x+Vt-Vf,f)df = -\lambda u(x,t) + \Bigg[V e^{-\lambda t} \frac{d}{dx} f(x+Vt)$$

$$- \lambda V e^{-\lambda t} \int_0^t \frac{d}{dx} f(x-V(t-2s))ds + \lambda^2 V e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f}$$

$$\times \frac{\partial}{\partial x} u(x+V(2s+f-t),f)dsdf\Bigg] + \lambda e^{-\lambda t} f(x+Vt)$$

$$- \lambda^2 e^{-\lambda t} \int_0^t e^{\lambda f} u(x+Vt-Vf,f)df,$$

$$\frac{\partial^2}{\partial t^2} u(x,t) = -\lambda \frac{\partial}{\partial t} u(x,t) - \lambda \Bigg[V e^{-\lambda t} \frac{d}{dx} f(x+Vt) - \lambda V e^{-\lambda t}$$

$$\times \int_0^t \frac{d}{dx} f(x-V(t-2s))ds + \lambda^2 V e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f} \frac{\partial}{\partial x} u(x+V(2s$$

$$+f-t),f)dsdf + \lambda e^{-\lambda t} f(x+Vt) - \lambda^2 e^{-\lambda t} \int_0^t e^{\lambda f} u(x+Vt-Vf,f)df \Bigg]$$

$$+ V^2 \Bigg[e^{-\lambda t} \frac{d^2}{dx^2} f(x+Vt) + \lambda e^{-\lambda t} \int_0^t \frac{d^2}{dx^2} f(x-V(t-2s))ds$$

$$- \lambda^2 e^{-\lambda t} \int_0^t \int_0^{t-f} e^{\lambda f} \frac{\partial^2}{\partial x^2} u(x+V(2s+f-t),f)dsdf \Bigg] - \lambda V e^{-\lambda t} \frac{d}{dx} f(x$$

$$+Vt) + \lambda^2 V e^{-\lambda t} \int_0^t e^{\lambda f} \frac{d}{dx} u(x+Vt-Vf,f)df + \lambda V e^{-\lambda t} \frac{d}{dx} f(x$$

$$+Vt) - \lambda^2 V e^{-\lambda t} \int_0^t e^{\lambda f} \frac{d}{dx} u(x+Vt-Vf,f)df + \lambda^2 e^{-\lambda t} e^{\lambda t} u(x,t)$$

$$= -\lambda \frac{\partial}{\partial t} u(x,t) - \lambda \Bigg[\frac{\partial}{\partial t} u(x,t) + \lambda u(x,t) \Bigg] + V^2 \frac{\partial^2}{\partial x^2} u(x,t) + \lambda^2 u(x,t)$$

$$= -2\lambda \frac{\partial}{\partial t} u(x,t) + V^2 \frac{\partial^2}{\partial x^2} u(x,t).$$

We thus obtained the Cauchy problem (1). The following theorem holds.

Theorem 1. *The function $u(x,t)$, satisfying (1) under the condition of integrability of $u(x,t)$ and $f(x)$, satisfies equation (2). The function $u(x,t)$, satisfying (1) under the condition of differentiability with respect to x and t, satisfies the Cauchy problem (1).*

The Cauchy problem (1) and equation (2) are equivalent in this sense.

Let us consider the question of whether there exist a solution of (2) in the space of functions

$$\phi(x,t) = \phi_0(x,t) + c, \tag{3}$$

where $c = \mathrm{const}, \phi_0(x) \to 0, x, t \to \infty$. This Banach space with sup-norm was studied in the works of V.S.Koroljuk and A.F. Turbin (see, for example, [4]).

Let us write (2) in the following form: $u(x,t) = Au(x,t)$, where $Au(x,t)$ is equal the right-hand side of (2). A acts in the space (3), when $f(x) = f_0(x) + c$. Indeed,

$$A(\phi_0 + c) = e^{-\lambda t}(f_0(x) + c) + \lambda e^{-\lambda t}\int_0^t [f_0(x)(x - V(t-2s)) + c]ds$$

$$+ \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}[\phi_0(x + V(s-l), t-(s+l)) + c]dlds = \Big\{ f_0(x)e^{-\lambda t}$$

$$+ \lambda e^{-\lambda t}\int_0^t f_0(x)(x - V(t-2s))ds + \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}\phi_0(x + V(s-l), t$$

$$- (s+l))dlds\Big\} + \Big\{ e^{-\lambda t}c + \lambda e^{-\lambda t}tc + \lambda^2\Big(-\frac{1}{\lambda}e^{-\lambda t}ct - \frac{1}{\lambda^2}(e^{-\lambda t} - 1)c\Big)\Big\}$$

$$= \Big\{ f_0(x)e^{-\lambda t} + \lambda e^{-\lambda t}\int_0^t f_0(x)(x - V(t-2s))ds + \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}$$

$$\times \phi_0(x + V(s-l), t-(s+l))dlds\Big\} + c,$$

where the expression in braces converges to 0 as $x, t \to \infty$. Indeed,

$$e^{-\lambda t}\int_0^t f(x)dx < e^{-\lambda t}\int_0^t \sup_x |f(x)|dx = \sup_x |f(x)| te^{-\lambda t} \to 0,$$

$$x, t \to \infty, t > 0,$$

$$\int_0^t \int_0^{t-s} e^{-\lambda(l+s)}\phi_0(x+V(s-l),t-(s+l))dlds$$

$$< \sup_{x,t}|\phi_0(x+V(s-l),t-(s+l))| \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}dlds$$

$$= \sup_{x,t}\int_0^t \left(-\frac{1}{\lambda}\right)\left[e^{-\lambda t}-e^{-\lambda s}\right]ds$$

$$= \sup_{x,t}\left[-\frac{1}{\lambda}te^{-\lambda t}-\frac{1}{\lambda^2}\left(e^{-\lambda t}-1\right)\right] \to 0, x, t \to \infty.$$

Let us show that A is a contraction. We have

$$\rho(A\phi_1, A\phi_2) = \sup_{x,t}|\lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}\phi_1(x+V(s-l),t-(s+l))dlds$$

$$-\lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}\phi_2(x+V(s-l),t-(s+l))dlds \le \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)}$$

$$\times |\phi_1(x+V(s-l),t-(s+l)) - \phi_2(x+V(s-l),t-(s+l))|dlds$$

$$= \rho(\phi_1,\phi_2)[-\lambda te^{-\lambda t}-e^{-\lambda t}+1],$$

where $[-\lambda te^{-\lambda t}-e^{-\lambda t}+1] < 1$, so A is indeed a contraction. By the fixed point theorem, a solution of (2) exists and it is unique.

Theorem 2. *For $f(x)$ from the space (3), equation (2) has a unique solution lying in (3).*

The solution is $\lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} Au_{n-1}(x,t)$, where $u_0(x,t)$ is an arbitrary function from (3). But convergence also takes place in other spaces.

Example 1. $f(x) = x$,

$$u_0(x,t) = 0, u_1(x,t) = e^{-\lambda t}(x+xt+Vt),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_n(x,t) = e^{-\lambda t}\left(x\sum_0^{2n-1}\frac{t^k}{k!}\lambda^k + \frac{V}{\lambda}\sum_0^{2n-1}\frac{(\lambda t)^k}{k!}\right),$$

where k is even in the second sum,

$$\lim_{n\to\infty} u_n(x,t) = x + \frac{V}{2\lambda}(1 - e^{-2\lambda t}),$$

which coincides with the first moment of the Markov random evolution found in [5].

It should be noted that this method can be used for hyperparabolic equations in R^n.

2. Nonlinear Hyperparabolic Equation

A fading telegraph process and a fading Markov random evolution defines a motion of a particle on a line in the field of gravity, if the particle is attracted to some point on the line. From the point of view of radioengineering a fading telegraph process defines a fading "rectangular wave".

When we consider a generalization of a telegraph process, a fading telegraph process, another integral equation appears,

$$u(V,x,t) = e^{-\lambda t} f(x+Vt) + \lambda e^{-\lambda t} \int_0^t f\left(x + Vs - \frac{V}{c}(t-s)\right)$$

$$+ \lambda^2 \int_0^t \int_0^{t-s} e^{-\lambda(l+s)} u\left(\frac{V}{c^2}, x + Vs - \frac{V}{c} l, t - (s+l)\right) dl ds. \tag{4}$$

Let us note that for $c = 1$, equation (4) coincides with (2), where $u(V,x,t) = u(x,t)$.

By making the changes of variables, similar to the one in Section 1, and differentiating we get a Cauchy problem corresponding to (4),

$$\frac{\partial^3}{\partial t^3} u(V,x,t) + 3\lambda \frac{\partial^2}{\partial t^2} u(V,x,t) + 3\lambda^2 \frac{\partial}{\partial t} u(V,x,t) - \lambda^2 \frac{\partial}{\partial t} u\left(\frac{V}{c^2}, x, t\right)$$

$$- \frac{V^2}{c^2} \frac{\partial^3}{\partial x^2 \partial t} u(V,x,t) - \frac{V^2 \lambda}{c^2} \frac{\partial^2}{\partial x^2} u(V,x,t)$$

$$- \frac{c-1}{c} \lambda^2 V u\left(\frac{V^2}{c^2}, x, t\right) + \lambda^3 u(V,x,t) - \lambda^3 u\left(\frac{V}{c^2}, x, t\right) = 0, \tag{5}$$

$$u(V,x,t) = f(x), \quad \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = V \frac{d}{dx} f(x),$$

$$\frac{\partial^2}{\partial t^2} u(x,t)\Big|_{t=0} = -\frac{c+1}{c} \lambda V \frac{d}{dx} f(x) + V^2 \frac{d^2}{dx^2} f(x).$$

For $c = 1$, we have

$$\frac{\partial^3}{\partial t^3}u(x,t) + 3\lambda\frac{\partial^2}{\partial t^2}u(x,t) + 2\lambda^2\frac{\partial}{\partial t}u(x,t) - V^2\frac{\partial^3}{\partial x^2\partial t}u(x,t) - V^2\lambda\frac{\partial^2}{\partial x^2}u(x,t) = 0 \quad (6)$$

or

$$\left\{\frac{\partial^3}{\partial t^3}u(x,t) + 2\lambda\frac{\partial^2}{\partial t^2}u(x,t) - V^2\frac{\partial^3}{\partial x^2\partial t}u(x,t)\right\} + \left\{\lambda\frac{\partial^2}{\partial t^2}u(x,t) + 2\lambda^2\frac{\partial}{\partial t}u(x,t) - V^2\lambda\frac{\partial^2}{\partial x^2}u(x,t)\right\} = 0,$$

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2}{\partial t^2}u(x,t) + 2\lambda\frac{\partial}{\partial t}u(x,t) - V^2\frac{\partial^2}{\partial x^2}u(x,t)\right\} + \lambda\left\{\frac{\partial^2}{\partial t^2}u(x,t) + 2\lambda\frac{\partial}{\partial t}u(x,t) - V^2\frac{\partial^2}{\partial x^2}u(x,t)\right\} = 0,$$

$$\left\{\frac{\partial}{\partial t} + \lambda\right\}\left\{\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - V^2\frac{\partial^2}{\partial x^2}\right\}u(x,t) = 0.$$

This is a factorized equation, one component of which coincides with (1). Correspondingly, if $u(x,t)$ satisfies (5) for $c = 1$, then it satisfies (1).

The proof of existence of a solution of (4) is similar to that in Section 1 for the space of functions $\phi(V,x,t) = \phi_0(V,x,t) + c$. The following theorem holds.

Theorem 3. *The Cauchy problem for nonlinear hyperparabolic equation (5) is equivalent to integral equation (4) that has a solution in the space of functions $\phi(V,x,t) = \phi_0(V,x,t) + c$, where $c = \text{const}, \phi_0(V,x,t) \to 0, V,x,t \to \infty$.*

As in Section 1, the method of successive approximations converges for the functions $f(x) = x^k$.

Example 2. $f(x) = x$,

$$u_0(V,x,t) = 0, u_1(V,x,t) = e^{-\lambda t}\left(x + xt + Vt - \frac{Vt^2}{2c} + \frac{Vt^2}{2}\right),$$

$$u_2(V,x,t) = u_1(V,x,t) + \lambda^2 e^{-\lambda t}\left(\frac{xt^2}{2} + \frac{xt^3}{6} + \frac{\lambda^2 t^3 V}{6}\frac{c^2-c+1}{c^2}\right.$$

$$\left. + \frac{\lambda^3 t^4 V}{24}\frac{c^3-c^2+c-1}{c^3}\right),$$

$$\lim_{n\to\infty} u_n(V,x,t) = x + \frac{Vc}{\lambda(c+1)}\left(1 - e^{-\lambda t\frac{1+c}{c}}\right).$$

For $c = 1$, we have $x + \frac{V}{2\lambda}\left(1 - e^{-2\lambda t}\right)$ (see Example 1).

REFERENCES

1. *Glinner E.B. and Koshliakov N.S.* Partial Differential Equations of Mathematical Physics [in Russian], Vysshaya Shkola, Moscow (1970).
2. *Levin B.R.* Theoretical Bases of Statistical Radio Engineering [in Russian], Soviet Radio, Moscow (1966).
3. *Kac M.* Probability and Related Questions in Physics [in Russian], Mir, Moscow (1965).
4. *Koroljuk V.S. and Turbin A.F.* Mathematical Foundation of State Lumping of Large Systems, Kluver Academic Press, Amsterdam (1990).
5. *Turbin A.F. and Samoilenko I.V.* "Probabilistic method of solution of telegraph equation with real-analytical initial conditions," Ukr. Mat. Zh., **52**, No. 8, 1127 – 1134 (2000).

Received 05.12.2000