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NONLINEAR FORCED VIBRATION OF CURVED MICROBEAM RESTING ON NONLINEAR FOUNDATION USING THE MODIFIED STRAIN GRADIENT THEORY

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Abstract. The nonlinear forced vibrations of a curved micro- beam resting on the nonlinear foundation are examined. The equations of motion are derived using the Hamilton's principle and the modified strain gradient theory which is capable to examine the size effects in the microstructures. The nonlinear partial differential equations of motion are reduced to a time-dependent ordinary differential equation containing quadratic and cubic nonlinear terms. A frequency response of the curved microbeam for the primary resonance is determined using multiple time scales perturbation method. From the application point of view, the frequency response curves may be useful to select the optimum values of design parameters. The effects of geometry parameters and foundation moduli on the vibration behavior of the curved microbeam are illustrated.

Key words: curved microbeam, forced vibration, modified strain gradient theory, multiple times scale perturbation, Visco-Pasternak elastic foundation.

1. Introduction.

The Micro-Electro-Mechanical and Nano-Electro-Mechanical systems (MEMS/NEMS) have an important role in many areas and are used in engineering applications; such as micro-switches and atomic force microscope (AFM). Microbeams are one of the most important element in MEMS/NEMS, so the study of dynamic and vibration behavior of microbeam has been investigated by several researches. The size-dependent phenomena which is observed in microstructures has been confirmed experimentally ([1], [2]), since the non-classical continuum theory such as modified couple stress theory (MCST) [3] and strain gradient theory (SGT) [4] have been developed to consider small scale effects in the theoretical study of microbeams.

Researchers have studied the static, free or forced vibration behavior of straight microbeams based on MSGT or MCST using Euler – Bernoulli or Timoshenko theory. Simsek studied the forced vibration of an embedded microbeam carrying a moving microparticle [5] and the static bending and free vibration of the microbeam resting on the nonlinear elastic foundation [6] based on the MCST and Euler – Bernoulli theory. Asghari et al. [7] presented a nonlinear size-dependent model based on Timoshenko theory and MCST to study the static bending and free vibration of microbeam. Ghayesh et al. [8] studied nonlinear resonant dynamics of a microbeam using a model developed on the basis of MCST. In order to constructed the frequency-response curves, the governing equation are solved numerically by means of the pseudo-arclength continuation technique. Akgöz and Civalek investigated vibration response of non-homogenous and non-uniform microbeams using MCST [9]. Some papers ([10], [11], [12]) studied the size effect on pull-in behavior of microbeams.

Several research papers examined the vibration behavior of composite and FG microbeam based on MCST. Roque et al. [13] employed a meshless method to study the bending of simply supported laminated composite beams subjected to the transverse loads. Thai et al. [14] examined the static bending, buckling and free vibration behaviors of FG sandwich microbeams based on Timoshenko beam theory. Al-Basyouni et al. [15] presented a novel size-dependent unified beam formulation based on MCST in order to examine the bending and free vibration responses of FG microbeams. Jia et al. [16] investigated the size effect on the free vibration of FG microbeams under the combined loads including electrostatic force, temperature change and Casimir force. They used Euler – Bernoulli theory and von Kármán geometric nonlinearity to model the beam and solved the equations using the differential quadrature method. Simsek [17] studied the large amplitude free vibration of axially FG Euler – Bernoulli microbeam with immovable ends.

The strain gradient theory was introduced by Mindlin [18] in general form and then Lam et al. [19] modified the theory and introduced it as the modified strain gradient theory. MSGT uses three material length scale parameters in the constitutive equations rather than one parameter in MCST. There are many research papers in which MSGT is used to consider the size-effects of microbeams. Ansari et al. [20] investigated the vibration response of FG microbeams. They assumed the material properties to be graded along the thickness on the basis of Mori-Tanaka approach. They also studied the nonlinear free vibration behavior of FG microbeams based on the von Karman geometric nonlinearity [21] and the bending, buckling and free vibration responses of FG Timoshenko beam based on the general form of strain gradient theory [22]. Asghari et al. [23] derived the geometrically nonlinear governing differential equations of motion and the corresponding boundary conditions to analyze the large deflection of Timoshenko microbeams. Kahrobaiyan et al. [24] and Tajalli et al. [25] derived the strain gradient formulation of FG Euler – Bernoulli and Timoshenko beams. Lie et al. [26] examined the static bending and vibration of a size-dependent FG beam based on the strain gradient theory and the sinusoidal shear deformation theory. Zhang et al. [27] developed a non-classical Timoshenko beam element based on the strain gradient theory to analyze the static, free vibration, and bucking behaviors of the microbeam. Li et al. [28] established an analytical model for the elastic bending of the bilayered microbeam. Akgöz and Civalek [29], based on Euler – Bernoulli model, studied the static behavior of the microbeam using strain gradient and modified couple stress theory. they [30] also used various beam theories to examine the bending response of non-homogenous microbeams resting on an elastic medium.

In recent years, the curved microbeam has been considered in MEMS/NEMS, because of it's features such as bistabilty nature and performance in large stroke. The curved microbeam can be used in micro-valves, electrical micro-relays and micro-switches [34]. Some research works, examined the vibration of circular curved microbeam using the linear models. Liu and Reddy [31] utilized the modified couple stress theory to examine the static bending and free vibration of a simply supported circular curved microbeam. Ansari et al. [32], studied free vibration of the FG curved microbeam based on the modified strain gradient theory. Zhang et al. [33] studied the static bending and free vibration of the FG curved microbeams based on the strain gradient elasticity theory and n^{th}-order shear deformation theory.

The aforementioned valuable papers have focused on the straight microbeam or linear model of curved microbeam. The above literature review has less attention on the effect of foundation on the vibration behavior of the microbeam. In this study, the nonlinear model of a curved microbeam is presented to study the primary resonance of microbeam subjected to an external harmonic distributed force and the effect of nonlinear foundation on forced vibration response is examined. Based on the modified strain gradient theory, the size dependent nonlinear governing partial differential equations of motion are derived using Hamilton's principle. The Galerkin technique is then applied to reduce the partial differential equations to the time-dependent second order nonlinear ordinary differential equations. The method of multiple time scales is then used in order to determine an implicit frequency response for primary resonance analysis of the curved microbeam.

2. The modified strain gradient theory.

In this research, the modified strain gradient theory is used to develop the governing equations of the curved microbeam. The theory expresses the potential energy as follows [33]

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m^s_{ij}\chi^s_{ij}\right)dV, \quad (1)$$

where ε_{ij} , γ_i , $\eta^{(1)}_{ijk}$ and χ^s_{ij} denote the strain tensor, the dilatation gradient vector, the deviatoric stretch gradient and the symmetric rotation gradient tensors, respectively, are defined as [20]

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,\mathrm{i}} + u_{i,j}u_{j,\mathrm{i}}); \quad (2)$$

$$\gamma_i = \varepsilon_{mm,i}; \quad (3)$$

$$\eta_{ijk} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) - \frac{1}{15}\delta_{\mathrm{ij}}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}) - $$
$$-\frac{1}{15}\left[\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})\right]; \quad (4)$$

$$\chi^s_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}), \quad (5)$$

where u_i and δ_{ij} are the displacement vector and the knocker delta, respectively. In addition, the rotation vector (θ_i) can be determined as

$$\theta_i = \left(\frac{1}{2}\mathrm{curl}(u)\right)_i. \quad (6)$$

The corresponding stress measures, respectively, are defined as [33]

$$\sigma_{ij} = \lambda tr\varepsilon\delta_{ij} + 2\mu\varepsilon_{ij}; \quad (7)$$

$$p_i = 2\mu l_0^2\gamma_i; \quad (8)$$

$$\tau^1{}_{ijk} = 2\mu l_1^2\eta^1_{ijk}; \quad (9)$$

$$m^s_{ij} = 2\mu l_2^2\chi^s_{ij}, \quad (10)$$

where λ and μ are the bulk and shear modulus, respectively and (l_0, l_1, l_2) are the independent material length scale parameters.

3. The governing equations of the curved microbeam.

Figure 1 shows the geometry of the curved microbeam of length *L*, radius *R* and thickness *h* with simply supported boundary conditions at both ends. The microbeam rests on a Visco-Pasternak medium which is modeled with both linear and nonlinear spring, damper and shear elements. In addition, the uniform distributed harmonic force is applied on the microbeam. The x and z axes of the curvilinear coordinate system coincide with the circumference and radial direction of the circular curved microbeam, respectively and the y axis is normal to the plane of beam.

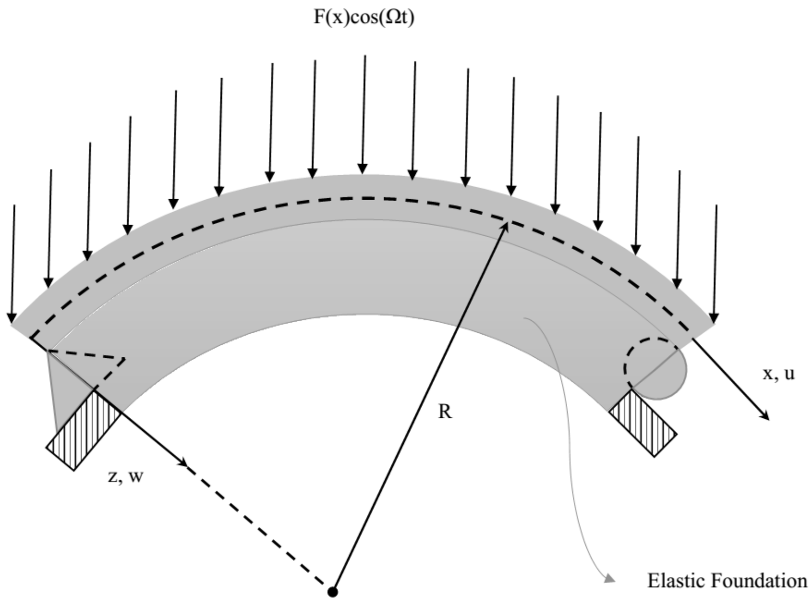


Figure 1. Geometry and coordinate system of the curved microbeam.

According to the Timoshenko beam theory the displacement field components of the circular curved microbeam based on the first order shear deformation theory can be assumed as [31]

$$u_x(x,z,t) = u(x,t) + z\varphi(x,t);\ u_y(x,z,t) = 0;\ u_z(x,z,t) = w(x,t), \tag{11}$$

where t denotes the time, u and w are the displacements of middle surface (i.e., displacement at $z = 0$), and φ describe the rotation of the microbeam cross section about the y-axis. Using Eqs. (2) – (11), the following relations can be obtained

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{w}{R} + z\frac{\partial \varphi}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \tag{12}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{u}{R} + \left(1 - \frac{z}{R}\right)\varphi\right); \tag{13}$$

$$\gamma_x = \frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}; \tag{14}$$

$$\gamma_z = \frac{\partial \varphi}{\partial x}; \tag{15}$$

$$\eta_{xxx}^{(1)} = \frac{2}{5}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{2R}\varphi + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right); \tag{16}$$

$$\eta_{zzz}^{(1)} = -\frac{1}{5}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x}\right); \tag{17}$$

$$\eta_{xxz}^{(1)} = \eta_{xzx}^{(1)} = \eta_{zxx}^{(1)} = \frac{4}{15}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x} \right); \tag{18}$$

$$\eta_{yyx}^{(1)} = \eta_{yxy}^{(1)} = \eta_{xyy}^{(1)} = -\frac{1}{5}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{15R}\varphi; \tag{19}$$

$$\eta_{yyz}^{(1)} = \eta_{yzy}^{(1)} = \eta_{zyy}^{(1)} = -\frac{1}{15}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x} \right); \tag{20}$$

$$\eta_{zzx}^{(1)} = \eta_{zxz}^{(1)} = \eta_{xzz}^{(1)} = -\frac{1}{5}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} \right) - \frac{4}{15}\varphi; \tag{21}$$

$$\chi_{xy}^{s} = \frac{1}{4}\left(\frac{\partial \varphi}{\partial x} - \frac{\partial^2 w}{\partial x^2} + \frac{1}{R}\frac{\partial u}{\partial x} + \frac{z}{R}\frac{\partial \varphi}{\partial x} \right); \tag{22}$$

$$\chi_{yz}^{s} = \frac{1}{4R}\varphi. \tag{23}$$

The classical and the non-classical stress tensors can be simplified as

$$\sigma_{xx} = (\lambda + 2\mu)\left(\frac{\partial u}{\partial x} + \frac{w}{R} + z\frac{\partial \varphi_y}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right); \tag{24}$$

$$\tau_{xz} = K_s \mu \left(\frac{\partial w}{\partial x} - \frac{u}{R} + \left(1 - \frac{z}{R}\right)\varphi \right); \tag{25}$$

$$p_x = 2\mu l_0^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \right); \tag{26}$$

$$p_z = 2\mu l_0^2 \gamma_z = 2\mu l_0^2 \frac{\partial \varphi}{\partial x}; \tag{27}$$

$$\tau_{xxx}^{(1)} = \frac{2}{5}\mu l_1^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{2R}\varphi + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \right); \tag{28}$$

$$\tau_{zzz}^{(1)} = -\frac{2}{5}\mu l_1^2 \left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x} \right); \tag{29}$$

$$\tau_{xxz}^{(1)} = \frac{8}{15}\mu l_1^2 \left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x} \right); \tag{30}$$

$$\tau_{yyx}^{(1)} = -\frac{2}{5}\mu l_1^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} \right) + \frac{2}{15R}\mu l_1^2 \varphi; \tag{31}$$

$$\tau_{yyz}^{(1)} = -\frac{2}{15}\mu l_1^2 \left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{1}{R}\frac{\partial u}{\partial x} + \left(1 - \frac{z}{R}\right)\frac{\partial \varphi}{\partial x} \right); \tag{32}$$

$$\tau_{zzx}^{(1)} = -\frac{2}{5}\mu l_1^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial w}{\partial x} + z\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - \frac{8}{15}\mu l_1^2 \varphi; \tag{33}$$

$$m_{xy}^s = \frac{\mu l_2^2}{2}\left(\frac{\partial \varphi}{\partial x} - \frac{\partial^2 w}{\partial x^2} + \frac{1}{R}\frac{\partial u}{\partial x} + \frac{z}{R}\frac{\partial \varphi}{\partial x}\right); \tag{34}$$

$$m_{yz}^s = \frac{\mu l_2^2}{2R}\varphi, \tag{35}$$

where K_s is the Timoshenko shear coefficient and defined as [31]

$$K_s = \frac{5+5\nu}{6+5\nu}. \tag{36}$$

3.1 Hamilton's principal. The nonlinear governing equations can be obtained by Hamilton's principle as follows

$$\int_0^t \left(\delta U - \left(\delta K + \delta W\right)\right) dt = 0, \tag{37}$$

where U, K and W are the total potential energy, the kinetic energy and the work done by the external forces, respectively. The total potential energy of the curved microbeam based on the modified strain gradient theory can be specified as

$$\begin{aligned} U = \frac{1}{2}\int_V \Big(&\sigma_{xx}\varepsilon_{xx} + 2\tau_{xz}\varepsilon_{xz} + p_x\gamma_x + p_z\gamma_z + \tau_{xxx}^{(1)}\eta_{xxx}^{(1)} + \tau_{zzz}^{(1)}\eta_{zzz}^{(1)} + 3\tau_{xxz}^{(1)}\eta_{xxz}^{(1)} + \\ &+3\tau_{yyx}^{(1)}\eta_{yyx}^{(1)} + 3\tau_{yyz}^{(1)}\eta_{yyz}^{(1)} + 3\tau_{zzx}^{(1)}\eta_{zzx}^{(1)} + 2m_{xy}^s\chi_{xy}^s + 2m_{yz}^s\chi_{yz}^s \Big) dV. \end{aligned} \tag{38}$$

The kinetic energy of the curved microbeam is obtained by

$$K = \frac{\rho}{2}\int_0^L \int_A \left(\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right) dA dx, \tag{39}$$

where ρ denotes the mass density of the microbeam.

Substituting Eqs. (38) and (39) into Eq. (37), integrating by the part and setting the coefficients of $\delta u, \delta w$ and $\delta\varphi$ equal to zero, the nonlinear governing partial differential equations of motion can be derived as

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{Q_{xz}}{R} - \frac{2}{5}\frac{\partial^2 T_{xxx}}{\partial x^2} + \frac{1}{5R}\frac{\partial T_{zzz}}{\partial x} - \frac{4}{5R}\frac{\partial T_{xxz}}{\partial x} + \\ &+\frac{3}{5}\frac{\partial^2 T_{yyx}}{\partial x^2} + \frac{1}{5R}\frac{\partial T_{yyz}}{\partial x} + \frac{3}{5}\frac{\partial^2 T_{zzx}}{\partial x^2} - \frac{\partial^2 P_x}{\partial x^2} + \frac{1}{2R}\frac{\partial Y_{xy}}{\partial x} = I_1\frac{\partial^2 u}{\partial t^2}; \end{aligned} \tag{40}$$

$$\begin{aligned} &-\frac{N_x}{R} + \frac{\partial Q_{xz}}{\partial x} + \frac{2}{5R}\frac{\partial T_{xxx}}{\partial x} + \frac{1}{5}\frac{\partial^2 T_{zzz}}{\partial x^2} - \frac{4}{5}\frac{\partial^2 T_{xxz}}{\partial x^2} - \frac{3}{5R}\frac{\partial T_{yyx}}{\partial x} + \\ &+\frac{1}{5}\frac{\partial^2 T_{yyz}}{\partial x^2} - \frac{3}{5R}\frac{\partial T_{zzx}}{\partial x} + \frac{1}{R}\frac{\partial P_x}{\partial x} + \frac{1}{2}\frac{\partial^2 Y_{xy}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right) + \\ &+\frac{2}{5}\frac{\partial}{\partial x}\left(T_{xxx}\frac{\partial^2 w}{\partial x^2}\right) - \frac{2}{5}\frac{\partial^2}{\partial x^2}\left(T_{xxx}\frac{\partial w}{\partial x}\right) + \frac{3}{5}\frac{\partial}{\partial x}\left(T_{yyx}\frac{\partial^2 w}{\partial x^2}\right) - \\ &-\frac{3}{5}\frac{\partial^2}{\partial x^2}\left(T_{yyx}\frac{\partial w}{\partial x}\right) + \frac{3}{5}\frac{\partial}{\partial x}\left(T_{zzx}\frac{\partial^2 w}{\partial x^2}\right) - \frac{3}{5}\frac{\partial^2}{\partial x^2}\left(T_{zzx}\frac{\partial w}{\partial x}\right) + \end{aligned}$$

$$+\frac{\partial}{\partial x}\left(P_x\frac{\partial^2 w}{\partial x^2}\right)-\frac{\partial^2}{\partial x^2}\left(P_x\frac{\partial w}{\partial x}\right)+q=I_1\frac{\partial^2 w}{\partial t^2}; \tag{41}$$

$$\begin{aligned}&\frac{\partial M_x}{\partial x}-Q_{xz}+\frac{1}{R}P_{xz}-\frac{2}{5}\frac{\partial^2 M_{xxx}}{\partial x^2}-\frac{1}{5R}T_{xxx}-\frac{2}{5}\frac{\partial T_{zzz}}{\partial x}+\\&+\frac{1}{5R}\frac{\partial M_{zzz}}{\partial x}+\frac{8}{5}\frac{\partial T_{xxz}}{\partial x}-\frac{4}{5R}\frac{\partial M_{xxz}}{\partial x}+\frac{3}{5}\frac{\partial^2 M_{yyx}}{\partial x^2}-\frac{1}{5R}T_{yyx}-\\&-\frac{2}{5}\frac{\partial T_{yyz}}{\partial x}+\frac{1}{5R}\frac{\partial M_{yyz}}{\partial x}+\frac{4}{5R}T_{zzx}+\frac{3}{5}\frac{\partial^2 M_{zzx}}{\partial x^2}-\frac{\partial^2 S_x}{\partial x^2}+\frac{\partial P_z}{\partial x}+\\&+\frac{1}{2}\frac{\partial Y_{xy}}{\partial x}+\frac{1}{2R}\frac{\partial X_{xy}}{\partial x}-\frac{1}{2R}Y_{yz}=I_2\frac{\partial^2\varphi}{\partial t^2},\end{aligned} \tag{42}$$

where, the moment of inertia (I_1, I_2) and the stress resultants are defined in appendix *A*. In Eq. (41) the distributed transverse force, q consists of both the reaction force of nonlinear Visco-Pasternak elastic foundation and distributed harmonic force which can be defined as follow

$$q=-k_L w+k_P\frac{\partial^2 w}{\partial x^2}-k_{NL}w^3-c_d\frac{\partial w}{\partial t}+F\cos(\Omega t), \tag{43}$$

where k_L, k_P, k_{NL} *and* c_d are the Winkler spring constant, Pasternak spring constant, nonlinear spring constant and damping constant, respectively. Also, F and Ω denote the amplitude and frequency of the distributed harmonic force. Substituting Eq. $(A-1)$ Error! Reference source not found. – $(A-25)$ Error! Reference source not found. into Eqs. (40) – (42), the governing equations yield as

$$\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x^2}+\frac{A_{11}}{R}\frac{\partial w}{\partial x}+A_{11}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{K_s}{R}\left(\frac{A_{22}}{R}\frac{\partial w}{\partial x}-\frac{A_{22}}{R}u+A_{22}\varphi\right)-\\&-\frac{8}{25}l_1^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+\frac{1}{2}\frac{A_{22}}{R}\frac{\partial^2\varphi}{\partial x^2}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)-\\&-\frac{8}{15}\frac{l_1^2}{R}\left(2A_{22}\frac{\partial^2\varphi}{\partial x^2}+A_{22}\frac{\partial^3 w}{\partial x^3}-\frac{A_{22}}{R}\frac{\partial^2 u}{\partial x^2}\right)-\\&-\frac{12}{25}l_1^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)-\\&-\frac{6}{25}\frac{l_1^2 A_{22}}{R}\frac{\partial^2\varphi}{\partial x^2}-2l_0^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)+\\&+\frac{1}{4}\frac{l_2^2}{R}\left(A_{22}\frac{\partial^2\varphi}{\partial x^2}-A_{22}\frac{\partial^3 w}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 u}{\partial x^2}\right)=I_1\frac{\partial^2 u}{\partial t^2};\end{aligned} \tag{44}$$

$$-\frac{1}{R}\left(A_{11}\frac{\partial u}{\partial x}+\frac{A_{11}}{R}w+\frac{1}{2}A_{11}\left(\frac{\partial w}{\partial x}\right)^2\right)+K_s\left(\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}-\frac{A_{22}}{R}\frac{\partial u}{\partial x}+A_{22}\frac{\partial \varphi}{\partial x}\right)+$$

$$+\frac{8}{25}\frac{l_1^2}{R}\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\frac{A_{22}}{R}\frac{\partial \varphi}{\partial x}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)-$$

$$-\frac{8}{15}l_1^2\left(2A_{22}\frac{\partial^3 \varphi}{\partial x^3}+A_{22}\frac{\partial^4 w}{\partial x^4}-\frac{A_{22}}{R}\frac{\partial^3 u}{\partial x^3}\right)-$$

$$-\frac{3}{5R}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)+\frac{2}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)-$$

$$-\frac{3}{5R}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)-\frac{8}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)+$$

$$+\frac{2l_0^2}{R}\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)+$$

$$+\frac{1}{4}l_2^2\left(A_{22}\frac{\partial^3 \varphi}{\partial x^3}-A_{22}\frac{\partial^4 w}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 u}{\partial x^3}\right)+\left(A_{11}\frac{\partial^2 u}{\partial x^2}+\frac{A_{11}}{R}\frac{\partial w}{\partial x}+\frac{1}{2}A_{11}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial x}+$$

$$+\left(A_{11}\frac{\partial u}{\partial x}+\frac{A_{11}}{R}w+\frac{1}{2}A_{11}\left(\frac{\partial w}{\partial x}\right)^2\right)\frac{\partial^2 w}{\partial x^2}- \tag{45}$$

$$-\frac{8}{25}\frac{l_1^2}{R}\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\frac{A_{22}}{R}\frac{\partial \varphi}{\partial x}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)\frac{\partial^2 w}{\partial x^2}-$$

$$-\frac{8}{25}\frac{l_1^2}{R}\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+\frac{1}{2}\frac{A_{22}}{R}\frac{\partial^2 \varphi}{\partial x^2}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)\frac{\partial w}{\partial x}+$$

$$+\frac{3}{5}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)+\frac{2}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}+$$

$$+\frac{3}{5}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)+\frac{2}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)\frac{\partial w}{\partial x}+$$

$$+\frac{3}{5R}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)-\frac{8}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}+$$

$$+\frac{3}{5R}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}+3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)-\frac{8}{15}\frac{l_1^2A_{22}}{R}\frac{\partial \varphi}{\partial x}\right)\frac{\partial w}{\partial x}-$$

$$-2l_0^2\left(A_{22}\frac{\partial^3 u}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 w}{\partial x^2}+A_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+A_{22}\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3}\right)\frac{\partial^2 w}{\partial x^2}-$$

$$-2l_0^2\left(A_{22}\frac{\partial^4 u}{\partial x^4}+\frac{A_{22}}{R}\frac{\partial^3 w}{\partial x^3}++3A_{22}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)\frac{\partial^2 w}{\partial x^2}-$$

$$-k_Lw+k_p\frac{\partial^2 w}{\partial x^2}-k_{NL}w^3-c_d\frac{\partial w}{\partial t}+F\cos(\Omega t)=I_1\frac{\partial^2 w}{\partial t^2};$$

$$
\begin{aligned}
&-\frac{4}{25}\frac{l_1^2}{R}\left(A_{22}\frac{\partial^2 u}{\partial x^2}+\frac{A_{22}}{R}\frac{\partial w}{\partial x}+\frac{1}{2}\frac{A_{22}}{R}\frac{\partial^2 \varphi}{\partial x^2}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)-\\
&-K_s\left(\frac{A_{22}}{R}\frac{\partial w}{\partial x}-\frac{A_{22}}{R}u+A_{22}\varphi\right)+\frac{16}{15}l_1^2\left(2A_{22}\frac{\partial^2 \varphi}{\partial x^2}+A_{22}\frac{\partial^3 w}{\partial x^3}-\frac{A_{22}}{R}\frac{\partial^2 u}{\partial x^2}\right)-\\
&-\frac{1}{5}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^2 u}{\partial x^2}+\frac{A_{22}}{R}\frac{\partial w}{\partial x}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)+\frac{2}{15}\frac{l_1^2 A_{22}}{R}\varphi\right)+\\
&+\frac{4}{5}\left(-\frac{2}{5}l_1^2\left(A_{22}\frac{\partial^2 u}{\partial x^2}+\frac{A_{22}}{R}\frac{\partial w}{\partial x}+A_{22}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)-\frac{8}{15}\frac{l_1^2 A_{22}}{R}\varphi\right)-\\
&-2l_0^2 D_{22}\frac{\partial^4 \varphi}{\partial x^4}+2l_0^2 A_{22}\frac{\partial^2 \varphi}{\partial x^2}+\frac{1}{4}l_2^2\left(A_{22}\frac{\partial^2 \varphi}{\partial x^2}-A_{22}\frac{\partial^3 w}{\partial x^3}+\frac{A_{22}}{R}\frac{\partial^2 u}{\partial x^2}\right)+\\
&+D_{11}\frac{\partial^2 \varphi}{\partial x^2}+\frac{1}{4}l_2^2\frac{D_{22}}{R^2}\frac{\partial^2 \varphi}{\partial x^2}-\frac{1}{4}\frac{l_2^2 A_{22}}{R^2}\varphi-\\
&-D_{22}\frac{\varphi}{R^2}-\frac{4}{5}l_1^2 D_{22}\frac{\partial^4 \varphi}{\partial x^4}+\frac{8}{15}l_1^2\frac{D_{22}}{R^2}\frac{\partial^2 \varphi}{\partial x^2}=I_2\frac{\partial^2 \varphi}{\partial t^2}.
\end{aligned} \tag{46}
$$

3.2 Galerkin technique. In this section, the approximate Galerkin technique is applied to convert the governing equations of motion to a set of time dependent nonlinear ordinary differential equations. In order to achieve the normalize form of governing equations for the parametric analysis, the following dimensionless parameters are introduced as

$$
\begin{gathered}
(U,W)=\frac{(u,w)}{h};\ \Phi=\varphi;\ \tau=\frac{t}{L}\sqrt{\frac{A_{11}}{I_1}};\ \xi=\frac{x}{L};\ \vartheta=\frac{R}{L};\ \kappa=\frac{L}{h};\ (\tilde{I}_1,\tilde{I}_2)=\left(\frac{I_1}{I_1},\frac{I_2}{I_1 h^2}\right);\\
(\tilde{A}_{22},\tilde{D}_{11},\tilde{D}_{22})=\left(\frac{A_{22}}{A_{11}},\frac{D_{11}}{A_{11}h^2},\frac{D_{22}}{A_{11}h^2}\right);\ (\tilde{l}_0,\tilde{l}_1,\tilde{l}_2)=\left(\frac{l_0}{L},\frac{l_1}{L},\frac{l_2}{L}\right);\ K_L=\frac{k_L L^2}{A_{11}}; \qquad (47)\\
K_{NL}=\frac{k_{NL}L^2 h^2}{A_{11}};\ K_P=\frac{k_P}{A_{11}};\ C_D=\frac{c_d L}{\sqrt{A_{11}I_1}};\ f=\frac{FL^2}{A_{11}h};\ \tilde{\Omega}=\Omega\sqrt{\frac{I_1 L^2}{A_{11}}}.
\end{gathered}
$$

The following assumptions for $U(\xi,\tau), W(\xi,\tau)$ and $\Phi(\xi,\tau)$ are considered

$$
U(\xi,\tau)=\lambda_1(\xi)\tilde{U}(\tau);\ W(\xi,\tau)=\lambda_2(\xi)\tilde{W}(\tau);\ \Phi(\xi,\tau)=\lambda_3(\xi)\tilde{\Phi}(\tau), \tag{48}
$$

where $\lambda_1(\xi),\lambda_2(\xi)$ and $\lambda_3(\xi)$ are the first eigen mode function of microbeam. Note considering the more terms in Fourier Transform results more accurate response, although for lack of complicity and develop an analytical formula, in this paper, one term is considered. Substituting Eq. into Eqs. (44) – (46), using the dimensionless parameters and applying the Galerkin technique, the equation are obtained as

$$
\Lambda_1\ddot{\tilde{U}}+\Lambda_2\tilde{U}+\Lambda_3\tilde{\Phi}+\Lambda_4\tilde{W}+\Lambda_5\tilde{W}^2=0; \tag{49}
$$

$$
\Pi_1\ddot{\tilde{W}}+\Pi_2\tilde{W}+\Pi_3\tilde{U}+\Pi_4\tilde{\Phi}+\Pi_5\tilde{U}\tilde{W}+\Pi_6\tilde{\Phi}\tilde{W}+\Pi_7\tilde{W}^2+\Pi_8\tilde{W}^3+\Pi_9\dot{\tilde{W}}=\hat{f}\cos(\tilde{\Omega}\tau); \tag{50}
$$

$$
\Upsilon_1\ddot{\tilde{\Phi}}+\Upsilon_2\tilde{\Phi}+\Upsilon_3\tilde{U}+\Upsilon_4\tilde{W}+\Upsilon_5\tilde{W}^2=0, \tag{51}
$$

where the coefficients Λ_i, Π_i and Υ_i are defined in the appendix A.

The effect of longitudinal inertia on the large amplitude flexural vibrations of the slender beams is negligible [35]. Note, in this paper it is assumed the length to thickness ratio is equal to 10. Furthermore, the coefficient of rotational inertia term (Υ_1) against other coefficients in Eq. (51) is too small and can be ignored. Accordingly, combining Eqs. (49) – (51) and making some mathematical simplifications, the second order ordinary differential equation with quadratic and cubic nonlinearities can be obtained as

$$\ddot{\tilde{W}} + \Gamma_1 \dot{\tilde{W}} + \Gamma_2 \tilde{W} + \Gamma_3 \tilde{W}^2 + \Gamma_4 \tilde{W}^3 = \tilde{f} \cos(\tilde{\Omega}\tau), \tag{52}$$

where the coefficients Γ_i and $\tilde{f}$ are defined in the appendix A.

4. Semi-analytical solution for curved microbeam.

In this paper, the semi-analytical method of multiple time scales perturbation is applied to determine the forced vibration response of the curved microbeam. The following assumptions are considered

$$\Gamma_1 = \zeta\varepsilon^2; \;\; \tilde{f} = f\varepsilon^3, \tag{53}$$

where ε denotes a dimensionless perturbation parameter. Note, this parameter has an order of the amplitude vibration. Eq. (51) by using Eq. (53) can be rewritten as

$$\ddot{\tilde{W}} + \varepsilon^2 \zeta \dot{\tilde{W}} + \Gamma_2 \tilde{W} + \Gamma_3 \tilde{W}^2 + \Gamma_4 \tilde{W}^3 = f\varepsilon^3 \cos(\tilde{\Omega}\tau). \tag{54}$$

According to this method, the following slow time scales are defined as [36]

$$T_n = \varepsilon^n \tau \Rightarrow \begin{cases} T_0 = \tau; \\ T_1 = \varepsilon\tau; \\ T_2 = \varepsilon^2\tau. \end{cases} \tag{55}$$

Applying the chain rule, the time derivative with respect to dimensionless time can be obtained as

$$\begin{gathered} \frac{d}{d\tau} = D_0 + \varepsilon D_1 + ... \\ \frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (2D_0 D_2 + D_1^2) + ... \end{gathered} \tag{56}$$

In which, the operator of D_n is defined as

$$D_n = \frac{\partial}{\partial T_n} \quad (n = 0, 1, 2, 3, ...). \tag{57}$$

The first approximation solution of Eq. (54) is assumed as

$$\tilde{W}(\tau;\varepsilon) = \varepsilon \tilde{W}_1(T_0, T_1, T_2) + \varepsilon^2 \tilde{W}_2(T_0, T_1, T_2) + \varepsilon^2 \tilde{W}_3(T_0, T_1, T_2) + ... \tag{58}$$

Inserting Eq. (58) in Eq. (54) and collecting the coefficients of same powers of ε leads to

$$D_0^2 \tilde{W}_1 + \Gamma_2 \tilde{W}_1 = 0; \tag{59}$$

$$D_0^2 \tilde{W}_2 + \Gamma_2 \tilde{W}_2 = -\Gamma_3 \tilde{W}_1^2 - 2(D_0 D_1 \tilde{W}_1); \tag{60}$$

$$\begin{gathered} D_0^2 \tilde{W}_3 + \Gamma_2 \tilde{W}_3 = -D_1^2 \tilde{W}_1 - 2(D_0 D_1 \tilde{W}_2) - 2(D_0 D_1 \tilde{W}_2) - \Gamma_4 \tilde{W}_1^3 - \\ -2\Gamma_3 \tilde{W}_1 \tilde{W}_2 - \zeta D_0 \tilde{W}_1 + \frac{1}{2}\Big(\exp(i\tilde{\Omega}T_0) + \exp(-i\tilde{\Omega}T_0)\Big). \end{gathered} \tag{61}$$

The solution of Eq. (59) can be expressed as

$$\tilde{W}_1\left(T_0,T_1,T_2\right)=\mathrm{Z}\left(T_1,T_2\right)\exp\left(i\sqrt{\Gamma_2}T_0\right)+\bar{\mathrm{Z}}\left(T_1,T_2\right)\exp\left(i\sqrt{\Gamma_2}T_0\right), \tag{62}$$

where $\bar{Z}$ stands for the complex conjugate of function Z. In order to examine the primary resonance of the curved microbeam, it is assumed the linear natural frequency of curved microbeam is nearly equal to the frequency of exciting force $(\tilde{\Omega})$ as following [36]

$$\tilde{\Omega}=\sqrt{\Gamma_2}+\varepsilon^2\sigma, \tag{63}$$

where σ denotes the detuning parameter. Inserting Eq. (62) into Eq. (60) leads to

$$\begin{aligned}D_0^2\tilde{W}_2+\Gamma_2\tilde{W}_2=-\Gamma_3Z^2\left(\exp(2i\sqrt{\Gamma_2}T_0)\right)-\Gamma_3\bar{Z}^2\left(\exp(-2i\sqrt{\Gamma_2}T_0)\right)-2\Gamma_3Z\bar{Z}-\\-2i\sqrt{\Gamma_2}\left(\exp(i\sqrt{\Gamma_2}T_0)\right)D_1Z+2i\sqrt{\Gamma_2}\left(\exp(-i\sqrt{\Gamma_2}T_0)\right)D_1\bar{Z}.\end{aligned} \tag{64}$$

Neglecting secular terms in Eq. (64) yields

$$-2i\sqrt{\Gamma_2}D_1\left(Z(T_1,T_2)\right)=0\Rightarrow Z=Z(T_2). \tag{65}$$

Eq. with regarding Eq. (65) be rewritten as

$$D_0^2\tilde{W}_2+\Gamma_2\tilde{W}_2=-\Gamma_3Z^2\left(\exp(2i\sqrt{\Gamma_2}T_0)\right)-\Gamma_3\bar{Z}^2\left(\exp(-2i\sqrt{\Gamma_2}T_0)\right)-2\Gamma_3Z\bar{Z}. \tag{66}$$

The particular solution of Eq. (66) is determined as

$$\tilde{W}_{2P}=\frac{\Gamma_3Z^2}{3\Gamma_2}\left(\exp(2i\sqrt{\Gamma_2}T_0)\right)+\frac{\Gamma_3\bar{Z}^2}{3\Gamma_2}\left(\exp(-2i\sqrt{\Gamma_2}T_0)\right)-\frac{2\Gamma_3Z\bar{Z}}{\Gamma_2}. \tag{67}$$

The following differential equation is obtained by inserting Eqs. (62) and (67) into Eq. (61) and omitting the secular term

$$-2i\sqrt{\Gamma_2}(D_2Z)-i\sqrt{\Gamma_2}\zeta Z+\frac{10}{3}\frac{\Gamma_3{}^2Z^2\bar{Z}}{\Gamma_2}-3\Gamma_4Z^2\bar{Z}+\frac{f}{2}\left(\exp(i\sigma T_2)\right)=0. \tag{68}$$

The solution of Eq. (68) can be expressed as

$$Z\left(T_2\right)=\frac{1}{2}\eta\left(T_2\right)\exp\left(i\psi\left(T_2\right)\right). \tag{69}$$

In which η and ψ are real function of T_2. Substituting Eq. (69) into Eq. (68) and separating the real and imaginary expressions, the following two differential equations are obtained

$$\eta\frac{d\psi}{dT_2}=-\frac{1}{8\sqrt{\Gamma_2}}\left(\frac{10}{3\Gamma_2}\Gamma_3^2-3\Gamma_4\right)\eta^3-\frac{f}{2\sqrt{\Gamma_2}}\cos(\sigma T_2-\psi); \tag{70}$$

$$\frac{d\eta}{dT_2}=-\frac{\zeta}{2}\eta+\frac{f}{2\sqrt{\Gamma_2}}\sin\left(\sigma T_2-\psi\right). \tag{71}$$

Assuming the steady state conditions, the time derivative with respect to T_2 will be equal to zero. The frequency response of curved microbeam is then obtained by combining Eqs. (70) and (71), as following

$$\left(\frac{\zeta}{2}\eta\right)^2-\left(\frac{1}{8\sqrt{\Gamma_2}}\left(\frac{10}{3\Gamma_2}\Gamma_3^2-3\Gamma_4\right)\eta^3+\sigma\eta\right)^2=\left(\frac{f}{2\sqrt{\Gamma_2}}\right)^2. \tag{72}$$

5. Result and discussion.

In order to validate the presented model, an isotropic curved microbeam with length to thickness ratio of $L/h=10$ and radius to length of $R/L=5$ is considered. The material properties are assumed will be same as Liu and Reddy [32]. The dimensionless fundamental natural frequency of the linear curved microbeam model based on the modified couple stress and classical theories is presented in Table. It should be noted, with assuming $\Gamma_1 = \Gamma_3 = = \Gamma_4 = \tilde{f} = 0$ i.e. neglecting damping, nonlinear stiffness and forcing effects, Eq. reduces to the linear curved microbeam model. The results have the excellent agreement with those calculated from linear curved microbeam model by Liu and Reddy [32]. The maximum difference of two models is 0,68 percent at $h/l = 10$.

Table. The dimensionless linear natural frequency of curved microbeam, $L = 10h$, $R = 5L, b = 2h$ and $K_L = K_P = K_{NL} = 0$.

h/l	Sources	CT	MCST
1	Ref. [32]	0,2752	0,4978
	Present	0,2771	0,5007
2	Ref. [32]	0,2752	0,3459
	Present	0,2771	0,3481
3	Ref. [32]	0,2752	0,3087
	Present	0,2771	0,3108
5	Ref. [32]	0,2752	0,2878
	Present	0,2771	0,2897
8	Ref. [32]	0,2752	0,2802
	Present	0,2771	0,2821
10	Ref. [32]	0,2752	0,2784
	Present	0,2771	0,2803

The material properties of the curved microbeam made of homogeneous epoxy material are $E = 1{,}44\,\text{GPa}$; $\nu = 0{,}38$; $\rho = 1{,}44\,\text{Kg/m}^3$. The three length scale parameters of the modified strain gradient theory are supposed to be equal to each other, $l_0 = l_1 = l_2 = l = 17{,}6\,\mu\text{m}$ ([23], [24]). In order to examine the effects of different parameters on the forced vibration characteristics of the curved microbeam, the frequency and force-response curves will be presented.

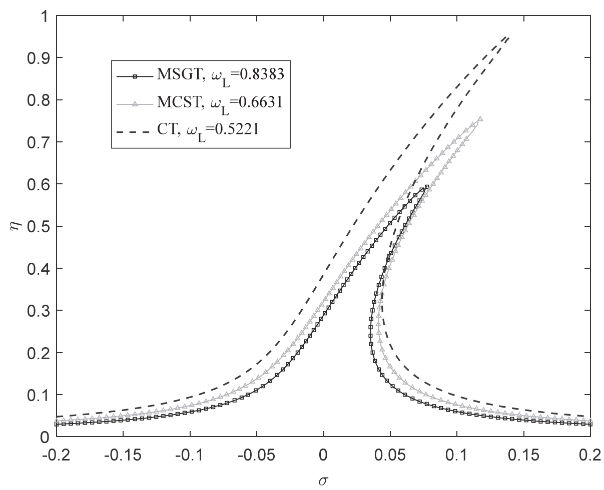
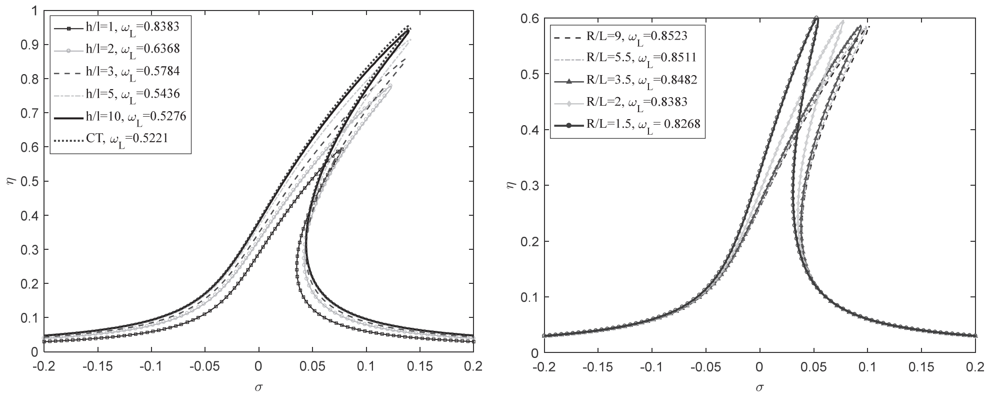


Figure 2. The frequency-response predicted by classical and non-classical theories ($K_L = K_{NL} = 0{,}1$; $K_P = 0{,}01$; $C_D = 0{,}02$; $L/h = 10$; $R/L = 2$; $h/l = 1$; $f = 0{,}01$).

Figure 2 shows the frequency-response of the curved microbeam calculated based on the modified strain gradient theory $(l_0 = l_1 = l_2 = l)$, the modified couple stress theory $(l_0 = l_l = 0, l_2 = l)$ and the classical theory $(l_0 = l_l = l_2 = 0)$. All theories predict hardening behavior for the curved microbeam in bending, however the higher hardening is predicted by MSGT and MCST relative to CT. The amplitude predicted by the non-classical models is less than classical one, for instance the amplitude peak from MSCT is about 58% less than CT.

Figures 3-a and 3-b present the frequency response for various values of h/l/l and R/L, respectively $(K_L = K_{NL} = 0,1; K_P = 0,01; C_D = 0,02; L/h = 10; R/L = 2; f = 0,01)$. With respect to figure 3-a, increasing of h/l ratio leads to increasing of the forced vibration amplitude. Moreover, the frequency response predicted by MSCT approaches to CT for $h/l \geq 5$ because the curved microbeam becomes size-independent. According to figure 3-b, the forced vibration amplitude decreases by increasing the R/L ratio. Note that the coefficient of quadratic nonlinear term in Eq. approaches to zero as the R/L ratio increases, consequently the cubic nonlinearity term becomes more effective than quadratic one.



(a) **(b)**

Figure. 3. The frequency-response for various values of **a)** thickness-to-material length scale ratio $(R/L = 2)$, **b)** radius-to-length ratio $(h/l = 1)$.

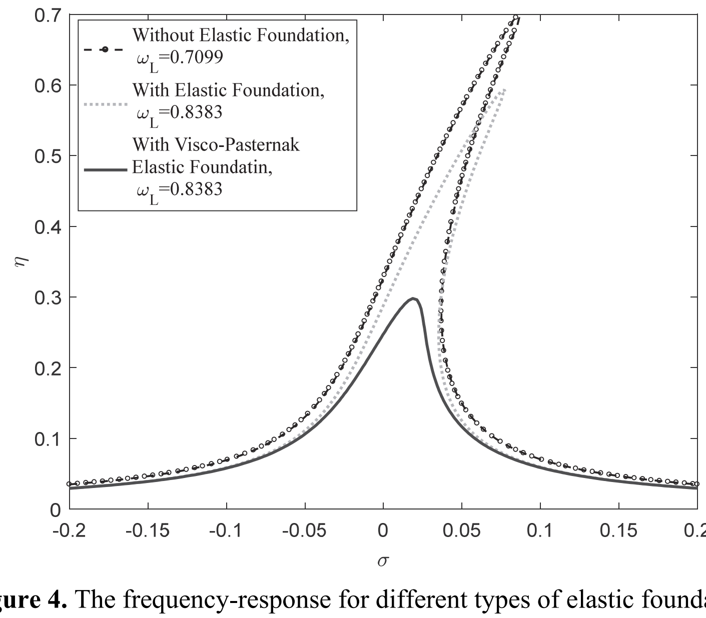


Figure 4. The frequency-response for different types of elastic foundation $(L/h = 10;\ R/L = 2;\ h/l = 1;\ f = 0,01)$.

To study the influence of foundation, the frequency-response curves are displayed in figure 4 for three cases of 1-without elastic foundation, 2-Winkler-Pasternak elastic foundation $(K_L = K_{NL} = 0{,}1\, K_P = 0{,}01; C_D = 0)$ and 3-Visco-Pasternak elastic foundation $(K_L = K_{NL} = 0{,}1;$ $K_P = 0{,}01; C_D = 0{,}04)$. Winkler-Pasternak and Visco-Pasternak elastic foundation decrease the forced vibration amplitude about 14% and 57%, respectively. The Visco-elastic foundation shifts the peak of curve to the right, so the response curve approaches to linear response one.

Figures 5 depicts the effect of nonlinear foundation on frequency response for $f = 0{,}01$. According to the figures 5-a, c and d, the increasing of K_L, K_P and C_D causes to decrease η. The effect of K_P on the response amplitude is more than K_L; as K_L changes from 0 to 0,5; the amplitude decreases about 26% and increasing of K_P from 0 to 0,2 leads to 24% decreasing of η. According to figure 5-b, K_{NL} has no effect on the maximum amplitude. However, increasing K_{NL} causes the frequency curves bend to right and the behavior of curved microbeam becomes hardener.

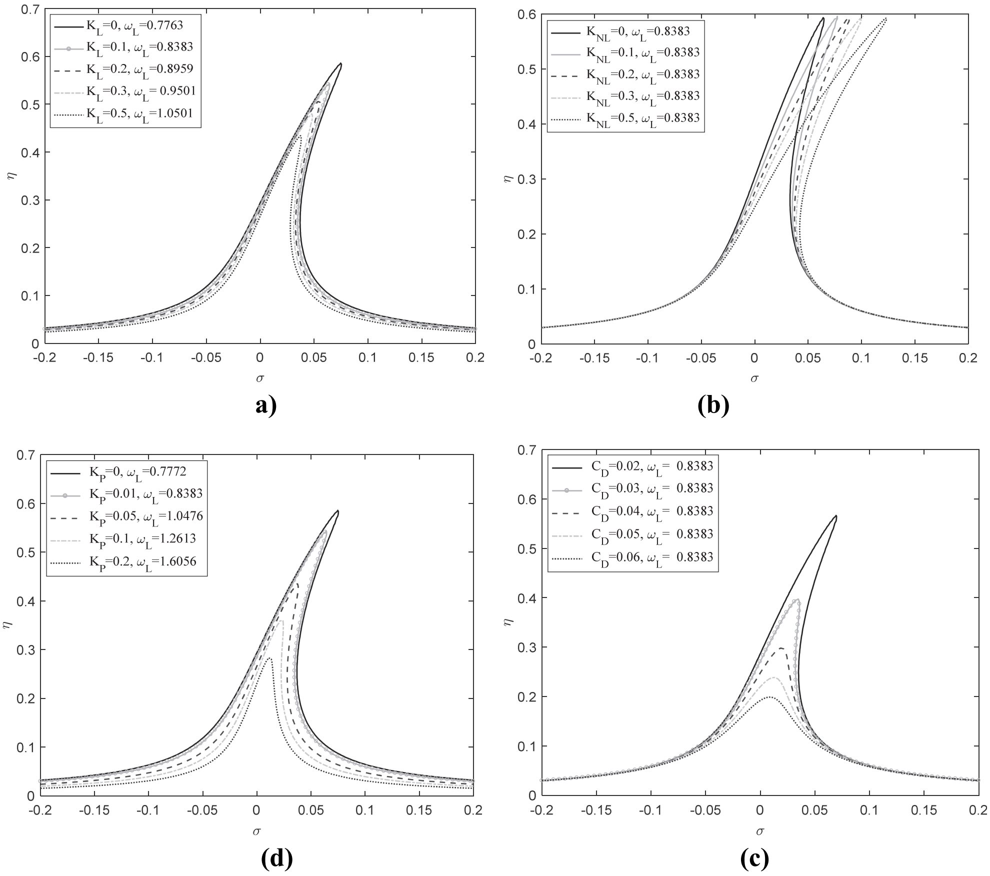


Figure 5. The frequency-response for various values of **a)** linear foundation modulus, **b)** nonlinear foundation modulus, **c)** Pasternak modulus, **d)** damping foundation modulus. $(L/h = 10; R/L = 2; h/l = 1;\ f = 0{,}01)$.

Figures 6 – 8 demonstrate the amplitude of response versus the amplitude of excitation force. Figure 6 shows the response amplitude for several values of detuning parameter. For $\sigma = 0{,}01$ the response is completely stable and single-value. As the detuning parameter increases two bifurcation points appear in the forced-response curves. Note the multivalued response curves are important from the physical point of view because of jump phenomenon. For instance, along the lower branch, for $\sigma = 0,2,$ when force amplitude slowly increases and reaches to about 0,125 the response amplitude suddenly jumps from 0,58 to 1,1. The zone of unstable response (dash line between two bifurcations) grows with increasing of detuning parameter.

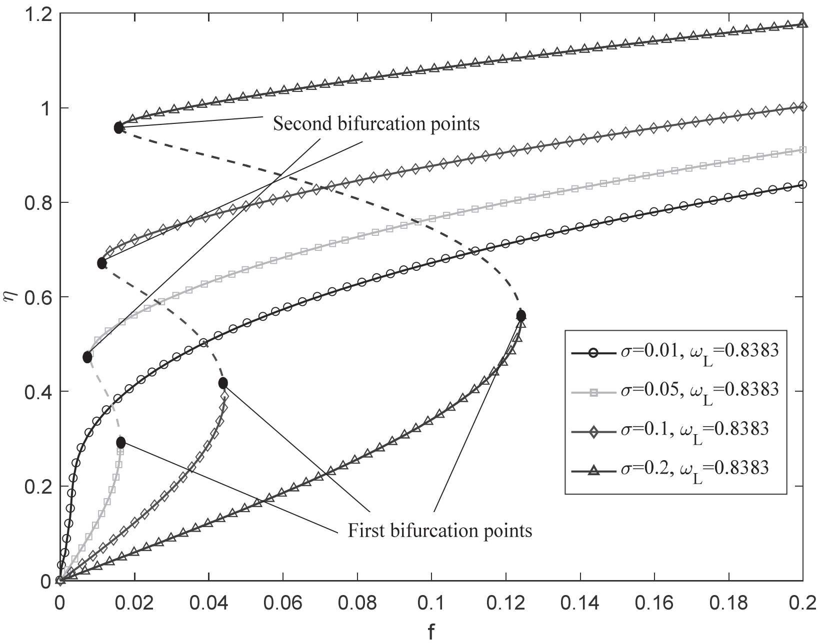
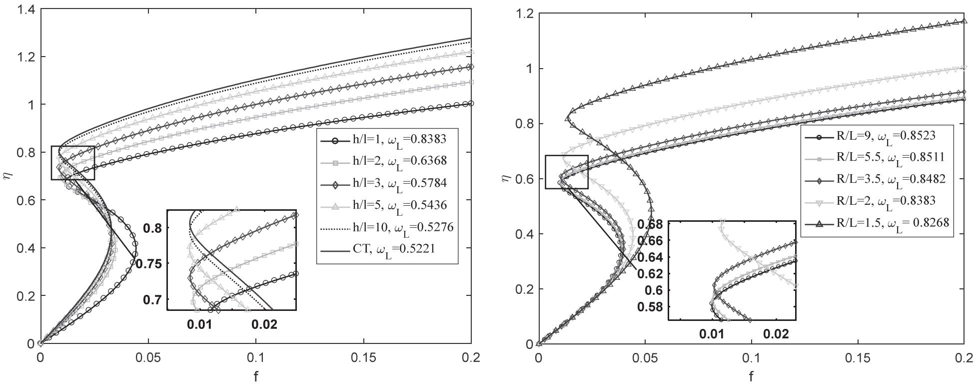


Figure 6. The amplitude of response vs. the amplitude of force for various values of detuning parameter $(K_L = K_{NL} = 0,1; K_P = 0,01; C_D = 0,02; L/h = 10; R/L = 2; h/l = 1)$.

The influence of h/l and R/L on the force-response are shown in figures 7. According to the figures the change rate of η on the lower branches is more than upper ones. Both bifurcation points shift to right when h/l or R/L decreases. With respect to figure 7-a, the response amplitude increases with increasing of h/l. The response is predicted by CT is nearly approaches to MSGT for h/l=10, however, the CT predicts the higher values of η relative to MSGT.



(a) (b)

Figure 7. The amplitude of response vs. the amplitude of force for various values of **a)** thickness-to-material length scale ratio, **b)** radius-to-length ratio $(K_L = K_{NL} = 0,1; K_P = 0,01; \ C_D = 0,02; L/h = 10; \sigma = 0,1)$.

Figures 8 depict the influence of foundation coefficients on the response amplitude of the curved microbeam at σ=0.1. According to figure 8-a, c and d, with increasing of K_L, K_P and C_D both bifurcation points shift to right, however with respect to figure 8-b, increasing of K_{NL} shifts the bifurcation points to the left and the zone of unstable response becomes smaller. The Pasternak modulus has different effect on η along lower and upper branches. On the lower branches, the amplitude response decreases with increasing K_P, however there is no specific pattern on the upper branches.

Figures 4, 5 and 8 may be helpful to understand the vibration behavior of the curved microbeam in order to select the optimum values of foundation moduli.

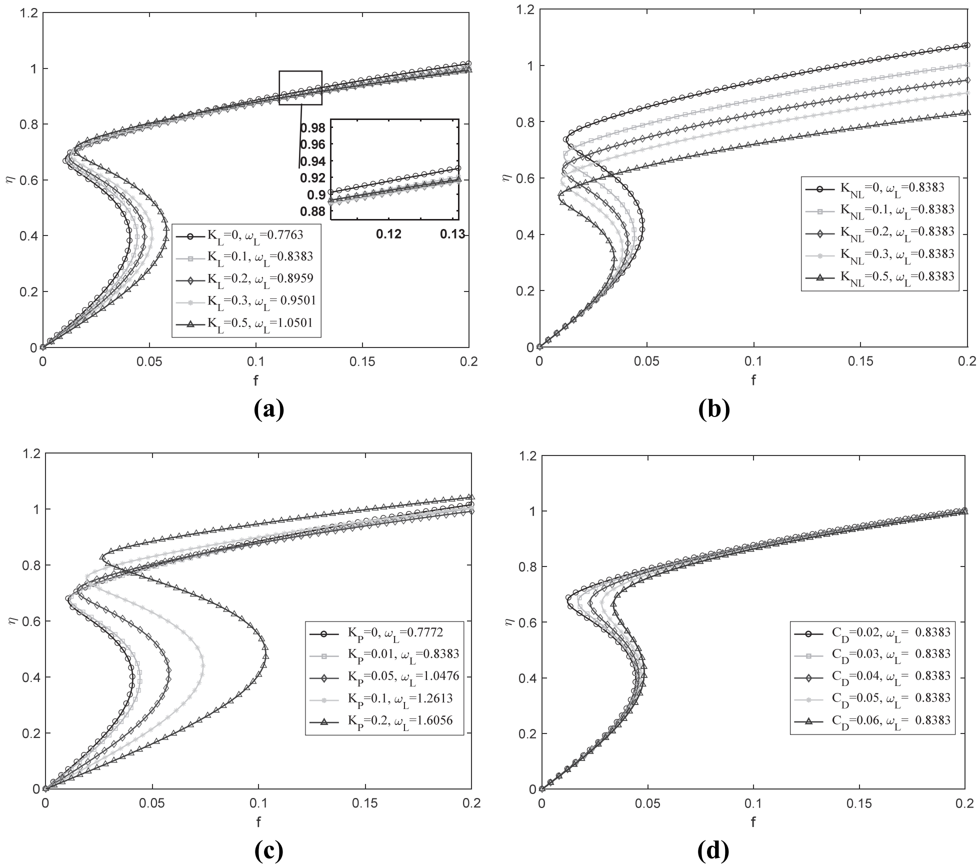


Figure 8. The amplitude of response vs. the amplitude of force for various values of a) linear foundation modulus, b) Pasternak modulus, c) nonlinear foundation modulus, d) damping foundation modulus, $(L/h=10; R/L=2; h/l=1; \sigma=0{,}1)$.

6. Conclusion.

In this paper, the forced vibration of the curved microbeam resting on the nonlinear foundation with simply supported boundary conditions at two ends is investigated. Based on Timoshenko beam theory and the modified strain gradient theory, the nonlinear governing equations of motion are derived using Hamilton's principle. The second order nonlinear ordinary differential equation is then developed using Galerkin technique and applying some simplifications. In order to determine the forced response of curved microbeam, the semi analytical method is utilized. The results are validated with the other researches in an especial case of linear model of the curved microbeam. The effect of geometric parameters and foundation moduli on frequency response curve are investigated. The results may be help choose the geometric parameters and foundation coefficients in order to achieve the desired application of the curved microbeam.

The results are predicted by the modified strain gradient theory approaches to classic theory as h/l increases and for $h/l < 5$ the size effect is nearly diminished. When the h/l decreases, the behavior of the curved microbeam will be stiffer and the jump value of amplitude decreases at bifurcation points.

The effect of nonlinear foundation including Winkler spring (K_L), Pasternak spring (K_P), nonlinear spring (K_{NL}) and damping (C_D) on the response of curved microbeam is also studied. As K_L, K_P and C_D increases, the response amplitude decreases and the maximum amplitude occurs at the lower value of excitation frequency in the frequency response curves. However, as K_{NL} increases the maximum amplitude occurs at the higher value of excitation frequency. For lower values of force amplitude ($f < 0{,}02$), K_{NL} has less effect on the response amplitude, however the increasing of K_{NL} from 0 to 0,5 leads to about 70% decrease of the response amplitude, on the upper branch of stable response.

Increasing the excitation frequency, both bifurcation points occur at higher value of force amplitude. On the upper stable branch by increasing the excitation frequency the forced vibration amplitude increases and on the lower stable branch decreases.

7. Appendix A.

The moment of inertia, in equations (40) – (42), are determined as

$$\left(I_1, I_2\right) = \int_{-h/2}^{h/2} \rho \left\{1, z^2\right\} dz. \tag{A-1}$$

The stress resultants, in equations (40) – (42) are expressed as

$$N_x = \int_s \sigma_{xx} dA = A_{11}\frac{\partial u}{\partial x} + A_{11}\frac{w}{R} + A_{11}\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \tag{A-2}$$

$$M_x = \int_s \sigma_{xx} z dA = D_{11}\frac{\partial \varphi}{\partial x}; \tag{A-3}$$

$$Q_{xz} = \int_s \tau_{xz} dA = k_s\left(A_{22}\frac{\partial w}{\partial x} - A_{22}\frac{u}{R} + A_{22}\varphi\right); \tag{A-4}$$

$$P_{xz} = \int_s \tau_{xz} z dA = -\frac{D_{22}}{R}\varphi; \tag{A-5}$$

$$P_x = \int_s p_x dA = 2l_0^2\left(A_{22}\frac{\partial^2 u}{\partial x^2} + A_{22}\frac{1}{R}\frac{\partial w}{\partial x} + A_{22}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right); \tag{A-6}$$

$$S_x = \int_s p_x z dA = 2l_0^2 D_{22}\frac{\partial^2 \varphi}{\partial x^2}; \tag{A-7}$$

$$P_z = \int_s p_z dA = 2l_0^2 A_{22}\frac{\partial \varphi}{\partial x}; \tag{A-8}$$

$$Y_{xy} = \int_s m_{xy} dA = \frac{l_2^2}{2}\left(A_{22}\frac{\partial \varphi}{\partial x} - A_{22}\frac{\partial^2 w}{\partial x^2} + A_{22}\frac{1}{R}\frac{\partial u}{\partial x}\right); \tag{A-9}$$

$$X_{xy} = \int_s m_{xy} z dA = \frac{l_2^2}{2} D_{22} \frac{1}{R} \frac{\partial \varphi}{\partial x}; \tag{A-10}$$

$$Y_{yz} = \int_s m_{yz} dA = \frac{l_2^2}{2R} A_{22} \varphi; \tag{A-11}$$

$$T_{xxx} = \int_s \tau_{xxx}^{(1)} dA = \frac{4}{5} \mu l_1^2 \left(A_{22} \frac{\partial^2 u}{\partial x^2} + A_{22} \frac{1}{R} \frac{\partial w}{\partial x} + A_{22} \frac{1}{2R} \varphi + A_{22} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right); \tag{A-12}$$

$$M_{xxx} = \int_s \tau_{xxx}^{(1)} z dA = \frac{4}{5} \mu l_1^2 D_{22} \frac{\partial^2 \varphi}{\partial x^2}; \tag{A-13}$$

$$T_{zzz} = \int_s \tau_{zzz}^{(1)} dA = -\frac{2}{5} \mu l_1^2 \left(2A_{22} \frac{\partial \varphi}{\partial x} + A_{22} \frac{\partial^2 w}{\partial x^2} - A_{22} \frac{1}{R} \frac{\partial u}{\partial x} \right); \tag{A-14}$$

$$M_{zzz} = \int_s \tau_{zzz}^{(1)} z dA = \frac{2}{5} l_1^2 \frac{D_{22}}{R} \frac{\partial \varphi}{\partial x}; \tag{A-15}$$

$$T_{xxz} = \int_s \tau_{xxz}^{(1)} dA = \frac{8}{15} l_1^2 \left(2A_{22} \frac{\partial \varphi}{\partial x} + A_{22} \frac{\partial^2 w}{\partial x^2} - A_{22} \frac{1}{R} \frac{\partial u}{\partial x} \right); \tag{A-16}$$

$$M_{xxz} = \int_s \tau_{xxz}^{(1)} z dA = -\frac{8}{15} l_1^2 \frac{D_{22}}{R} \frac{\partial \varphi}{\partial x}; \tag{A-17}$$

$$T_{yyx} = \int_s \tau_{yyx}^{(1)} dA = -\frac{2}{5} l_1^2 \left(A_{22} \frac{\partial^2 u}{\partial x^2} + A_{22} \frac{1}{R} \frac{\partial w}{\partial x} + A_{22} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + \frac{2}{15R} l_1^2 A_{22} \varphi; \tag{A-18}$$

$$M_{yyx} = \int_s \tau_{yyx}^{(1)} z dA = -\frac{2}{5} l_1^2 D_{22} \frac{\partial^2 \varphi}{\partial x^2}; \tag{A-19}$$

$$T_{yyz} = \int_s \tau_{yyz}^{(1)} dA = -\frac{2}{15} l_1^2 \left(2A_{22} \frac{\partial \varphi_y}{\partial x} + A_{22} \frac{\partial^2 w}{\partial x^2} - A_{22} \frac{1}{R} \frac{\partial u}{\partial x} \right); \tag{A-20}$$

$$M_{yyz} = \int_s \tau_{yyz}^{(1)} z dA = \frac{2}{15} l_1^2 \frac{D_{22}}{R} \frac{\partial \varphi}{\partial x}; \tag{A-21}$$

$$T_{zzx} = \int_s \tau_{zzx}^{(1)} dA = -\frac{2}{5} l_1^2 \left(A_{22} \frac{\partial^2 u}{\partial x^2} + A_{22} \frac{1}{R} \frac{\partial w}{\partial x} + A_{22} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - \frac{8}{15} l_1^2 A_{22} \varphi; \tag{A-22}$$

$$M_{zzx} = \int_s \tau_{zzx}^{(1)} z dA = -\frac{2}{5} l_1^2 D_{22} \frac{\partial^2 \varphi}{\partial x^2}, \tag{A-23}$$

where, the stiffness components of A_{ii} and $D_{ii} (i = 1,2)$ are given by

$$\left(A_{11}, D_{11}\right)=\int_{-h/2}^{h/2}(\lambda+2\mu)\left\{1, z^2\right\}dz; \tag{A-24}$$

$$\left(A_{22}, D_{22}\right)=\int_{-h/2}^{h/2}\mu\left\{1, z^2\right\}dz. \tag{A-25}$$

The coefficients of equations of - are defined as

$$\Lambda_1=-\frac{1}{2}\tilde{I}_1; \tag{A-26}$$

$$\Lambda_2=-\frac{2}{5}\pi^4\tilde{A}_{22}\tilde{l}_1^2-\pi^4\tilde{A}_{22}\tilde{l}_0^2-\frac{1}{2}\pi^2-\frac{4}{15}\frac{\tilde{l}_1^2\pi^2\tilde{A}_{22}}{\vartheta^2}-\frac{1}{8}\frac{\tilde{l}_2^2\pi^2\tilde{A}_{22}}{\vartheta^2}-\frac{1}{2}\frac{K_s\tilde{A}_{22}}{\vartheta^2}; \tag{A-27}$$

$$\Lambda_3=\frac{1}{2}\frac{\kappa K_s\tilde{A}_{22}}{\vartheta}+\frac{11}{15}\frac{\kappa\pi^2\tilde{A}_{22}\tilde{l}_1^2}{\vartheta}-\frac{1}{8}\frac{\kappa\pi^2\tilde{A}_{22}\tilde{l}_2^2}{\vartheta}; \tag{A-28}$$

$$\Lambda_4=\frac{1}{2}\frac{K_s\tilde{A}_{22}}{\vartheta}+\frac{2}{3}\frac{\pi^3\tilde{A}_{22}\tilde{l}_1^2}{\vartheta}+\frac{\pi^3\tilde{A}_{22}\tilde{l}_0^2}{\vartheta}+\frac{\pi^3\tilde{A}_{22}\tilde{l}_2^2}{8\vartheta}+\frac{\pi}{2\vartheta}; \tag{A-29}$$

$$\Lambda_5=-\frac{16}{3}\frac{\pi^4\tilde{A}_{22}\tilde{l}_0^2}{\kappa}-\frac{32}{15}\frac{\pi^4\tilde{A}_{22}\tilde{l}_0^2}{\kappa}-\frac{2}{3}\frac{\pi^2}{\kappa}; \tag{A-30}$$

$$\Pi_1=-\frac{1}{2}\tilde{I}_1; \tag{A-31}$$

$$\begin{gathered}\Pi_2=-\frac{\pi^2\tilde{A}_{22}\tilde{l}_0^2}{\vartheta^2}-\frac{2}{5}\frac{\pi^2\tilde{A}_{22}\tilde{l}_1^2}{\vartheta^2}-\frac{1}{2\vartheta^2}-\frac{1}{2}K_s\tilde{A}_{22}-\frac{4}{15}\pi^4\tilde{A}_{22}\tilde{l}_1^2- \\ -\frac{1}{8}\pi^4\tilde{A}_{22}\tilde{l}_2^2-\frac{1}{2}K_L-\frac{1}{2}\pi^2K_p;\end{gathered} \tag{A-32}$$

$$\Pi_3=-\frac{\pi K_s\tilde{A}_{22}}{2\vartheta}+\frac{2}{3}\frac{\pi^3\tilde{A}_{22}\tilde{l}_1^2}{\vartheta}+\frac{\pi^3\tilde{A}_{22}\tilde{l}_0^2}{\vartheta}+\frac{1}{8}\frac{\pi^3\tilde{A}_{22}\tilde{l}_2^2}{\vartheta}+\frac{\pi}{2\vartheta}; \tag{A-33}$$

$$\Pi_4=-\frac{1}{5}\frac{\pi\kappa\tilde{A}_{22}\tilde{l}_1^2}{\vartheta^2}-\frac{1}{2}\pi\kappa K_s\tilde{A}_{22}-\frac{8}{15}\pi^3\kappa\tilde{A}_{22}\tilde{l}_1^2+\frac{1}{8}\pi^3\kappa\tilde{A}_{22}\tilde{l}_2^2; \tag{A-34}$$

$$\Pi_5=\frac{2}{3}\frac{\pi^2}{\kappa}+\frac{4}{3}\frac{\pi^4\tilde{A}_{22}\tilde{l}_0^2}{\kappa}+\frac{8}{15}\frac{\pi^4\tilde{A}_{22}\tilde{l}_1^2}{\kappa}; \tag{A-35}$$

$$\Pi_6=-\frac{4}{15}\frac{\pi^2\tilde{A}_{22}\tilde{l}_1^2}{\vartheta}; \tag{A-36}$$

$$\Pi_8=-\frac{\pi}{\kappa\vartheta}; \tag{A-37}$$

$$\Pi_9=-\frac{1}{2}\frac{\pi^6\tilde{A}_{22}\tilde{l}_0^2}{\kappa^2}-\frac{3}{16}\frac{\pi^4}{\kappa^2}-\frac{1}{5}\frac{\pi^6\tilde{A}_{22}\tilde{l}_1^2}{\kappa^2}-\frac{3}{8}K_{NL}; \tag{A-38}$$

$$\Pi_9 = -\frac{1}{2} C_D; \tag{A-39}$$

$$\Upsilon_1 = -\frac{1}{2} \tilde{I}_2; \tag{A-40}$$

$$\Upsilon_2 = -\frac{1}{2}\pi^2 \tilde{D}_{11} - \frac{2}{5}\pi^4 \tilde{D}_{22}\tilde{l}_1^2 - \frac{4}{15}\frac{\pi^2 \tilde{D}_{22}\tilde{l}_1^2}{\vartheta^2} - \pi^4 \tilde{D}_{22}\tilde{l}_1^2 - \frac{1}{8}\frac{\pi^2 \tilde{D}_{22}\tilde{l}_2^2}{\vartheta^2} - \frac{1}{8}\frac{\kappa^2 \tilde{A}_{22}\tilde{l}_2^2}{\vartheta^2} - \\ - \frac{4}{15}\frac{\kappa^2 \tilde{A}_{22}\tilde{l}_1^2}{\vartheta^2} - \frac{16}{15}\pi^2\kappa^2 \tilde{A}_{22}\tilde{l}_1^2 - \frac{1}{8}\pi^2\kappa^2 \tilde{A}_{22}\tilde{l}_2^2 - \frac{1}{2}K_s\kappa^2 \tilde{A}_{22} - \pi^2\kappa^2 \tilde{A}_{22}\tilde{l}_0^2 - \frac{1}{2}\frac{\tilde{D}_{22}}{\vartheta^2}; \tag{A-41}$$

$$\Upsilon_3 = \frac{1}{2}\frac{K_s\kappa\tilde{A}_{22}}{\vartheta} + \frac{11}{15}\frac{\pi^2\kappa\tilde{A}_{22}\tilde{l}_1^2}{\vartheta} - \frac{1}{8}\frac{\pi^2\kappa\tilde{A}_{22}\tilde{l}_2^2}{\vartheta}; \tag{A-42}$$

$$\Upsilon_4 = -\frac{1}{5}\frac{\pi\kappa\tilde{A}_{22}\tilde{l}_1^2}{\vartheta^2} - \frac{1}{2}\pi K_s\kappa\tilde{A}_{22} - \frac{8}{15}\pi^3\kappa\tilde{A}_{22}\tilde{l}_1^2 + \frac{1}{8}\pi^3\kappa\tilde{A}_{22}\tilde{l}_2^2; \tag{A-43}$$

$$\Upsilon_5 = \frac{4}{15}\frac{\pi^2 \tilde{A}_{22}\tilde{l}_1^2}{\vartheta}. \tag{A-44}$$

The coefficients of equation of are defined as

$$\Gamma_1 = \frac{\Pi_9}{\Pi_1}; \tag{A-45}$$

$$\Gamma_2 = \frac{\Pi_2}{\Pi_1} + \frac{\Pi_3(\Lambda_4\Upsilon_5 - \Lambda_5\Upsilon_3)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)} - \frac{\Pi_4(\Lambda_3\Upsilon_5 - \Lambda_5\Upsilon_4)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)}; \tag{A-46}$$

$$\Gamma_3 = \frac{\Pi_7}{\Pi_1} + \frac{\Pi_3(\Lambda_4\Upsilon_6 - \Lambda_6\Upsilon_3)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)} - \frac{\Pi_4(\Lambda_3\Upsilon_6 - \Lambda_6\Upsilon_4)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)} + \\ + \frac{\Pi_5(\Lambda_4\Upsilon_5 - \Lambda_5\Upsilon_3)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)} - \frac{\Pi_6(\Lambda_3\Upsilon_5 - \Lambda_5\Upsilon_4)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)}; \tag{A-47}$$

$$\Gamma_4 = \frac{\Pi_8}{\Pi_1} + \frac{\Pi_5(\Lambda_4\Upsilon_6 - \Lambda_6\Upsilon_3)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)} - \frac{\Pi_6(\Lambda_3\Upsilon_6 - \Lambda_6\Upsilon_4)}{\Pi_1(\Lambda_3\Upsilon_3 - \Lambda_4\Upsilon_4)}; \tag{A-48}$$

$$\tilde{f} = \frac{\hat{f}}{\Pi_1}. \tag{A-49}$$

РЕЗЮМЕ. Розглянуто нелінійні змушені коливання викривленої мікробалки, яка лежить на нелінійній основі. Рівняння руху отримані на основі принципу Гамільтона і модифікованої теорії градієнтів деформації. що дає змогу вивчити ефекти розміру в мікроструктурі. Нелінійні рівняння руху з частинними похідними зведено до залежного від часу рівняння, яке містить квадратично і кубічно нелінійні члени. Визначено частотну характеристику викривленої балки для першого резонансу, для чого використано метод багатомасштабних у часі збурень. З прикладної точки зору, криві частотної характеристики можуть бути корисними для вибору оптимальних значень параметрів при проектуванні. Проілюстровано вплив геометричних параметрів модулів основи на коливання викривленої мікробалки.

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