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CLOSED POLYNOMIALS AND SATURATED SUBALGEBRAS OF POLYNOMIAL ALGEBRAS

ЗАМКНЕНІ ПОЛІНОМИ ТА НАСИЧЕНІ ПІДАЛГЕБРИ ПОЛІНОМІАЛЬНИХ АЛГЕБР

The behavior of closed polynomials, i.e., polynomials $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ such that the subalgebra $\Bbbk[f]$ is integrally closed in $\Bbbk[x_1, \ldots, x_n]$, is studied under extensions of the ground field. Using some properties of closed polynomials, we prove that every polynomial $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ after shifting by constants can be factorized in a product of irreducible polynomials of the same degree. Some types of saturated subalgebras $A \subset \Bbbk[x_1, \ldots, x_n]$ are considered, i.e., such that for any $f \in A \setminus \Bbbk$ a generative polynomial of f is contained in A.

Досліджено поведінку замкнених поліномів, тобто таких поліномів $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$, що підалгебра $\Bbbk[f]$ є інтегрально замкненою в $\Bbbk[x_1, \ldots, x_n]$, у випадку розширень основного поля. З використанням деяких властивостей замкнених поліномів доведено, що кожен поліном $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ після зсувів на константи може бути розкладений у добуток незвідних поліномів одного й того ж степеня. Розглянуто деякі типи насичених підалгебр $A \subset \Bbbk[x_1, \ldots, x_n]$, тобто таких алгебр, що для будь-якого $f \in A \setminus \Bbbk$ породжуючий поліном для f міститься в A.

1. Introduction. Recall that a polynomial $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ is called *closed* if the subalgebra $\Bbbk[f]$ is integrally closed in $\Bbbk[x_1, \ldots, x_n]$. It turns out that a polynomial f is closed if and only if f is *non-composite*, i.e., f cannot be presented in the form $f = F(g)$ for some $g \in \Bbbk[x_1, \ldots, x_n]$ and $F(t) \in \Bbbk[t],\ \deg(F) > 1$. Because any polynomial in n variables can be obtained from a closed polynomial by taking a polynomial in one variable from it, the problem of studying closed polynomials is of interest. Besides, closed polynomials in two variables appear in a natural way as generators of rings of constants of non-zero derivations.

Let us go briefly through the content of the paper. In Section 2 we collect numerous characterizations of closed polynomials (Theorem 1). A major part of these characterizations is contained in the union of [1 – 4], etc, but some results seem to be new. In particular, implication (i) $\Rightarrow$ (iv) in Theorem 1 over any perfect field and Proposition 1 solve a problem stated in [1] (Section 8).

Define a generative polynomial h of a polynomial $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ as a closed polynomial such that $f = F(h)$ for some $F \in \Bbbk[t]$. Clearly, a generative polynomial exists for any f. Moreover, a generative polynomial is unique up to affine transformations (Corollary 1).

The above-mentioned results allow us to prove that over an algebraically closed field $\Bbbk$ for any $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ and for all but finite number $\mu \in \Bbbk$ the polynomial $f + \mu$ can be decomposed into a product $f + \mu = \alpha \cdot f_{1\mu} \cdot f_{2\mu} \ldots f_{k\mu}$, $\alpha \in \Bbbk^{\times}$, $k \geqslant 1$, of irreducible polynomials $f_{i\mu}$ of the same degree d not depending on μ and such that

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$f_{i\mu} - f_{j\mu} \in \mathbb{k}$, $i, j = 1, \ldots, k$ (Corollary 2). This result may be considered as an analogue of the Fundamental Theorem of Algebra for polynomials in many variables.

Moreover, Stein – Lorenzini – Najib's Inequality (Theorem 2) implies that the number of "exceptional" values of μ is less then $\deg(f)$. The same inequality gives an estimate of the number of irreducible factors in $f + \mu$ for exceptional μ, see Theorem 3.

Section 4 is devoted to saturated subalgebras $A \subset \mathbb{k}[x_1, \ldots, x_n]$, i.e., such that for any $f \in A \setminus \mathbb{k}$ a generative polynomial of f is contained in A. Clearly, any subalgebra that is integrally closed in $\mathbb{k}[x_1, \ldots, x_n]$ is saturated. On the other hand, it is known that for monomial subalgebras these two conditions are equivalent. In Theorem 4 we characterize subalgebras of invariants $A = \mathbb{k}[x_1, \ldots, x_n]^G$, where G is a finite group acting linearly on $\mathbb{k}[x_1, \ldots, x_n]$, with A being saturated. This result provides many examples of saturated homogeneous subalgebras that are not integrally closed in $\mathbb{k}[x_1, \ldots, x_n]$.

2. Characterizations of closed polynomials. Let $\mathbb{k}$ be an arbitrary field.

Proposition 1. *Let $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ and $\mathbb{k} \subset L$ be a separable extension of fields. Then f is closed over $\mathbb{k}$ if and only if f is closed over L.*

Proof. If $f = F(h)$ over $\mathbb{k}$, then the same decomposition holds over L.

Now assume that f is closed over $\mathbb{k}$. Consider an element $g \in L[x_1, \ldots, x_n]$ integral over $L[f]$. We shall prove that $g \in L[f]$. Since the number of non-zero coefficients of g is finite, we may assume that L is a finitely generated extension of $\mathbb{k}$. Then there exists a finite separable transcendence basis of L over $\mathbb{k}$, i.e., a finite set $\{\xi_1, \ldots, \xi_m\}$ of elements in L that are algebraically independent over $\mathbb{k}$ and L is a finite separable algebraic extension of $L_1 = \mathbb{k}(\xi_1, \ldots \xi_m)$.

Let us show that f is closed over L_1. The subalgebra $\mathbb{k}[f][\xi_1, \ldots, \xi_m]$ is integrally closed in $\mathbb{k}[x_1, \ldots, x_n][\xi_1, \ldots, \xi_m]$ [5] (Chapter V.1, Proposition 12). Let T be the set of all non-zero elements of $\mathbb{k}[\xi_1, \ldots, \xi_m]$. Then the localization $T^{-1}\mathbb{k}[f][\xi_1, \ldots, \xi_m]$ is integrally closed in $T^{-1}\mathbb{k}[x_1, \ldots, x_n][\xi_1, \ldots \xi_m]$ [5] (Chapter V.1, Proposition 16). This proves that $L_1[f]$ is integrally closed in $L_1[x_1, \ldots, x_n]$.

Fix a basis $\{\omega_1, \ldots, \omega_k\}$ of L over L_1. With any element $l \in L$ one may associate an L_1-linear operator $M(l)\colon L \to L$, $M(l)(\omega) = l\omega$. Let $tr(l)$ be the trace of this operator. It is known that there exists a basis $\{\omega_1^\star, \ldots, \omega_k^\star\}$ of L over L_1 such that $tr(\omega_i \omega_j^\star) = \delta_{ij}$ [5] (Chapter V.1.6). Assume that $g = \sum_i \omega_i a_i$ with $a_i \in L_1[x_1, \ldots, x_n]$. Any $\omega_j^\star$ is integral over L_1 and thus over $L_1[f]$. This shows that $g\omega_j^\star$ is integral over $L_1[f]$. Set $K = L_1(x_1, \ldots, x_n)$. The element $g\omega_j^\star$ determines a K-linear map $L \otimes_K K \to L \otimes_K K$, $b \to g\omega_j^\star b$. Since $g\omega_j^\star$ is integral over $L_1[f]$, the trace of this K-linear operator is also integral over $L_1[f]$ [5] (Chapter V.1.6). Note that $tr(g\omega_j^\star) = \sum_i a_i \, tr(\omega_i \omega_j^\star)$. On the other hand, the elements $\{\omega_1 \otimes 1, \ldots, \omega_k \otimes 1\}$ form a basis of $L \otimes_K K$ over K. Hence $\mathrm{tr}\,(\omega_i \omega_j^\star) = \delta_{ij}$ and $\mathrm{tr}\,(g\omega_j^\star) = a_j$ is integral over $L_1[f]$. This shows that $a_j \in L_1[f]$ for any j and thus $g \in L[f]$.

The proposition is proved.

Let $\mathcal{M}$ be the set of all subalgebras $\mathbb{k}[f]$, $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$, partially ordered by inclusion.

In the next Theorem various characterizations of closed polynomials are collected (see [1 – 4], etc). A new result here is the implication (i) $\Rightarrow$ (iv).

Theorem 1. *The following conditions on a polynomial $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ are equivalent*:

(i) *f is non-composite*;

(ii) *$\mathbb{k}[f]$ is a maximal element of $\mathcal{M}$*;

(iii) *f is closed*;

(iv) ($\mathbb{k}$ *is a perfect field*) *$f+\lambda$ is irreducible over $\overline{\mathbb{k}}$ for all but finitely many $\lambda\in\overline{\mathbb{k}}$*;

(v) ($\mathbb{k}$ *is a perfect field*) *there exists $\lambda\in\overline{\mathbb{k}}$ such that $f+\lambda$ is irreducible over $\overline{\mathbb{k}}$*;

(vi) ($\mathrm{char}\,\mathbb{k}=0$) *there exists a* (*finite*) *family of derivations $\{D_i\}$ of the algebra $\mathbb{k}[x_1,\ldots,x_n]$ such that $\mathbb{k}[f]=\cap_i\mathrm{Ker}\,D_i$.*

Proof. (i) $\Rightarrow$ (iv). Let us assume that $\mathbb{k}=\overline{\mathbb{k}}$. Consider a morphism $\phi\colon\mathbb{k}^n\to\mathbb{k}^1$, $\phi(x_1,\ldots,x_n)=f(x_1,\ldots,x_n)$. We should prove that all fibers of this morphism except for finitely many are irreducible. But it follows from the first Bertini theorem (see, for example, [6, p. 139]).

If a perfect field $\mathbb{k}$ is non-closed, then Proposition 1 shows that $f\in\mathbb{k}[x_1,\ldots,x_n]$ is closed over $\mathbb{k}$ implies that f is closed over $\overline{\mathbb{k}}$.

The theorem is proved.

Example **1** [1]. If the field $\mathbb{k}$ is not perfect, then we can not guarantee that a polynomial f which is closed over $\mathbb{k}$, will be closed over $\overline{\mathbb{k}}$ as well. Indeed, let $F=\mathbb{k}(\eta)$ with $\eta\notin\mathbb{k}$, $\eta^p\in\mathbb{k}$. The polynomial $f(x_1,x_2)=x_1^p+\eta^p x_2^p$ is closed over $\mathbb{k}$. However, one has a decomposition $f=(x_1+\eta x_2)^p$ over F. The same example works for (i) $\not\Rightarrow$ (iv) in this case.

Now we are going to show that a generative polynomial is unique up to affine tarnsformations. Here we need two auxiliary lemmas.

Lemma 1. *For any $f\in\mathbb{k}[x_1,\ldots,x_n]\setminus\mathbb{k}$, the integral closure A of $\mathbb{k}[f]$ in $\mathbb{k}[x_1,\ldots,x_n]$ has the form $A=\mathbb{k}[h]$ for some closed $h\in\mathbb{k}[x_1,\ldots,x_n]$.*

Proof. Since $\mathrm{tr.deg}_{\mathbb{k}}Q(A)=1$, we have by the theorem of Gordan (see for example [4, p. 15]) $Q(A)=\mathbb{k}(h)$ for some rational function h. The subfield $Q(A)$ contains non-constant polynomials, so by the theorem of E. Noether (see for example [4, p. 16]) the generator h of the subfield $Q(A)$ can be chosen as a polynomial. Note that $\mathbb{k}(h)\cap\mathbb{k}[x_1,\ldots,x_n]=\mathbb{k}[h]$ because any rational function (but polynomial) of a non-constant polynomial cannot be a polynomial. Therefore $A\subseteq\mathbb{k}[h]$. Since the element h is integral over A and A is integrally closed in $\mathbb{k}[x_1,\ldots,x_n]$, we have $h\in A$ and $A=\mathbb{k}[h]$.

The lemma is proved.

Note that in the case $char\,\mathbb{k}=0$ this lemma follows immediately from the result of Zaks [7].

Lemma 2. *Let $\mathbb{k}$ be a field. Polynomials $f,g\in\mathbb{k}[x_1,\ldots,x_n]\setminus\mathbb{k}$ are algebraically dependent* (*over $\mathbb{k}$*) *if and only if there exists a closed polynomial $h\in\mathbb{k}[x_1,\ldots,x_n]$ such that f, $g\in\mathbb{k}[h]$.*

Proof. Assume that f, g are algebraically dependent. By the Noether Normalization Lemma, there exists an element $r\in\mathbb{k}[f,g]$ such that $\mathbb{k}[r]\subset\mathbb{k}[f,g]$ is an integral extension. By Lemma 1, the integral closure of $\mathbb{k}[r]$ in $\mathbb{k}[x_1,\ldots,x_n]$ has a form $\mathbb{k}[h]$ for some closed polynomial h.

Conversely, if f, $g\in k[h]$ then these polynomials are obviously algebraically dependent.

The lemma is proved.

***Corollary* 1.** *Let $f \in \Bbbk[x_1,\ldots,x_n]\setminus\Bbbk$. The integral closure of the subalgebra $\Bbbk[f]$ in $\Bbbk[x_1,\ldots,x_n]$ coincides with $\Bbbk[h]$, where h is a generative polynomial of f. In particular, a generative polynomial of f exists and is unique up to affine transformations.*

3. A factorization theorem. Let us assume in this section that the ground field $\Bbbk$ is algebraically closed. Theorem 1 states that for a closed polynomial $h \in \Bbbk[x_1,\ldots,x_n]$ the polynomial $h+\lambda$ may be reducible only for finitely many $\lambda\in\Bbbk$. Denote by $E(h)$ the set of $\lambda\in\Bbbk$ such that $h+\lambda$ is reducible and by $e(h)$ the cardinality of this set. Stein's inequality claims that

$$e(h) < \deg h.$$

Now for any $\lambda\in\Bbbk$ consider a decomposition

$$h+\lambda = \prod_{i=1}^{n(\lambda,h)} h_{\lambda,i}^{d_{\lambda,i}}$$

with $h_{\lambda,i}$ being irreducible. A more precise version of Stein's inequality is given in the next theorem.

Theorem 2 [Stein – Lorenzini – Najib's inequality]**.** *Let $h \in \Bbbk[x_1,\ldots,x_n]$ be a closed polynomial. Then*

$$\sum_{\lambda}(n(\lambda,h)-1) < \min_{\lambda}\Big(\sum_i \deg(h_{\lambda,i})\Big).$$

This inequality has rather long history. Stein [8] proved his inequality in characteristic zero for $n=2$. For any n over $\Bbbk=\mathbb{C}$ this inequality was proved in [9]. In 1993, Lorenzini [10] obtained the inequality as in Theorem 2 in any characteristic, but only for $n=2$ (see also [11] and [12]). Finally, in [13] the proof for an arbitrary n was reduced to the case $n=2$.

Now take any $f\in\Bbbk[x_1,\ldots,x_n]\setminus\Bbbk$, $\mu\in\Bbbk$ and consider a decomposition

$$f+\mu = \alpha\cdot\prod_{i=1}^{n(\mu,f)} f_{\mu,i}^{d_{\mu,i}}$$

with $\alpha\in\Bbbk^\times$ and $f_{\mu,i}$ being irreducible.

Let us state the main result of this section.

Theorem 3. *Let $f\in\Bbbk[x_1,\ldots,x_n]\setminus\Bbbk$. There exists a finite subset $E(f) = \{\mu_1,\ldots,\mu_{e(f)} \mid \mu_i\in\Bbbk\}$ with $e(f)<\deg f$ such that*:

(1) *for any $\mu\notin E(f)$ one has $f+\mu=\alpha\cdot f_{\mu,1}\cdot f_{\mu,2}\ldots f_{\mu,k}$, where all $f_{\mu,i}$ are irreducible and $f_{\mu,i}-f_{\mu,j}\in\Bbbk$*;

(2) *$f_{\mu,i}-f_{\nu,j}\in\Bbbk^\times$ for any $\mu,\nu\notin E(f)$ with $\nu\neq\mu$; in particular, the degree $d=\deg(f_{\mu,i})$ does not depend on i and μ*;

(3) *$\deg(f_{\mu,i})\le d$ for any $\mu\in\Bbbk$*;

(4) $\sum_\mu \left(n(\mu,f)-\dfrac{\deg(f)}{d}\right) < \min\limits_{\mu}\left(\sum_{i=1}^{n(\mu,f)}\deg(f_{\mu,i})\right)$.

Proof. Let h be the generative polynomial of f and $f=F(h)$. Then

$$F(h)+\mu=\alpha\cdot(h+\lambda_{\mu,1})\ldots(h+\lambda_{\mu,k})$$

for some $\lambda_{\mu,1},\ldots,\lambda_{\mu,k}\in\Bbbk$. Hence for any μ with $\lambda_{\mu,1},\ldots,\lambda_{\mu,k}\notin E(h)$ we have a decomposition of $f+\mu$ as in (1). Note that $\lambda_{\mu,i}\neq\lambda_{\nu,j}$ for $\mu\neq\nu$. This proves (2) with $d=\deg(h)$ and gives the inequalities

$$e(f)\le e(h)<\deg(h)\le\deg(f).$$

Any $f_{\mu,i}$ is a divisor of some $h+\lambda$. This implies (3).

Finally, (4) may be obtained as:

$$\sum_{\mu}\left(n(\mu,f)-\frac{\deg(f)}{d}\right)\leq\sum_{\lambda}(n(\lambda,h)-1)<$$

$$<\min_{\lambda}\left(\sum_{i}\deg(h_{\lambda,i})\right)\leq\min_{\mu}\left(\sum_{j}\deg(f_{\mu,j})\right).$$

The theorem is proved.

***Remark* 1.** It follows from the proof of Theorem 3 that

$$E(f)=\{-F(-\lambda)\mid\lambda\in E(h)\};$$

if f is not closed, then $e(f)<\frac{1}{2}\deg(f)$.

***Corollary* 2.** *Let $f\in\Bbbk[x_1,\ldots,x_n]\setminus\Bbbk$. Then for all but finite number $\mu\in\Bbbk$, the polynomial $f+\mu$ can be decomposed into the product*

$$f+\mu=\alpha\cdot f_{1\mu}\cdot f_{2\mu}\ldots f_{k\mu},\qquad\alpha\in\Bbbk^{\times},\quad k\geqslant 1,$$

of irreducible polynomials $f_{i\mu}$ of the same degree d not depending on the number μ and such that $f_{i\mu}-f_{j\mu}\in\Bbbk$, $i,j=1,\ldots,k$. The number of exceptional μ's for which such a decomposition does not exist is at most $\deg f-1$.

***Example* 2.** Take $f(x_1,x_2)=x_1^2x_2^4-2x_1^2x_2^3+x_1^2x_2^2+2x_1x_2^3-2x_1x_2^2+x_2^2+1$.

Here $h=x_1x_2(x_2-1)+x_2$ and $F(t)=t^2+1$. It is easy to check that $E(h)=$ $=\{0,-1\}$, thus $E(f)=\{-1,-2\}$. We have decompositions:

$\mu=-1\colon f-1=x_2^2(x_1x_2-x_1+1)^2$;

$\mu=-2\colon f-2=(x_2-1)(x_1x_2+1)(x_1x_2(x_2-1)+x_2+1)$;

$\mu\neq-1,-2\colon f+\mu=\big(x_1x_2(x_2-1)+x_2+\lambda\big)\big(x_1x_2(x_2-1)+x_2-\lambda\big)$, $\lambda^2=-1-\mu$.

In this case $\deg(f)=6$, $d=3$, $\sum_{\mu}(n(\mu,f)-2)=1$ and

$$\min_{\mu}\left(\sum_{i}\deg(f_{\mu,i})\right)=\min\{3,6,6\}=3.$$

4. Saturated subalgebras and invariants of finite groups. Let $\Bbbk$ be a field.

Definition 1. *A subalgebra $A\subseteq\Bbbk[x_1,\ldots,x_n]$ is said to be saturated if for any $f\in A\setminus\Bbbk$ the generative polynomial of f is contained in A.*

Clearly, the intersection of a family of saturated subalgebras in $\Bbbk[x_1,\ldots,x_n]$ is again a saturated subalgebra. So we may define *the saturation* $S(A)$ of a subalgebra A as the minimal saturated subalgebra containing A.

If A is integrally closed in $\Bbbk[x_1,\ldots,x_n]$, then A is saturated. By Theorem 1, if $A=\Bbbk[f]$, then the converse is true. Moreover, the converse is true if A is a monomial subalgebra. In order to prove it, consider a submonoid $P(A)$ in $\mathbb{Z}^n_{\geq 0}$ consisting of multidegrees of all monomials in A. Then monomials corresponding to elements of the "saturated" semigroup $P'(A)=(\mathbb{Q}_{\geq 0}P(A))\cap\mathbb{Z}^n_{\geq 0}$ are generative elements of A. On the other hand, it is a basic fact of toric geometry that the monomial subalgebra corresponding to $P'(A)$ is integrally closed in $\Bbbk[x_1,\ldots,x_n]$, see for example [14] (Section 2.1).

Now we come from monomial to homogeneous saturated subalgebras. The degree of monomials $\deg(\alpha x_1^{i_1}\ldots x_n^{i_n})=i_1+\ldots+i_n$ defines a $\mathbb{Z}_{\geq 0}$-grading on the polynomial algebra $\Bbbk[x_1,\ldots,x_n]$. Recall that a subalgebra $A\subset\Bbbk[x_1,\ldots,x_n]$ is called *homogeneous* if for any element $a\in A$ all its homogeneous components belong to A.

Consider a subgroup $G\subset GL_n(\Bbbk)$. The linear action $G\colon \Bbbk[x_1,\ldots,x_n]\to\Bbbk[x_1,\ldots,x_n]$ determines the homogeneous subalgebra $\Bbbk[x_1,\ldots,x_n]^G$ of G-invariant polynomials.

Theorem 4. *Let $G\subseteq GL_n(\Bbbk)$ be a finite subgroup. The subalgebra $A=\Bbbk[x_1,\ldots,x_n]^G$ is saturated in $\Bbbk[x_1,\ldots,x_n]$ if and only if G admits no non-trivial homomorphisms $G\to\Bbbk^\times$.*

Proof. Assume that there is a non-trivial homomorphism $\phi\colon G\to\Bbbk^\times$. Let G_ϕ be the kernel of ϕ and $G^\phi=G/G_\phi$. Then G^ϕ is a finite cyclic group of some order k and it may be identified with a subgroup of $\Bbbk^\times$.

Lemma 3. *Let H be a cyclic subgroup of order k in $\Bbbk^\times$. Then any finite dimensional (over $\Bbbk$) H-module W is a direct sum of one-dimensional submodules.*

Proof. The polynomial X^k-1 annihilates the linear operator P in $GL(W)$ corresponding to a generator of H. By assumption, X^k-1 is a product of k non-proportional linear factors in $\Bbbk[X]$. This shows that the operator P is diagonalizable.

Lemma 4. *Let $H\subset G$ be a proper subgroup. Then $\Bbbk[x_1,\ldots,x_n]^H\neq\Bbbk[x_1,\ldots,x_n]^G$.*

Proof. Let K be a field and G a finite group of its automorphisms. By Artin's Theorem [15] (Section 2.1, Theorem 1.8), $K^G\subset K$ is a Galois extension and $[K\colon K^G]=|G|$. This implies $\Bbbk(x_1,\ldots,x_n)^H\neq\Bbbk(x_1,\ldots,x_n)^G$. The implication

$$\frac{f}{h}\in\Bbbk(x_1,\ldots,x_n)^G \implies \frac{f\prod_{g\in G, g\neq e} g\cdot f}{h\prod_{g\in G, g\neq e} g\cdot f}\in\Bbbk(x_1,\ldots,x_n)^G$$

shows that $\Bbbk(x_1,\ldots,x_n)^G$ (resp. $\Bbbk(x_1,\ldots,x_n)^H$) is the quotient field of $\Bbbk[x_1,\ldots,x_n]^G$ (resp. $\Bbbk[x_1,\ldots,x_n]^H$), thus $\Bbbk[x_1,\ldots,x_n]^H\neq\Bbbk[x_1,\ldots,x_n]^G$.

The lemma is proved.

Now we may take a finite-dimensional G-submodule $W\subset\Bbbk[x_1,\ldots,x_n]^{G_\phi}$ which is not contained in $\Bbbk[x_1,\ldots,x_n]^G$. Then W is a G^ϕ-module. By Lemma 3, one may find a G^ϕ-eigenvector $h\in W$, $h\notin\Bbbk[x_1,\ldots,x_n]^G$. Then $h^k\in\Bbbk[x_1,\ldots,x_n]^G$ and $\Bbbk[x_1,\ldots,x_n]^G$ is not saturated.

Conversely, assume that any homomorphism $\chi\colon G\to\Bbbk$ is trivial. If h is a generative element of a polynomial $f\in\Bbbk[x_1,\ldots,x_n]^G$, then for any $g\in G$ the element $g\cdot h$ is also a generative element of f. By Corollary 1, the generative element is unique up to affine transformation. Without loss of generality we can assume that the constant term of h is zero. Then the element $g\cdot h$ has obviously zero constant term and by Corollary 1 this element is proportional to h for any $g\in G$. Thus G acts on the line $\langle h\rangle$ via some character. But any character of G is trivial, so $h\in\Bbbk[x_1,\ldots,x_n]^G$, and $\Bbbk[x_1,\ldots,x_n]^G$ is saturated.

The theorem is proved.

***Remark* 2.** Since all coefficients of the polynomial

$$F_f(T)=\prod_{g\in G}(T-g\cdot f)$$

are in $\Bbbk[x_1,\ldots,x_n]^G$, any element $f\in\Bbbk[x_1,\ldots,x_n]$ is integral over $\Bbbk[x_1,\ldots,x_n]^G$. Thus Theorem 4 provides many saturated homogeneous subalgebras that are not integrally closed in $\Bbbk[x_1,\ldots,x_n]$.

***Corollary* 3.** *Assume that* $\Bbbk$ *is algebraically closed and* $char\,\Bbbk = 0$.

(1) *The subalgebra* $\Bbbk[x_1, \ldots, x_n]^G$ *is saturated in* $\Bbbk[x_1, \ldots, x_n]$ *if and only if* G *coincides with its commutant.*

(2) *The saturation of* $\Bbbk[x_1, \ldots, x_n]^G$ *is* $\Bbbk[x_1 \ldots, x_n]$ *if and only if* G *is solvable.*

***Example* 3.** In general, the saturation $S(A)$ is not generated by generative elements of elements of A. Indeed, take any field $\Bbbk$ that contains a primitive root of unit of degree six. Let $G = S_3$ be the permutation group acting naturally on $\Bbbk[x_1, x_2, x_3]$ and $A_3 \subset S_3$ be the alternating subgroup. The proof of Theorem 4 shows that any generative element of an S_3-invariant is an S_3-semiinvariant and thus belongs to $\Bbbk[x_1, x_2, x_3]^{A_3}$. On the other hand, $S(\Bbbk[x_1, x_2, x_3]^{S_3}) = \Bbbk[x_1, x_2, x_3]$.

***Example* 4.** It follows from Theorem 4 that the property of a subalgebra to be saturated is not preserved under field extensions. Let us give an explicit example of this effect.

Let $\Bbbk = \mathbb{R}$ and G be the cyclic group of order three acting on $\mathbb{R}^2$ by rotations. We begin with calculation of generators of the algebra of invariants $\mathbb{R}[x, y]^G$. Consider the complex polynomial algebra $\mathbb{C}[x, y] = \mathbb{R}[x, y] \oplus \mathrm{i}\mathbb{R}[x, y]$ with the natural G-action. Then $\mathbb{C}[x, y]^G = \mathbb{R}[x, y]^G \oplus \mathrm{i}\mathbb{R}[x, y]^G$. Put $z = x + \mathrm{i}y$, $\overline{z} = x - \mathrm{i}y$. Clearly, $\mathbb{C}[x, y] = \mathbb{C}[z, \overline{z}]$, and G acts on $z, \overline{z}$ as $z \to \epsilon z$, $\overline{z} \to \overline{\epsilon z}$, where $\epsilon^3 = 1$. This implies $\mathbb{C}[z, \overline{z}]^G = \mathbb{C}[f_1, f_2, f_3]$ with $f_1 = z^3$, $f_2 = \overline{z}^3$ and $f_3 = z\overline{z}$. Finally, $\mathbb{R}[x, y]^G = \mathbb{R}\big[\mathrm{Re}(f_i), \mathrm{Im}(f_i);\, i = 1, 2, 3\big] = \mathbb{R}\big[x^3 - 3xy^2,\, y^3 - 3x^2y,\, x^2 + y^2\big]$.

By Theorem 4, the subalgebra $\mathbb{R}[x, y]^G$ is saturated in $\mathbb{R}[x, y]$. On the other hand, the subalgebra $\mathbb{C}\big[x^3 - 3xy^2,\, y^3 - 3x^2y,\, x^2 + y^2\big]$ contains $x^3 - 3xy^2 + \mathrm{i}(y^3 - 3x^2y) = (x - \mathrm{i}y)^3$.

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