KINETICS OF THE ELECTRON DISTRIBUTION FUNCTION FORMATION AND ITS SELF-SIMILARITY IN THE COURSE OF HEATING

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In many important cases one should treat plasma transport kinetically. In more general sense the solutions of the Landau-Fokker-Planck (LFP) equation, which is one of the key ingredient of plasma kinetic equation, have much broader interest ranging from plasma physics to stellar dynamics. We examine the interplay of the non-linear collisional operator with the different heating terms: mono kinetic; hot ions, DC electrical field and a quasi-linear diffusion operator that models interaction of RF waves with a plasma. Throughout the paper obtained analytical asymptotic results are confirmed with high accuracy by the numerical computing of the non-linear kinetic equation and vice versa.

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1. PRELIMINARIES

In many important cases one should treat plasma transport kinetically. Examples are: the electron heat transport in inertial confinement fusion; propagation of the heat bursts, caused by edge-localized mode (ELM), into scrape-off layer (SOL) of tokamaks. In more general sense the solutions of the Landau-Fokker-Planck (LFP) equation, which is one of the key ingredient of plasma kinetic equation, have much broader interest ranging from plasma physics to stellar dynamics (e.g., see [1-4] and the references therein). In certain cases, it worthwhile to analyse simpler models of plasma transport problems, solution of which, nevertheless, exhibit some important features of the problem of interest and also help to benchmark complex kinetic codes.

The non-linear LFP equation reads

$$\frac{1}{\Gamma}\frac{\partial f_\alpha}{\partial t} = -\frac{\partial}{\partial v_i}\left\{ f_\alpha \frac{\partial h_\alpha}{\partial v_i} + \frac{1}{2}\frac{\partial}{\partial v_j}\left(f_\alpha \frac{\partial^2 g_\alpha}{\partial v_i \partial v_j}\right)\right\};$$

$$h_\alpha = \sum_\beta \left(1 + m_\alpha / m_\beta\right) \int dw\, f_\beta(w,t)\left|v-w\right|^{-1},$$

$$g_\alpha = \int dw\, f_\beta(w,t)\left|v-w\right|, \quad \Gamma = 2\pi e^4\, L / m^2,$$

where L is the Coulomb logarithm, $\alpha = e, i$.

A complexity of the LFP equation makes it almost impossible to use analytical methods for applied problems. Therefore the role of numerical methods for this equation is very important. The well-known specific property of nonlinear kinetic equations is that a single equation has several conservation laws (mass, energy, momentum). In case of isotropic LFP equation this means that equation can be written in two equivalent divergent forms: conservation of mass and conservation of energy. We have used implicit schemes with iterations that guaranteed energy conservation with high accuracy (till the order of machine errors, if necessary). For linear equation when particle exchange energy with the bulk plasma the only collision invariant is particle mass (density).

We consider the isotropic electron distribution $f(v,t)$ and study the electron heating with a simple model: $\partial f / \partial t = I(f,f) + H(f)$, where $I(f,f)$ is the LFP collisional integral and $H(f)$ is the heating source. We examine the interplay of the non-linear collisional operator with the different heating terms: mono kinetic distribution (the energy (particle) source and sink can be provided by ion beams, neutral injection, etc.); hot ions (two component plasma), and a quasi-linear diffusion operator that models interaction of RF waves with a plasma. We shortly review some selected results of our works on the subject with a heating source localized in the velocity space. Then a broader class of the heating terms resulting in enhancement of the tail of the distribution function is analytically analyzed. Analytical treatment of the nonlinear kinetic equation usually deals with rigorous simplifications (linearization of the equation, taking into account small parameters, such as mass ratio $\rho = m_e / m_i << 1$, etc.). Throughout the paper obtained analytical asymptotic results are confirmed with high accuracy by the numerical computing of the non-linear kinetic equation and vice versa.

The investigation is mainly concentrated on the evolution of the distribution function tails in high velocity region $v \to \infty, t \to \infty$. To characterize the tail formation we use the following presentation of the distribution where $f(\xi,t) = G(\xi,t)\cdot f_M(\xi)$, where $\xi = v^2 / v_{th}^2$, v_{th} is the thermal velocity, and Maxwell distribution is $f_M \approx e^{-\xi}$. It is known that the Coulomb diffusion influence is the utmost in the cold energetic region $0 \le \xi < 1$. In the high-velocity region $\xi >> 1$ the LFP non-linear parabolic equation degenerates because of known Rutherford cross-section dependence on velocity and acquires more pronounced hyperbolic type when the transport term (the first derivative) becomes more important $I(f) \sim \xi^{-1/2}(f_\xi + f)_\xi$. This circumstance leads to inevitable retarding of the distribution tail formation comparing to the relaxation in the thermal velocity region $\xi \sim 1, t = \tilde{t} / t_e \sim 1$, t_e – the collision time.

The particle density n and temperature T (average energy $e(t)$) expressed in energetic units are defined through integrals

$$\int_0^\infty d\xi\, \xi^{1/2} f(\xi,t) = 1, \quad \int_0^\infty d\xi\, \xi^{3/2} f(\xi,t) = e(t).$$

In the absence of energy (particle) sources the density and the thermal velocity (energy) are constant. The unique equilibrium solution of the problem is the Maxwell distribution function. In this case the following results were obtained from asymptotic analysis of the distribution function behavior in the region $\xi >> 1$ and for time intervals larger than collision time $t >> 1$. The initial function is located in the thermal velocity region and equals zero in high velocity region. The time period when the relaxation in the bulk of the distribution is finished is characterized by $G(\xi, t) \sim 1$ (Fig.1). Asymptotic analysis shows that $G(\xi, t)$ for $\xi, t >> 1$ can be analytically expressed in terms of error function, has a character of a propagating wave, which front moves under the law $\xi_f = (3t)^{2/3}$ with the constant front width $\Delta_f(t) = \sqrt{\pi}$ and can be roughly estimated as $\sim \exp(-\xi^{5/2})$.

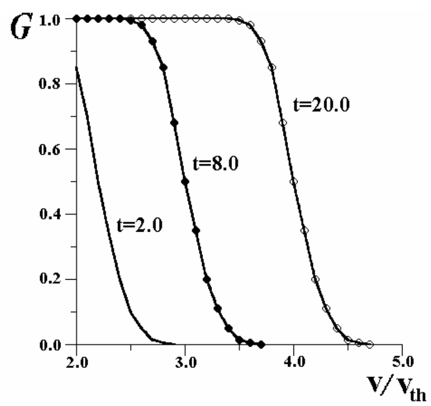

Fig.1. The graph of the function G(v,t) for different time instants

2. HEATING LOCALIZED IN THE VELOCITY SPACE

2.1. HEATING BY THE SOURCE OF HOT ELECTRONS

Let us include into consideration the heating of the electrons by the source of the hot particles $H(f)$, which is localized in a high energetic region $\xi_+ >> 1$. In a cold region the distribution is supposed Maxwellian. The next simplified assumption is that the heating source has small intensity $\sigma << 1$, so the density and the energy of the system practically do not undergo noticeable changes during the process under consideration. Then asymptotic analysis shows that if the source is localized at $\xi \sim \log(1/\sigma)$ then for the time period $1 << t << (\ln(1/\sigma))^{-1}$ a non-equilibrium quasi steady-state local distribution is formed. It is located inside the momentum interval between the energy (particle) source and the bulk of the particle distribution and reads

$$f(\xi) \approx f_M(\xi) + \int_{-\infty}^{\xi} dx x^{1/2} H(x).$$

This non-equilibrium distribution has the form of plateau (Fig.2). In general the functional dependence of the quasi steady-state electron distribution is insensitive to the extent to which the source and sink are located in momentum space or the sink if any is localized at ξ_- and $\xi_+ < \xi_-$. If, in particular, the source and the sink are mono kinetic distributions (like δ-type functions)

$$H(f) \cong \sigma \cdot \xi^{-1/2} \cdot [\delta(\xi - \xi_+) - \delta(\xi - \xi_-)],$$

then $f(\xi) = C \cdot e^{-\xi} + \sigma \cdot \Delta(\xi)$, where

$$\Delta(\xi) = \eta[\xi_+ - \xi] + \eta[(\xi - \xi_+)] \cdot \exp[-(\xi - \xi_+)]$$
$$- \eta[\xi_- - \xi] + \eta[(\xi - \xi_-)] \cdot \exp[-(\xi - \xi_-)]$$

and $\eta[y]$ is a unit function. The non-equilibrium distribution may differ from the equilibrium by tens of orders in magnitude. Such extended knees of the distribution functions can be observed in laboratory experiments (additional heating in tokamaks, afterglow gas discharge with metastable atoms), as well in magnetoshperic plasmas.

However, the distribution tail is under heated in comparison to the Maxwellian. Its formation is described by aforesaid formulas in the region $\xi >> \xi_+$, $t >> 1$ having wave-like character.

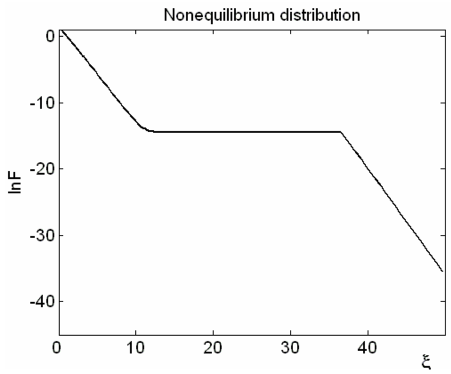

Fig.2. Logarithm of the steady-state non equilibrium distribution function vs. squared velocity (arbitrary units). The computations were carried out for the source located at $\xi = 36$ for theintensity $\sigma = 10^{-6}$

2.2. HEATING BY THE SOURCE OF HOT IONS

Now let pass to the classical problem of electron and ion temperature relaxation considering a system of electron and ion LFP equations, with $H(f) = 0$, $T_e(t) + T_i(t) = 2T_{eq}$. We use two asymptotic parameters $\rho = m_e / m_i << 1$ and $\varepsilon(t) = \rho T_i / T_e << 1$ for analytical analysis. We introduce the self-similar variables $f_\alpha(\xi, t) \cong v_{th}^{-3} f(v^2 / v_\alpha^2, t)$ then the equation for the electron function reads:

$$T_e^{3/2} \frac{df_e}{dt} = \frac{1}{\xi^{1/2}} \frac{\partial}{\partial \xi} \Big[\frac{1}{\rho} I(f_e, f_e) + \left(\frac{T_i}{T_e} \frac{\partial f_e}{\partial \xi} + f_e \right) + T_e^{1/2} \frac{dT_e}{dt} \xi^{3/2} f_e \Big].$$

For the temperature changing we have

$$T_e^{3/2} \frac{dT_e}{dt} = \frac{2}{3} \left[T_i f_e(0, t) - T_e \int_0^{\infty} d\xi \, f_e(\xi, t) \right].$$

The remarkable property of the above equation is that the electron distribution is independent of the ion distribution and depends only on the ion energy. $T_i > T_e$ The hot ions interact primarily with cold electrons at $\xi \approx 0$. From asymptotic analysis (singular perturbation theory) it is obtained that the perturbation of the electron distribution function in a cold region has a character of a boundary layer with a width of $\sim \varepsilon(t)^{2/3}$. The electron function achieves its maximum absolute deviation from its local (in time) Maxwell distribution at $\xi \cong 0$:

$$f(\xi,t) \cong e^{-\xi} \cdot [1 + 2.9\,\varepsilon^{2/3}(T_e - T_i)/T_i].$$

The applicability condition of the known formula for temperatures $T_e^{/2}\, T_t \cong C(T_i - T_e)$ is $\varepsilon << 1$ that is hundred times less rigorous than the condition $\rho << 1$. From numerical simulation the condition is estimated as $\varepsilon \le 0.1$. Note, from the formula the time dependence of the temperature is $T \sim t^{2/3}$. The relative deviation of the electron distribution $G(\xi,t) = f_e(v,t)/f_M(v,t)$ from the equilibrium is much larger in the tail region. It has a wave character with a stable front, which propagates in high velocity region being described by the above formulas in the case without heating. For hot electrons $T_e > T_i$ the electron tail is cooling while relaxation in the high energetic region having the same character of a propagating wave with $v_f^3 \sim t$ (Fig.3).

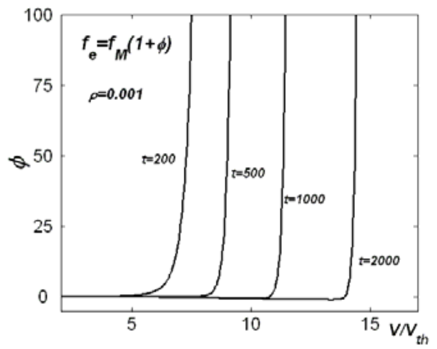

Fig.3. The graph of the deviation of the electron function φ for different time instants

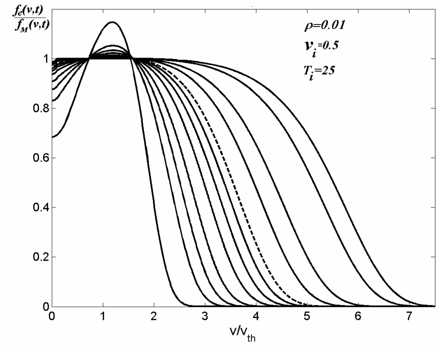

Fig.4. The graph of the function G(v,t) for different time instants

Now we consider the example when the initial electron and ion distribution functions are $\delta(v-1)$ and $\delta(v-0.5)$, correspondingly, and the ion function is constant in time. For this case the plots demonstrate the comparison of the numerical simulation results with analytical results. First, the spreading of the function G in high velocity region is presented (Fig.4). From the time, when the parameter $\varepsilon \le 0.05$ the curves do not change their slope. Then the dependence of the energy, the wave front velocity and the width front on time is given (Figs.5-7). As can be seen the analytical and numerical results are a good match.

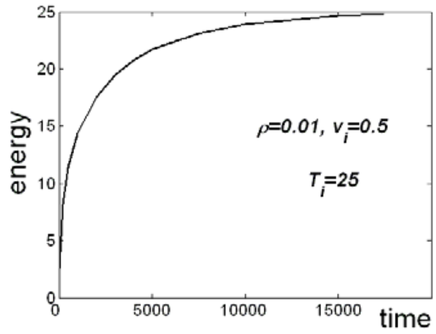

Fig.5. The electron energy dependence on time

The deviation of the width front at the beginning is understandable: the system "remembers the initial state". Another example considered here is interaction of RF waves with plasma that is simulated by the quasi-linear operator (usually 2D in velocity space) acting within corresponding resonant region

$$H(f) = \xi^{-1/2}[D_{ql} \cdot f_\xi]_\xi, \quad D_{ql} \ne 0 \;\; if \;\; \xi_1 \le \xi \le \xi_2 .$$

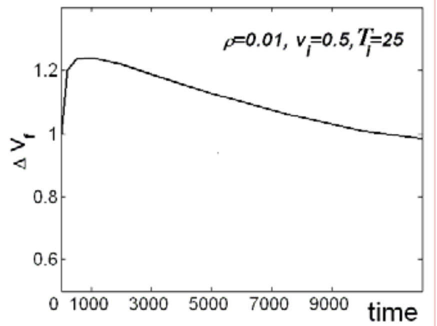

Fig.6. The plot of the front width on time

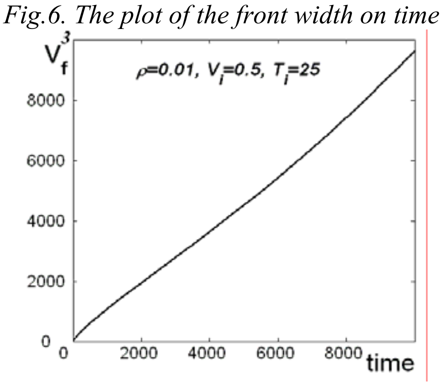

Fig.7. The wave front velocity dependence on time

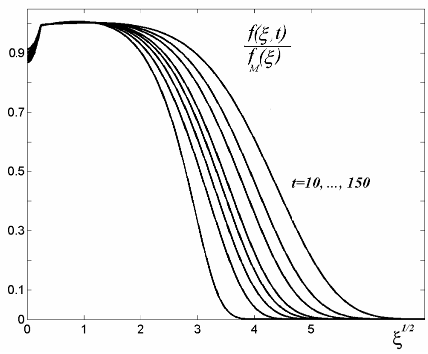

Fig.8. The function $G(\xi,t)$ for the different time moments in a case of the quasi-linear operator action within the energetic region $0.05 \le \xi \le 0.75$

For this case Fig.8 demonstrates the numerical result of the temporal tail formation that has a character of a propagating wave with the slope slightly changing in time because of constant heating.

The same behavior of the distribution function tails seems valid for any localized in the momentum space heat source having time dependence as $T(t) \sim t^{2/3}$.

2.3. HEATING AND ACCELERATION BY DC ELECTRICAL FIELD

The problem of runaway electrons is connected with the solution of the 2D in the velocity space LFP equation with the DC electrical field action. The influence of an electrical field which is small in comparison with that of Dreiser $\gamma = E/E_D << 1$ is taken into account as follows

$$\partial f / \partial t = I(f, f) + \gamma \cdot f / \partial v_z .$$

In the direction of the electrical field the distribution has the enhanced tail and it is depleted in the opposite direction so the density of particles is preserved. During the process of constant heating the thermal region of the distribution function is close to the Maxwell distribution because the parameter γ is small. Otherwise the maximum of the distribution corresponds to the velocity

$$\bar{v} = \int_{-1}^{1} d\mu \int_{0}^{\infty} dv v^3 \mu \cdot f(v, \mu) .$$

For the numerical simulation the boundary condition $f(v,t) \to 0$ for $v \to \infty$ is used, so that the numerical distribution function is equal to machine zero. Fig.9 shows the electron distribution function that formed under the DC field action when the initial temperature rises two times, $\gamma = 0.01$. The distribution has an accelerated tail in the electrical field direction up to the critical velocity $v_{cr} \cong \gamma^{-1/2}$ ($v_{cr}^2 / v_{th}^2 \approx 100$). Further the curve slightly changes the functional dependence. Thus even with the additional transport term in the LFP equation the tail of the electron distribution is under heated.

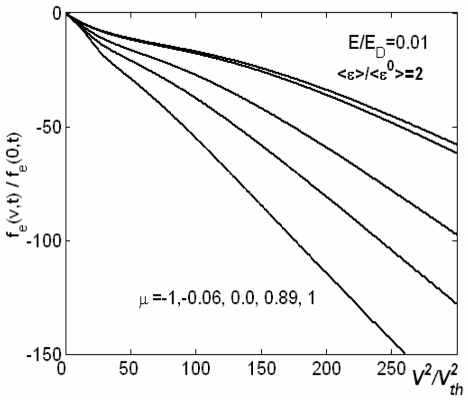

Fig.9. The graph of the electron distribution function for 2D case for different $\mu = \cos(\vec{v}\ \vec{E})$

3. SELF-SIMILAR SOLUTIONS WITH ACCELERATED TAILS

Unlike the cases considered above the situation changes drastically when coefficient of quasi-linear diffusion increases with velocity increasing. We consider a special class of functions $D_{ql} = D_0 \cdot \xi^{p-3/2} \cdot T(t)^{-1/2}$, where D_0 is the normalization constant and $5/2 > p \geq 0$ is an adjustable parameter, for which it is possible to construct a self-similar solution in high velocity region

$$f(\xi \to \infty) \sim \exp\left[-\frac{\xi^{5/2-p}}{5/2-p} \right].$$

Here the variable $\xi = v^2 / v_h^2$ is changing in time.

Two cases ought to be specified: for $p = 3/2$ the solution is the Maxwellian and for $p = 5/2$ we have a power-law tail $f \to \xi^{-5/2}$. The time dependence of the temperature is as the above $T(t) \sim t^{2/3}$.

In numerical modeling the initial distribution is approximated on the mesh in the usual way, that is, it differs from zero at one point. Very rapidly, the solution acquires a quasi equilibrium form in the thermal velocity region at the instant $t_e \sim 1$ that corresponds to the collision time. In this region the distribution functions are close to each other throughout the entire relaxation process for different coefficients of quasi-linear diffusion. The main difference is observed in the region of the distribution tails for $\xi >> 1$. For the case p=3/2 the solution is Maxwellian. Numerical results show that at the beginning the tail has Coulombian character and then since the time $t \sim 1/D_0$ it spreads into super thermal region following the diffusion action (Fig.10).

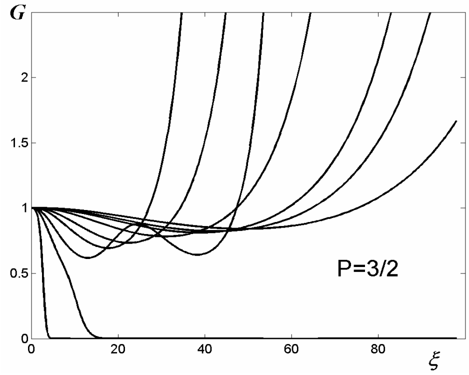

Fig.10. The graph of the function $G(\xi, t)$ *for the quasi-linear coefficient* $D_0 = 0.0015$ *for different time moments* $t = 10, 50, 100, ..., 1000$

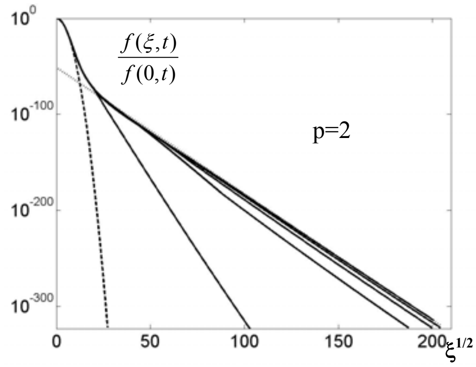

Fig.11. Temporal evolution of the electron distribution function for p=3/2. In agreement with analytic results, the distribution function approaches Maxwellian distribution (dotted line)

Fig.11 shows the logarithm of the distribution function $f(\xi,t)$ normalized on its value at zero velocity $f(0,t)$ for different time moments for $p=2$. It displays the transition region between the Maxwellian part and the enhanced tail. In the region $0 \le v \sim 4v_{th}$ the distribution is visibly close to Maxwellian. The obtained results can be used for the assessment of the impact of ELMs on heat transport and sheath parameters.

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КИНЕТИКА ФОРМИРОВАНИЯ ФУНКЦИИ РАСПРЕДЕЛЕНИЯ ЭЛЕКТРОНОВ И ЕЕ САМОПОДОБИЕ В ПРОЦЕССЕ НАГРЕВА

И.Ф. Потапенко

Во многих важных случаях перенос плазмы необходимо рассматривать кинетически. В более общем смысле решения уравнения Ландау-Фоккера-Планка, являющегося одним из ключевых компонентов кинетического уравнения плазмы, представляют более широкий интерес: от физики плазмы до динамики звезд. Мы исследуем взаимосвязь нелинейного столкновительного оператора с различными составляющими нагрева: однокомпонентная кинетика, горячие ионы, постоянное электрическое поле и квазилинейный диффузионный оператор, который моделирует взаимодействие ВЧ-волн с плазмой. Полученные в работе аналитические асимптотические результаты подтверждаются с высокой точностью численным моделированием нелинейного кинетического уравнения и наоборот.

КІНЕТИКА ФОРМУВАННЯ ФУНКЦІЇ РОЗПОДІЛУ ЕЛЕКТРОНІВ ТА ЇЇ САМОПОДІБНІСТЬ У ПРОЦЕСІ НАГРІВУ

І.Ф. Потапенко

У багатьох важливих випадках транспорт плазми треба розглядати кінетично. У більш загальному розумінні розв'язки рівняння Ландау-Фокера-Планка, яке є одним з ключових компонентів кінетичного рівняння плазми, становлять більш широку цікавість: від фізики плазми до динаміки зірок. Ми дослідимо взаємозв'язок нелінійного столкновительного оператора зіткнень з різними складовими нагріву: однокомпонентна кінетика, гарячі іони, постійне електричне поле та квазілінійний дифузійний оператор, який моделює взаємодію ВЧ-хвиль з плазмою. Отримані у роботі аналітичні асимптотичні результати підтверджуються з високою точністю числовим моделюванням нелінійного кінетичного рівняння та навпаки.