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ON SOME NONCOERCIVE VARIATIONAL INEQUALITIES*

ДЕЯКІ НЕКОЕРЦИТИВНІ ВАРІАЦІЙНІ НЕРІВНОСТІ

We study existence and regularity of solutions of noncoercive variational inequalities.

Вивчаються питання про існування та регулярність розв'язків некоерцитивних варіаційних нерівностей.

Introduction. In this paper we study existence and regularity of solutions of two variational inequalities that now we define. Let Ω_1 and Ω_2 be open sets of R^n with Ω_2 a bounded and connected C^0 set; let us suppose $\Omega_1 = \Omega_2$ or $\overline{\Omega}_2 \subset \Omega_1$. Moreover, let us assume $a \in L^\infty(\Omega_1)$ with $a \geq a_0$ almost everywhere (a. e.) on Ω_1 ($a_0 = \text{const} > 0$) and, for $l = 1, 2$, $A_l = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^l \frac{\partial}{\partial x_i}\right)$ being $a_{ij}^l \in L^\infty(\Omega_l)$ uniformly elliptic operator on Ω_l. Let us set $V_l = H_0^1(\Omega_l)$ or $V_l = H^1(\Omega_l)$ and let us denote by $\langle \cdot, \cdot \rangle_l$ the pairing between V_l' (dual of V_l) and V_l. Given $f_l \in V_l'$ and denoted

$$K = \{(v_1, v_2) \in V_1 \times V_2 : v_1 \geq v_2 \text{ a. e. on } \Omega_2\},$$

let us consider the following variational inequality

$$(u_1, u_2) \in K: \sum_{i,j=1}^{n} \int_{\Omega_1} a_{ij}^1 (u_1)_{x_i} (v_1 - u_1)_{x_j} dx + \int_{\Omega_1} a u_1 (v_1 - u_1) dx +$$

$$+ \sum_{i,j=1}^{n} \int_{\Omega_2} a_{ij}^2 (u_2)_{x_i} (v_2 - u_2)_{x_j} dx \geq \langle f_1, v_1 - u_1 \rangle_1 + \langle f_2, v_2 - u_2 \rangle_2 \quad \forall (v_1, v_2) \in K. \quad (1)$$

Let Ω be a bounded, connected, open C^0 set of R^2. Let T be a nontrivial triangle with vertices x^1, x^2, x^3 and $T \subset \Omega$. Let us denote by Γ_0 the side of triangle with extremes x^1 and x^2, and by Γ_i, $i = 1, 2$, the side with extremes x^i and x^3; let us set $\operatorname{int} \Gamma_0 = \Gamma_0 - \{x^1, x^2\}$, $\operatorname{int} \Gamma_i = \Gamma_i - \{x^i, x^3\}$ and let us suppose

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$\Gamma_0 \subset \{x \in R^2,\ x_2 = 0\}$, $T - \Gamma_0 \subset \{x \in R^2,\ x_2 > 0\}$. Moreover let us consider an open set $\Omega_0 \subset \overset{0}{T}$, where $\overset{0}{T}$ is topological interior of T, and the operators $A = \sum_{\substack{|r|=2\\|s|=2}} D^s(a_{rs}D^r)$, $B = \sum_{\substack{|i|=m\\|j|=m}} (-1)^m D^j(b_{ij}D^i)$, $m \in N$, under the assumptions

$$x^3 \in \partial\Omega_0, \quad \partial\Omega_0 \cap \Gamma_0 \neq \varnothing, \quad \partial\Omega_0 \cap \Gamma_0 \subset \operatorname{int}\Gamma_0, \quad \partial\Omega_0 \cap \operatorname{int}\Gamma_i = \varnothing \quad \text{for } i = 1, 2,$$

$$a_{rs} \in L^\infty(\Omega), \ \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r v D^s v\,dx \geq \alpha_0 \sum_{|s|=2} \int_\Omega \left|D^s v\right|^2 dx \ \ \forall v \in H^2(\Omega), \ \alpha_0 = \text{const} > 0,$$

$$b_{ij} \in L^\infty(\Omega), \ \sum_{\substack{|i|=m\\|j|=m}} \int_\Omega b_{ij} D^i v D^j v\,dx \geq \beta_0 \sum_{|i|=m} \int_\Omega \left|D^i v\right|^2 dx \ \ \forall v \in H^m(\Omega), \ \beta_0 = \text{const} > 0.$$

Given $f \in (H^2(\Omega))'$, $b \in L^\infty(\Omega)$ with $b \geq b_0$ a. e. on Ω ($b_0 = \text{const} > 0$) and denoted by

$$K = \Big\{(v_1, v_2) \in H^2(\Omega) \times H^m(\Omega) \colon v_1 \leq v_2 \text{ on } \Omega_0 \text{ (a.e. if } m = 1), \ v_1 \leq v_2 \text{ on } \Gamma_0 \text{ (in the trace sense if } m = 1)\Big\},$$

let us consider the following variational inequality

$$(u_1, u_2) \in K\colon \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s (v_1 - u_1)\,dx + \sum_{\substack{|i|=m\\|j|=m}} \int_\Omega b_{ij} D^i u_2 D^j (v_2 - u_2)\,dx +$$
$$+ \int_\Omega b u_2 (v_2 - u_2)\,dx \ \geq \ \langle f, v_1 - u_1 \rangle \quad \forall (v_1, v_2) \in K, \tag{2}$$

$\langle \cdot, \cdot \rangle$ being the pairing between $(H^2(\Omega))'$ and $H^2(\Omega)$.

We observe that the inequality (1) with $V_2 = H^1(\Omega_2)$ and the inequality (2) fall in the so-called class of the noncoercive variational inequalities which was studied, for example, in [1] and more recently in [2–5]. The mentioned authors obtained existence theorems for noncoercive variational inequalities that we can apply to our problems only in few particular cases, that we shall immediately point out.

The inequality (1) was studied in [6] but only when $\Omega_1 = \Omega_2 = \Omega$, $V_1 = V_2 = H_0^1(\Omega)$; our situation is more general and it needs to make significant changes in technique used in [6] and the result obtained there is included in ours. About (1), found the necessary condition for its solvability, we analyse the case when there is uniqueness of solution and we study its H^2-regularity (Theorem 1). The following Theorem 2 characterises the solution of the (1).

About inequality (2), that is absolutely new in literature, found the necessary conditions for the existence of a solution, we establish that under suitable hypotheses on the data they are also sufficient except when $\langle f, 1 \rangle > 0$ and

$$x^0 = \left(\frac{\langle f, x_1 \rangle}{\langle f, 1 \rangle}, \frac{\langle f, x_2 \rangle}{\langle f, 1 \rangle} \right) \in \Gamma_i \quad i = 1, 2,$$

when (2) has no solution for $m = 1$ (Theorem 4). Except the case when $\langle f, 1 \rangle > 0$ and $x^0 \in \overset{0}{T}$, where only the existence of solution is guaranteed (Theorem 3), in the other cases the solutions of (2) are infinite and some classes of these are obtained using additional solvable variational equations and inequalities (Theorems 5, 6, 9).

Particularly when $\langle f,1\rangle > 0$ and $x^0 \in \operatorname{int}\Gamma_0$ we recur to the variational inequality.

$$(u_1,u_2)\in K_0:\ \sum_{\substack{|r|=2\\|s|=2}}\int_\Omega a_{rs}D^r u_1 D^s(v_1-u_1)\,dx + \sum_{\substack{|i|=m\\|j|=m}}\int_\Omega b_{ij}D^i u_2 D^j(v_2-u_2)\,dx +$$
$$+\int_\Omega b u_2(v_2-u_2)\,dx \ \geq\ \langle f, v_1-u_1\rangle \quad \forall (v_1,v_2)\in K_0, \tag{3}$$

where $K_0 = \{(v_1,v_2)\in H^2(\Omega)\times H^m(\Omega):\ v_1\leq v_2 \text{ on } \Gamma_0$ (in the trace sense if $m=1$)$\}$.

About the inequality above, let us note that when $m>1$ it is easy to find solutions of (2) starting from a solution (u_1,u_2) of (3), thanks to the local liptschitzianity of u_1 and u_2; this property fails for u_2 when $m=1$. We get over this obstacle with Theorems 7 and 8 which, in different hypotheses, assure, among other things, that the difference u_2-u_1 is locally near every point of $\operatorname{int}\Gamma_0$ greater than a lipschitz function equal to zero on Γ_0.

1. In order to (1), first of all we note that if $V_2\equiv H_0^1(\Omega_2)$ there exists a unique solution [1] and if $V_2\equiv H^1(\Omega_2)$ the inequality $\langle f_2,1\rangle_2\geq 0$ is necessary condition so that (1) has a solution.

If $V_2\equiv H^1(\Omega_2)$ and $\langle f_2,1\rangle_2=0$, we consider the variational equations:

$$u\in V_1:\ \sum_{i,j=1}^{n}\int_{\Omega_1} a_{ij}^1 u_{x_i}v_{x_j}\,dx + \int_{\Omega_1} a u v\,dx = \langle f_1,v\rangle_1 \quad \forall v\in V_1, \tag{4}$$

$$u\in H^1(\Omega_2):\ \sum_{i,j=1}^{n}\int_{\Omega_2} a_{ij}^2 u_{x_i}v_{x_j}\,dx = \langle f_2,v\rangle_2 \quad \forall v\in H^1(\Omega_2). \tag{5}$$

The equation (4) has unique solution; the (5), in the above stated conditions about Ω_2, admits infinite solutions which are different two by two in a real constant. Let u_1 and u_2, respectively, be the solution of (4) and a solution of (5), it is trivial that (1) is solvable if and only if

$$\inf_{\Omega_2}(u_1-u_2) > -\infty \tag{6}$$

and that, if (6) is verified, all and only the solutions of (1) are the pairs

$$(u_1,\ u_2+c) \quad \text{with} \quad c\leq \inf_{\Omega_2}(u_1-u_2).$$

The (6) is, for example, verified when $n\in\{2,3\}$, $f_1\in L^2(\Omega_1)$, $f_2\in L^2(\Omega_2)$ and Ω_2 is $C^{1,1}$ set because [7] both u_1 and u_2 belong to the space $H^2(\Omega_2)$ and consequently they are continuous on $\overline{\Omega}_2$.

In the case $\langle f_2,1\rangle_2>0$, the (1) has unique solution; the existence of this solution is given by a theorem presented in [1, 8], the uniqueness is obvious. Following we will suppose $\langle f_2,1\rangle_2>0$, when $V_2\equiv H^1(\Omega_2)$. Let (u_1,u_2) be the solution of (1) and remarking the upper limitation

$$\|u_1\|_{H^1(\Omega_1)}+\|u_2\|_{H^1(\Omega_2)} \leq c\left(\|f_1\|_{V_1'}+\|f_2\|_{V_2'}+\|u_2\|_{L^2(\Omega_2)}\right) \quad \left(c=c(a_{ij}^l,a)\right), \tag{7}$$

we now show the following regularity theorem.

Theorem 1. *If Ω_2 is $C^{1,1}$ set, $a_{ij}^l\in C^{0,1}(\overline{\Omega}_l)$, $f_l\in L^2(\Omega_l)$, it follows that:*

α_1) *for any open set G with $\overline{G}\subset\Omega_2$, it results: $u_l\in H^2(G)$, $\|u_1\|_{H^2(G)}+$ $+\|u_2\|_{H^2(G)}\leq c\left(\|f_1\|_{L^2(\Omega_1)}+\|f_2\|_{L^2(\Omega_2)}+\|u_2\|_{L^2(\Omega_2)}\right)$ $(c=c(a_{ij}^l,a,G,\Omega_2))$;*

α_2) *if* $\overline{\Omega}_2 \subset \Omega_1$, *then* $u_1 \in H^2(\Omega_2) \cap H^2(G)$ *for every bounded open set* G *with* $\overline{G} \subset \Omega_1 - \Omega_2$, $u_2 \in H^2(\Omega_2)$, $\|u_1\|_{H^2(\Omega_2)} + \|u_1\|_{H^2(G)} + \|u_2\|_{H^2(\Omega_2)} \le$ $\le c\left(\|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|u_2\|_{L^2(\Omega_2)}\right)$ $(c = c(a_{ij}^l, a, G, \Omega_2))$;

α_3) *if* $\Omega_1 = \Omega_2 = \Omega$ *and at least one of the spaces* V_1 *and* V_2 *is* $H_0^1(\Omega)$, *then* $u_l \in H^2(\Omega)$, $\|u_1\|_{H^2(\Omega)} + \|u_2\|_{H^2(\Omega)} \le c\left(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\right)$ $(c = c(a_{ij}^l, a, \Omega))$;

α_4) *if* $\Omega_1 = \Omega_2 = \Omega$, $a_{ij}^1 = a_{ij}^2 = a_{ij}$ *and* $V_l = H^1(\Omega)$, *then* $u_l \in H^2(\Omega)$, $\|u_1\|_{H^2(\Omega)} + \|u_2\|_{H^2(\Omega)} \le c\left(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\right)$ $(c = c(a_{ij}, a, \Omega))$.

Proof. The property α_1) is a consequence of a result obtained in [9]. Let us show α_2) when $V_l = H^1(\Omega_l)$; similarly in the other cases. Taking into account the α_1) and since

$$u_1 \in H^2(G), \tag{8}$$

$$\|u_1\|_{H^2(G)} \le c\left(\|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|u_2\|_{L^2(\Omega_2)}\right) \quad (c = c(a_{ij}^l, a, G, \Omega_l)) \tag{9}$$

for any bounded open set G with $\overline{G} \subset \Omega_1 - \overline{\Omega}_2$. By virtue of equality $A_1 u_1 + a u_1 = f_1$ in the sense of $D'(\Omega_1 - \overline{\Omega}_2)$ and of inequality (7) restricted to (9), we have only to study the regularity of u_1 and u_2 in a neighbourhood of $\partial\Omega_2$.

Let $\bar{x}$ be a point of $\partial\Omega_2$. The assumption about Ω_2 gives the existence of an open neighbourhood U of $\bar{x}$, with $\overline{U} \subset \Omega_1$, of a sphere S_r of R^n, with the centre in the origin and radius r, and of an invertible $C^{1,1}$ application Φ of S_r on U, having $C^{1,1}$ inverse Φ^{-1} and unit Jacobian, such that $\Phi(\Sigma_r^+) = U^+$, $\Phi(\{y \in S_r : y_n = 0\}) = \partial\Omega_2 \cap U$, where $\Sigma_r^+ = \{y \in S_r : y_n > 0\}$, $U^+ = \Omega_2 \cap U$. Let us denote with $H(\Sigma_r^+)$ the closure with respect to the norm of $H^1(\Sigma_r)$ of the space of the functions $\varphi \in C^\infty(\overline{\Sigma}_r^+)$ satisfying the condition: $\exists r_\varphi \in]0, r[: \varphi(y) = 0$ for $|y| > r_\varphi$; let us suppose that for every $v \in H_0^1(S_r)$ [resp. $v \in H(\Sigma_r^+)$] $v \circ \Phi^{-1}$ is extended to zero over Ω_1 [resp. Ω_2] and let us put: $\tilde{u}_l = u_l \circ \Phi$, $\tilde{a} = a \circ \Phi$, $\tilde{f}_l = f_l \circ \Phi$, $\Sigma_r^- = \{y \in S_r : y_n < 0\}$,

$$K_1 = \left\{(v_1 + v_2) \in H_0^1(S_r) \times H(\Sigma_r^+) : u_1 + v_1 \circ \Phi^{-1} \ge u_2 + v_2 \circ \Phi^{-1} \text{ a. e. on } U^+\right\}.$$

Setting $x = \Phi(y)$, we get the equalities:

$$\sum_{i,j=1}^n \int_U a_{ij}^1 v_{x_i} w_{x_j} \, dx = \sum_{h,k=1}^n \int_{S_r} b_{hk}^1 (v \circ \Phi)_{y_h} (w \circ \Phi)_{y_k} \, dy \quad \forall v, w \in H^1(U),$$
$$\sum_{i,j=1}^n \int_{U^+} a_{ij}^2 v_{x_i} w_{x_j} \, dx = \sum_{h,k=1}^n \int_{\Sigma_r^+} b_{hk}^2 (v \circ \Phi)_{y_h} (w \circ \Phi)_{y_k} \, dy \quad \forall v, w \in H^1(U^+), \tag{10}$$

where b_{hk}^1 [resp. b_{hk}^2] is $C^{0,1}$ on $\overline{S}_r$ [resp. $\overline{\Sigma}_r^+$] and it depends of Φ and the functions a_{ij}^1 [resp. a_{ij}^2]; moreover the operator $B_l = -\sum_{h,k=1}^n \frac{\partial}{\partial y_k}\left(b_{hk}^l \frac{\partial}{\partial y_h}\right)$ is uniformly elliptic.

Let us observe

$$\sum_{h,k=1}^{n} \int_{S_r} b_{hk}^1 (\tilde{u}_1)_{y_h} (v_1)_{y_k} \, dy + \sum_{h,k=1}^{n} \int_{\Sigma_r^+} b_{hk}^2 (\tilde{u}_2)_{y_h} (v_2)_{y_k} \, dy \geq$$

$$\geq \int_{S_r} \tilde{f}_1 v_1 \, dy + \int_{\Sigma_r^+} \tilde{f}_2 v_2 \, dy - \int_{S_r} \tilde{a} \tilde{u}_1 v_1 \, dy \quad \forall (v_1, v_2) \in K_1. \tag{11}$$

If we choose $r' \in]0, r[$ and $\chi \in C_0^\infty(S_r)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $\overline{S}_{r'}$, for $h \in \{1,\ldots,n\}$ and $s \in \{1,\ldots,n-1\}$, we have:

$$(\chi \tilde{u}_1)_{y_h y_s} \in L^2(S_r), \qquad (\chi \tilde{u}_2)_{y_h y_s} \in L^2(\Sigma_r^+), \tag{12}$$

$$\left\| (\chi \tilde{u}_1)_{y_h y_s} \right\|_{L^2(S_r)} + \left\| (\chi \tilde{u}_2)_{y_h y_s} \right\|_{L^2(\Sigma_r^+)} \leq c \left(\| f_1 \|_{L^2(\Omega_1)} + \| f_2 \|_{L^2(\Omega_2)} + \| u_2 \|_{L^2(\Omega_2)} \right) \tag{13}$$

$$(c = c(a_{ij}^l, a, \chi, \Phi)).$$

Namely, considering the functions

$$D_s^t (\chi \tilde{u}_l)(y) = \frac{(\chi \tilde{u}_l)(y + t^s) - (\chi \tilde{u}_l)(y)}{t},$$

$$D_s^{-t} D_s^t (\chi \tilde{u}_l)(y) = \frac{(\chi \tilde{u}_l)(y + t^s) + (\chi \tilde{u}_l)(y - t^s) - 2(\chi \tilde{u}_l)(y)}{t^2},$$

where t is a nonzero real number with sufficiently small modulus and $t^s = (0,\ldots,\underset{s}{t},\ldots,0)$, observing that

$$D_s^t (\chi \tilde{u}_1) \in H_0^1(S_r), \qquad D_s^t (\chi \tilde{u}_2) \in H(\Sigma_r^+),$$

$$\left(\varepsilon \chi D_s^{-t} D_s^t (\chi \tilde{u}_1),\ \varepsilon \chi D_s^{-t} D_s^t (\chi \tilde{u}_2) \right) \in K_1 \quad \text{for} \quad 0 < \varepsilon < \frac{t^2}{2},$$

$$\sum_{h,k=1}^{n} \int_{S_r} b_{hk}^1 (D_s^t (\chi \tilde{u}_1))_{y_h} (D_s^t (\chi \tilde{u}_1))_{y_k} \, dy \leq c \| u_1 \|_{H^1(\Omega_1)} \left(\sum_{h=1}^{n} \left\| D_s^t \left((\chi \tilde{u}_1)_{y_h} \right) \right\|_{L^2(S_r)}^2 \right)^{1/2} +$$

$$+ \sum_{h,k=1}^{n} \int_{S_r} b_{hk}^1 (\tilde{u}_1)_{y_h} (-\chi D_s^{-t} D_s^t (\chi \tilde{u}_1))_{y_k} \, dy \qquad (c = c(a_{ij}^1, \Phi)),$$

$$\sum_{h,k=1}^{n} \int_{\Sigma_r^+} b_{hk}^2 (D_s^t (\chi \tilde{u}_2))_{y_h} (D_s^t (\chi \tilde{u}_2))_{y_k} \, dy \leq c \| u_2 \|_{H^1(\Omega_2)} \left(\sum_{h=1}^{n} \left\| D_s^t \left((\chi \tilde{u}_2)_{y_h} \right) \right\|_{L^2(\Sigma_r^+)}^2 \right)^{1/2} +$$

$$+ \sum_{h,k=1}^{n} \int_{\Sigma_r^+} b_{hk}^2 (\tilde{u}_2)_{y_h} (-\chi D_s^{-t} D_s^t (\chi \tilde{u}_2))_{y_k} \, dy \qquad (c = c(a_{ij}^2, \Phi)),$$

and using (7), (11), we come to the inequality

$$\left(\sum_{h=1}^{n} \left\| D_s^t \left((\chi \tilde{u}_1)_{y_h} \right) \right\|_{L^2(S_r)}^2 \right)^{1/2} + \left(\sum_{h=1}^{n} \left\| D_s^t \left((\chi \tilde{u}_2)_{y_h} \right) \right\|_{L^2(\Sigma_r^+)}^2 \right)^{1/2} \leq$$

$$\leq c\left(\| f_1 \|_{L^2(\Omega_1)} + \| f_2 \|_{L^2(\Omega_2)} + \| u_2 \|_{L^2(\Omega_2)}\right) \quad (c = c(a_{ij}^l, a, \chi, \Phi))$$

from which we obtain (12) and (13).

Let us show that

$$\chi \tilde{u}_1 \in H^2(\Sigma_r^-), \tag{14}$$

$$\chi \tilde{u}_1 \in H^2(\Sigma_r^+), \tag{15}$$

$$\chi \tilde{u}_2 \in H^2(\Sigma_r^+), \tag{16}$$

$$\| \chi \tilde{u}_1 \|_{H^2(\Sigma_r^+)} + \| \chi \tilde{u}_1 \|_{H^2(\Sigma_r^-)} + \| \chi \tilde{u}_2 \|_{H^2(\Sigma_r^+)} \leq$$

$$\leq c\left(\| f_1 \|_{L^2(\Omega_1)} + \| f_2 \|_{L^2(\Omega_2)} + \| u_2 \|_{L^2(\Omega_2)}\right) \quad (c = c(a_{ij}^l, a, \chi, \Phi)). \tag{17}$$

First of all, setting

$$\eta = \chi \circ \Phi^{-1}, \qquad K_2 = \left\{ (v_1, v_2) \in H_0^1(S_r) \times H(\Sigma_r^+) \colon v_1 > v_2 \text{ a.e. on } \Sigma_r^+ \right\},$$

$$F_l = \eta f_l - \sum_{i,j=1}^{n} \left[a_{ij}^l (u_l)_{x_i} \eta_{x_j} - (a_{ij}^l u_l \eta_{x_i})_{x_j} \right], \quad \tilde{F}_l = F_l \circ \Phi,$$

$$\nu = (\nu_1, \ldots, \nu_n) = \text{the unit outward normal vector to } \partial\Omega_2,$$

$$\sigma = \text{surface measure on } \partial\Omega_2,$$

taking into account (10), we have

$$\sum_{h,k=1}^{n} \int_{S_r} b_{hk}^1 (\chi \tilde{u}_1)_{y_h} (v_1 - \chi \tilde{u}_1)_{y_k} dy + \int_{S_r} \tilde{a} \chi \tilde{u}_1 (v_1 - \chi \tilde{u}_1) dy +$$

$$+ \sum_{h,k=1}^{n} \int_{\Sigma_r^+} b_{hk}^2 (\chi \tilde{u}_2)_{y_h} (v_2 - \chi \tilde{u}_2)_{y_k} dy - \sum_{i,j=1}^{n} \int_{\partial\Omega_2} a_{ij}^2 u_2 \eta_{x_i} (v_2 \circ \Phi^{-1} - \eta u_2) \nu_j \, d\sigma \geq$$

$$\geq \int_{S_r} \tilde{F}_1 (v_1 - \chi \tilde{u}_1) dy + \int_{\Sigma_r^+} \tilde{F}_2 (v_2 - \chi \tilde{u}_2) dy \qquad \forall (v_1, v_2) \in K_2; \tag{18}$$

from (18), owing α_1) and (8), it follows that:

$$\chi \tilde{u}_1 \in H^2_{\mathrm{loc}}(\Sigma_r^-), \quad B_1(\chi \tilde{u}_1) = \tilde{F}_1 - \tilde{a} \chi \tilde{u}_1 \quad \text{a.e. on } \Sigma_r^-, \tag{19}$$

$$\chi \tilde{u}_2 \in H^2_{\mathrm{loc}}(\Sigma_r^+), \quad B_2(\chi \tilde{u}_2) \leq \tilde{F}_2 \quad \text{a.e. on } \Sigma_r^+, \tag{20}$$

$$B_1(\chi \tilde{u}_1) + B_2(\chi \tilde{u}_2) = \tilde{F}_1 + \tilde{F}_2 - \tilde{a} \chi \tilde{u}_1, \quad \text{a.e. on } \Sigma_r^+. \tag{21}$$

The relation (19) and the first one of (12) give (14). From (20) we obtain the following relations

$$B_1(\chi \tilde{u}_2) \in L^2_{\mathrm{loc}}(\Sigma_r^+), \tag{22}$$

$$B_1(\chi \tilde{u}_2) \leq \frac{b_{nn}^1}{b_{nn}^2} \left[\tilde{F}_2 + \sum_{(h,k) \neq (n,n)} (b_{hk}^2 (\chi \tilde{u}_2)_{y_h})_{y_k} + (b_{nn}^2)_{y_n} (\chi \tilde{u}_2)_{y_n} \right] -$$

$$- (b_{nn}^1)_{y_n} (\chi \tilde{u}_2)_{y_n} - \sum_{(h,k) \neq (n,n)} (b_{hk}^1 (\chi \tilde{u}_2)_{y_h})_{y_k} \quad \text{a.e. on } \Sigma_r^+,$$

i.e.

$$(B_1(\chi \tilde{u}_2))^+ \in L^2(\Sigma_r^+) \tag{23}$$

because of the second of (12).

Extending $\chi \tilde{u}_1$, to zero over the half-space $y_n < 0$, let us consider the function

$$\psi(y) = \begin{cases} (\chi\tilde{u}_1)(y_1,\ldots,y_n) & \text{if} \quad y_n < 0; \\ \sum_{i=1}^{3} \lambda_i (\chi\tilde{u}_1)(y_1,\ldots,y_{n-1},-i\,y_n) & \text{if} \quad y_n > 0, \end{cases}$$

where $(\lambda_1, \lambda_2, \lambda_3) \in R^3$ is the solution of the system $\sum_{i=1}^{3} (-i)^j \lambda_i = 1 \quad \forall j \in \{0,1,2\}$. By virtue of (19) we have

$$\psi \in H_0^2(S_r), \tag{24}$$

$$\|\psi\|_{H_0^2(S_r)} \leq c\|\chi\tilde{u}_1\|_{H^2(\Sigma_r^-)} \quad (c = \text{const} > 0 \text{ independent on } \chi\tilde{u}_1). \tag{25}$$

Introducing the convex set $K_3 = \left\{v \in H_0^1(\Sigma_r^+) \colon v \geq \chi\tilde{u}_1 - \psi \text{ a.e. on } \Sigma_r^+\right\}$, whose elements are supposed extended to zero on S_r, the relation (18) implies that $w = \chi\tilde{u}_1 - \psi$ on Σ_r^+ is the solution of the variational inequality

$$w \in K_3\colon \sum_{h,k=1}^{n} \int_{\Sigma_r^+} b_{hk}^1 w_{y_h} (v-w)_{y_k} dy \geq \int_{\Sigma_r^+} (\tilde{F}_1 - \tilde{a}\chi\tilde{u}_1 - B_1\psi)(v-w)\,dy \quad \forall v \in K_3.$$

Then, owing (22), (23), (24), it results [14]:

$$w \in H^2(\Sigma_r^+), \tag{26}$$

$$\|w\|_{H^2(\Sigma_r^+)} \leq c\left(\left\|\tilde{F}_1\right\|_{L^2(\Sigma_r^+)} + \|\tilde{a}\chi\tilde{u}_1\|_{L^2(\Sigma_r^+)} + \left\|(B_1(\chi\tilde{u}_2))^+\right\|_{L^2(\Sigma_r^+)} + \|B_1\psi\|_{L^2(\Sigma_r^+)}\right) \tag{27}$$

$$(c = c(b_{hk}^1)).$$

Then using (24) and (26) we obtain (15). The relation (16) is deduced from (21) using the second one of (12) and (15). The upper limitations (7), (13), (25), (27) lead to (17). The α_1) and (8), (9), (14)–(17) are obviously sufficient to get α_2). The α_3), which is known when $V_1 = V_2 = H_0^1(\Omega)$ [6], can be established by topics similar enough to those used above.

Finally, let us verify α_4). The relation

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(u_1)_{x_i}(v_1-u_1)_{x_j} dx + \int_\Omega a u_1(v_1-u_1)\,dx + \sum_{i,j=1}^{n} \int_\Omega a_{ij}(u_2)_{x_i}(v_2-u_2)_{x_j} dx \geq$$

$$\geq \int_\Omega f_1(v_1-u_1)\,dx + \int_\Omega f_2(v_2-u_2)\,dx \qquad \forall (v_1, v_2) \in K \tag{28}$$

implies that $\sum_{i,j=1}^{n} \int_\Omega a_{ij}(u_1+u_2)_{x_i} v_{x_j} dx = \int_\Omega (f_1 + f_2 - au_1)v\,dx \quad \forall v \in H^1(\Omega)$; then [7]

$$u_1 + u_2 \in H^2(\Omega), \qquad \|u_1+u_2\|_{H^2(\Omega)} \leq c\left(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\right) \tag{29}$$

$$(c = c(a_{ij}, a, \Omega)),$$

and

$$\int_\Omega (f_1 + f_2 - au_1)\,dx = 0. \tag{30}$$

Setting $K_4 = \{v \in H^1(\Omega)\colon v > 0 \text{ a.e. on } \Omega\}$, putting in (28) $v_1 = u_2 + v$ and $v_2 = u_2$, $v_1 = u_1$ and $v_2 = u_1 - v$ with $v \in K_4$, we note that $w = u_1 - u_2$ is solution of the variational inequality:

$$w \in K_4: \quad \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} w_{x_i} (v-w)_{x_j} dx \ \geq \ \int_{\Omega} (f_1 + f_2 - a u_1)(v-w)\, dx \quad \forall v \in K_4.$$

Since because of (30) also $w+1$ is solution of the same inequality and since for $v \in C^1(\overline{\Omega}) - \{0\}$ and $0 < \varepsilon < \left(\max_{\overline{\Omega}} |v|\right)^{-1}$ $w+1 \pm \varepsilon v \in K_4$, we get:

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} w_{x_i} v_{x_j} dx \ = \ \int_{\Omega} (f_1 + f_2 - a u_1) v\, dx \quad \forall v \in C^1(\overline{\Omega}).$$

It follows that [7] $w \in H^2(\Omega)$, $\| w \|_{H^2(\Omega)} \leq c\left(\| f_1 \|_{L^2(\Omega)} + \| f_2 \|_{L^2(\Omega)} + \| u_2 \|_{L^2(\Omega)}\right)$, where $(c = c(a_{ij}, \Omega))$ and then the $\alpha_4)$ is true by virtue of (29).

***Remark* 1.** About $\alpha_1)$, the belonging of u_l to the space $H^2(G)$ does not depend on the fact that Ω_2 is $C^{1,1}$ set but it depends only on the following conditions

$$a_{ij}^{l} \in L^\infty(\Omega_l) \cap C^{0,1}_{\mathrm{loc}}(\Omega_2), \quad f_1 \in V_1' \cap C^2_{\mathrm{loc}}(\Omega_2), \quad f_2 \in V_2' \cap C^2_{\mathrm{loc}}(\Omega_2).$$

Dealing with $\alpha_2)$, taking in account the assumptions about a_{ij}^{2}, Ω_2 and f_2, the relations

$$u_1 \in H^2(\Omega_2) \cap H^2(G), \quad u_2 \in H^2(\Omega_2)$$

hold admitting, more in general, $a_{ij}^{1} \in L^\infty(\Omega_1) \cap C^{0,1}_{\mathrm{loc}}(\Omega_1)$ and $f_1 \in V_1' \cap L^2_{\mathrm{loc}}(\Omega_1)$. Let us add that, if Ω_1 is bounded and $C^{1,1}$ set, since

$$\sum_{i,j=1}^{n} \int_{\Omega_1 - \overline{\Omega}_2} a_{ij}^{1} (u_1)_{x_i} v_{x_j} dx + \int_{\Omega_1 - \overline{\Omega}_2} a u_1 v\, dx \ = \ \langle f_1, v \rangle_1 \quad \forall v \in C^1(\overline{\Omega}_1), \text{ with } \operatorname{supp} v \subset \overline{\Omega}_1 - \Omega_2,$$

the hypotheses of Theorem 1 assure that $u_1 \in H^2(\Omega_1 - \overline{\Omega}_2)$.

We complete the study of (1) with a characterisation of the solution when suitable hypotheses on the data occur. For every $(v_1, v_2) \in V_1 \times V_2$ let $\Omega_2(v_1, v_2)$ be the set of the points $x \in \overline{\Omega}_2$ satisfying the condition: there exist $c_x > 0$ and a neighbourhood I_x of x such that $v_1 - v_2 \geq c_x$ a. e. on $I_x \cap \Omega_2$. In the case $\overline{\Omega}_2 \subset \Omega_1$, $V_l = H^1(\Omega_l)$ we give the following theorem.

Theorem 2. *Let hypotheses of Theorem* 1, *if* Ω_1 *is bounded and* $C^{1,1}$ *set, the pair* $(u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ *is solution of* (1) *if and only if*

$$u_1 \in H^2(\Omega_2) \cap H^2(\Omega_1 - \overline{\Omega}_2), \quad u_2 \in H^2(\Omega_2), \quad u_1 \geq u_2 \quad \textit{a. e. on } \Omega_2,$$

$$A_1 u_1 + A_2 u_2 \ = \ f_1 + f_2 - a u_1 \quad \textit{a. e. on } \Omega_2,$$

$$A_1 u_1 \ \geq \ f_1 - a u_1 \quad \textit{a. e. on } \Omega_2,$$

$$A_1 u_1 \ = \ f_1 - a u_1 \quad \textit{a. e. on } \Omega_2 (u_1, u_2) \cup (\Omega_1 - \overline{\Omega}_2),$$

$$\sum_{i,j=1}^{n} a_{ij}^{1} (\overline{u}_1)_{x_i} \nu_j^2 - \sum_{i,j=1}^{n} a_{ij}^{1} (\overline{\overline{u}}_1)_{x_i} \nu_j^2 \ = \ \sum_{i,j=1}^{n} a_{ij}^{2} (u_2)_{x_i} \nu_j^2 \quad \sigma_2\textit{-a. e. on } \partial\Omega_2,$$

$$\sum_{i,j=1}^{n} a_{ij}^{1} (\overline{\overline{u}}_1)_{x_i} \nu_j^1 \ = \ 0 \quad \sigma_2\textit{-a. e. on } \partial\Omega_1,$$

with $\overline{u}_1$ [resp. $\overline{\overline{u}}_1$] *is the restriction of* u_1 *to* Ω_2 [*resp.* $\Omega_1 - \overline{\Omega}_2$], ν_j^l *the*

j^{th} *component of the unit outward normal vector to* $\partial\Omega_l$, σ_l *the surface measure on* $\partial\Omega_l$.

Leaving the easy proof of Theorem 2, we only add that characterisations of the same kind are also possible in the other cases.

2. Passing to (2), let $\not\in_1$ be the space of real polynomials at most of first degree and let us observe that (2) can admit solution only when one of the following cases holds:

$$\langle f, 1\rangle = 0 \quad \text{and} \quad \langle f, x_1\rangle = \langle f, x_2\rangle = 0, \tag{31}$$

$$\langle f, 1\rangle > 0 \quad \text{and} \quad x^0 = \left(\frac{\langle f, x_1\rangle}{\langle f, 1\rangle}, \frac{\langle f, x_2\rangle}{\langle f, 1\rangle}\right) \in T. \tag{32}$$

Besides, if (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$ are solutions of (2), it results $\overline{u}_1 = u_1 + p$, $\overline{u}_2 = u_2$ with $p \in \not\in_1$ and $p(x^0) = 0$.

When the case (31) holds, if we consider the variational equation

$$u \in H^2(\Omega): \quad \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u D^s v\,dx = \langle f, v\rangle \quad \forall v \in H^2(\Omega), \tag{33}$$

which admits infinite solutions, different two by two for a polynomial of $\not\in_1$, the solutions of (2) are all and only the pairs $(u, 0)$ with u nonpositive on $\Omega_0 \cup \Gamma_0$ solution of (33).

In the case (32) with $x^0 \in \overset{0}{T}$, the problem (2) is solvable because [1, 8] for each $p \in \not\in_1 - \{0\}$ with $p \le 0$ on $\Omega_0 \cup \Gamma_0$ $\langle f, p\rangle = p(x^0)\langle f, 1\rangle < 0$. Then we have theorem.

Theorem 3. *If* $\langle f, 1\rangle > 0$ *and* $x^0 \in \overset{0}{T}$ *the problem* (2) *admits at least a solution.*

In regard to the case

$$\langle f, 1\rangle > 0 \quad \text{and} \quad x^0 \in \Gamma_i, \quad i = 1, 2, \tag{34}$$

we previously give the following theorem.

Theorem 4. *With the assumptions* (34) *for* $m = 1$ *the problem* (2) *has no solution.*

Proof. Arguing by contradiction, let (u_1, u_2) a solution of (2). We have:

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\,dx \le \langle f, \varphi\rangle \quad \forall \varphi \in C_0^\infty(R^2) \quad \text{with} \quad \varphi \ge 0, \tag{35}$$

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\,dx + \sum_{i,j=1}^{2} \int_\Omega b_{ij} (u_2)_{x_i} \varphi_{x_j} dx + \int_\Omega b u_2 \varphi\,dx = \langle f, \varphi\rangle \tag{36}$$

$$\forall \varphi \in C_0^\infty(R^2).$$

Relation (35) holds the distribution on R^2

$$L(\varphi) = \langle f, \varphi\rangle - \sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\,dx \qquad \forall \varphi \in C_0^\infty(R^2)$$

is nonnegative; then there exists a Radon measure μ on R^2 such that

$$L(\varphi) = \int_{R^2} \varphi\,d\mu \qquad \forall \varphi \in C_0^\infty(R^2). \tag{37}$$

Using (37) with $\varphi = 1$ on $\overline{\Omega}$, we get:

$$\mu(R^2) = \langle f, 1 \rangle. \tag{38}$$

Besides we add

$$\int_{R^2} \varphi \, d\mu = 0 \quad \forall \varphi \in C_0^\infty(R^2) \quad \text{with } \operatorname{supp} \varphi \subset R^2 - \{x^i, x^3\}. \tag{39}$$

Relation (39) is obvious if $\operatorname{supp} \varphi \subset R^2 - \{\overline{\Omega}_0 \cup \Gamma_0\}$; else, taking $p \in \notin_1$ equal to zero on Γ_i and positive on $T - \Gamma_i$, the foregoing relation can be got from the following

$$(u_1 - p \pm \varepsilon\varphi,\ u_2) \in K \qquad \text{for } 0 < \varepsilon < \frac{\min\limits_G p}{\max\limits_G |\varphi|} \qquad G = (\overline{\Omega}_0 \cup \Gamma_0) \cap \operatorname{supp} \varphi.$$

From (39) it follows that

$$\mu(R^2) = \mu\big(\{x^i\}\big) + \mu\big(\{x^3\}\big) \tag{40}$$

then relation (37) becomes $L(\varphi) = \mu(\{x^i\})\,\varphi(x^i) + \mu(\{x^3\})\,\varphi(x^3)$. So, taking into account the (36), it results:

$$\sum_{i,j=1}^{2} \int_\Omega b_{ij}(u_2)_{x_i} \varphi_{x_j} dx + \int_\Omega b u_2 \varphi \, dx = \mu(\{x^i\})\,\varphi(x^i) + \mu(\{x^3\})\,\varphi(x^3).$$

This last relation is false for $u_2 \in H^1(\Omega)$, in virtue of (38) and (40).

The following considerations let us to find infinite solutions of (2) when $m > 1$. With $p_i \in \notin_1$ such that $p_i(x^0) = 0$ and $p_i(x) > 0 \ \ \forall x \in T - \{x^0\}$ if $x^0 = x^i, x^3$,

$$p_i(x) = 0 \quad \forall x \in \Gamma_i \text{ and } p_i(x) > 0 \quad \forall x \in T - \Gamma_i \quad \text{if } x^0 \in \operatorname{int} \Gamma_i,$$

let us consider the variational equations

$$u_1 \in H^2(\Omega): \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^r u_1 D^s v \, dx = \langle f, v \rangle - v(x^0)\langle f, 1 \rangle \quad \forall v \in H^2(\Omega), \tag{41}$$

$$u_2 \in H^m(\Omega): \sum_{\substack{|i|=m \\ |j|=m}} \int_\Omega b_{ij} D^i u_2 D^j v \, dx + \int_\Omega b u_2 v \, dx = v(x^0)\langle f, 1 \rangle \quad \forall v \in H^m(\Omega), \tag{42}$$

and, setting $K_i = \big\{(v_1, v_2) \in H^2(\Omega) \times H^m(\Omega) \colon v_1(x^i) \le v_2(x^i) \text{ and } v_1(x^3) \le v_2(x^3)\big\}$, let us consider the variational inequality

$$(u_1, u_2) \in K_i: \sum_{\substack{|r|=2 \\ |s|=2}} \int_\Omega a_{rs} D^r u_1 D^s (v_1 - u_1)\, dx + \sum_{\substack{|i|=m \\ |j|=m}} \int_\Omega b_{ij} D^i u_2 D^j (v_2 - u_2)\, dx +$$

$$+ \int_\Omega b u_2 (v_2 - u_2)\, dx \ge \langle f, v_1 - u_1 \rangle \qquad \forall (v_1, v_2) \in K_i. \tag{43}$$

Since $\langle f, p \rangle - p(x^0)\langle f, 1 \rangle = 0 \ \ p_i \in \notin_1$, equation (41) admits infinite solutions which are different two by two for a polynomial of $\notin_1$. Equation (42) has unique solution. If $x^0 \in \operatorname{int}\Gamma_i$, equation (43), whose resolvent cone is made up by the pairs $(\lambda p_i, 0)$ with $\lambda \in R$, has at least a solution (u_1, u_2) [4] and it is obvious that all the pairs $(u_1 + \lambda p_i,\ u_2)$ with $\lambda \in R$ are solutions of (43). Under the following assumptions

$$a_{rs} \in C^3_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \qquad b_{ij} \in C^3_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \quad \text{if} \quad m = 2,$$
$$f \in \left(H^{1,p}_{\text{loc}}(\Omega)\right)' \cap \left(H^2(\Omega)\right)' \tag{44}$$

with $2 < p < +\infty$, remarking the continuity of the embedding from $H^{1,p}_{\text{loc}}(\Omega)$ to $C^0_{\text{loc}}(\Omega)$, in relation to the solutions u_1, of (41) and to the solution u_2 of (42), we get [10]:

$$u_1 \in H^{3,p'}_{\text{loc}}(\Omega) \quad \text{with} \quad p' = \frac{p}{p-1}, \quad u_2 \in H^{3,p'}_{\text{loc}}(\Omega) \quad \text{if} \quad m = 2; \tag{45}$$

in particular u_1, and u_2 are $C^{0,1}_{\text{loc}}(\Omega)$.

The considerations above are also valid for the components of the solutions of (43) because it is obvious that there exists a Radon measure μ on R^2 such that

$$\mu(R^2) = \mu\left(\{x^i\}\right) + \mu\left(\{x^3\}\right) = \langle f, 1\rangle,$$

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\, dx = \langle f, \varphi\rangle - \mu\left(\{x^i\}\right)\varphi(x^i) + \mu\left(\{x^3\}\right)\varphi(x^3) \quad \forall \varphi \in C_0^\infty(\Omega),$$

$$\sum_{\substack{|i|=m\\|j|=m}} \int_\Omega b_{ij} D^i u_2 D^j \varphi\, dx + \int_\Omega b u_2 \varphi\, dx = \mu\left(\{x^i\}\right)\varphi(x^i) + \mu\left(\{x^3\}\right)\varphi(x^3) \quad \forall \varphi \in C_0^\infty(\Omega).$$

Since for $x^0 = x^i,\ x^3$, we have

$$\textit{for each } \ \nu \in C^{0,1}_{\text{loc}}(\Omega), \quad \textit{with} \quad \nu(x^0) \ge 0, \quad \inf_{\Omega_0 \cup \operatorname{int}\Gamma_0} \frac{\nu}{p_i} > -\infty, \tag{46}$$

let us suppose the following condition verified when $x^0 \in \operatorname{int}\Gamma_i$:

there exist a neighbourhood S *of* x^3 *and* $\varepsilon \in]0,1[$ *such that* (47)

$\forall x \in S \cap \Omega_0 \ \ |x - \tilde{x}| \ge \varepsilon\left|x - x^3\right|$ *where* $\tilde{x} =$ *orthogonal projection of* x *on* Γ_i,

which yields that

$$\textit{for each } \ \nu \in C^{0,1}_{\text{loc}}(\Omega) \ \textit{with} \ \nu(x^i) \ge 0 \ \textit{and} \ \nu(x^3) \ge 0 \quad \inf_{\Omega_0 \cup \operatorname{int}\Gamma_0} \frac{\nu}{p_i} > -\infty. \tag{48}$$

In virtue of (45), (46), (48) we can give the following theorem.

Theorem 5. *For* $m > 1$ *under the assumptions* (34), (44), *it follows that:*

β_1) *if* $x^0 = x^i,\ x^3$, *let* u_1 *and* u_2 *respectively be the solution of* (41), *equal to zero in* x^0, *and the solution of* (42), *all the pairs*

$$(u_1 + u_2(x^0) + \lambda p_i,\ u_2) \ \textit{with} \ \lambda \le \inf_{\Omega_0 \cup \operatorname{int}\Gamma_0} \frac{u_2 - (u_1 + u_2(x^0))}{p_i}$$

are solutions of (2);

β_2) *if* $x^0 \in \operatorname{int}\Gamma_0$ *and taking into account* (47), *let* (u_1, u_2) *be a solution of* (43), *all the pairs*

$$(u_1 + \lambda p_i,\ u_2) \ \textit{with} \ \lambda \le \inf_{\Omega_0 \cup \operatorname{int}\Gamma_0} \frac{u_2 - u_1}{p_i}$$

are solutions of (2).

Finally in order to study the case

$$\langle f, 1\rangle > 0 \quad \text{and} \quad x^0 \in \Gamma_0 \tag{49}$$

let us setting $p_0(x) = x_2 \ \ \forall x = (x_1, x_2) \in R^2$. Relation (49) assures that the (3) has at least a solution (u_1, u_2) since [4] its resolvent cone is made up by the pairs $(\lambda p_0, 0)$ with $\lambda \in R$. Moreover, all the pairs $(u_1 + \lambda p_0,\ u_2)$ with $\lambda \in R$ are solutions of (3). Since there exists a Radon measure μ on R^2 such that $\mu(R^2) = \mu(\Gamma_0) = \langle f, 1\rangle$,

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\, dx = \langle f, \varphi\rangle - \int_\Omega \varphi\, d\mu \qquad \forall \varphi \in C_0^\infty(R^2),$$

$$\sum_{\substack{|i|=m\\|j|=m}} \int_\Omega b_{ij} D^i u_2 D^j \varphi\, dx + \int_\Omega b u_2 \varphi\, dx = \int_\Omega \varphi\, d\mu \qquad \forall \varphi \in C_0^\infty(R^2),$$

for $m \geq 2$ and under the conditions (44) the regularity properties expressed by (45) are true also for the components of the solutions of (3). Then, since

$$\textit{if } U \textit{ is an open rectangle containing } \partial\Omega_0 \cap \Gamma_0,$$

$$\textit{for} \quad v \in C^{0,1}(U) \quad \textit{with} \quad v \geq 0 \quad \textit{on} \quad U \cap \Gamma_0 \ \inf_{U\cap\Omega_0} \frac{v}{p_i} > -\infty, \tag{50}$$

we have the following statement.

Theorem 6. *For $m > 1$ under the hypotheses* (44), (49), *if (u_1, u_2) is solution of* (3), *all the pairs*

$$(u_1 + \lambda p_0,\ u_2) \qquad \textit{with} \quad \lambda \leq \inf_{\Omega_0} \frac{u_2 - u_1}{p_0}$$

are solutions of (2).

The statement (50) is very important in order to find solutions of (2) when $m \geq 2$. Its efficiency is due to the local Lipschitzianity of the components of the solution of (3). If $m = 1$ the previous reasoning is not able to state this property for the second component; however the following two theorems let us still use (50).

Remarking that for each $\bar{x} \in \operatorname{int}\Gamma_0$ for each $r > 0$ and for $1 < t < +\infty$

$$S_r = \{x \in R^2 : |x - \bar{x}| < r\}, \qquad \Sigma_r^+ = \{x \in S_r : x_2 > 0\},$$

$$\Sigma_r^- = \{x \in S_r : x_2 < 0\}, \qquad \Gamma_{0r} = \{x \in S_r : x_2 = 0\}, \qquad t' = \frac{t}{t-1},$$

let us start by proving the theorem.

Theorem 7. *For $m = 1$ under the hypothesis* (49),

$$a_{rs} \in C^3_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega), \quad b_{ij} = b_{ji} \in C^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega), \quad f \in \left(H^{1,p}_{\mathrm{loc}}(\Omega)\right)' \cap \left(H^2(\Omega)\right)'$$

with $2 < p < 4$, if (u_1, u_2) is solution of (3) *we have:*

$\gamma_1)$ $u_1 \in H^{3,p'}_{\mathrm{loc}}(\Omega)$,

$\gamma_2)$ *for each $\bar{x} \in \operatorname{int}\Gamma_0$ there exist S_r, with $\overline{S}_r \subset \Omega - \{x^1, x^2\}$, and $\Psi \in H^{2,q}(\Sigma_r^+) \cap H^{2,q}(\Sigma_r^-) \cap C^0(\overline{S}_r)$ such that $\Psi(x) = 0 \ \ \forall x \in \Gamma_{0r}$, $u_2 \in H^{1,q}(S_r)$, $u_2(x) - u_1(x) \geq \Psi(x) \quad \forall x \in \overline{S}_r$, being $q = \dfrac{p'}{2 - p'}$ (> 2).*

Proof. The equality

$$\sum_{\substack{|r|=2\\|s|=2}} \int_\Omega a_{rs} D^r u_1 D^s \varphi\, dx + \sum_{i,j=1}^{2} \int_\Omega b_{ij} (u_2)_{x_i} \varphi_{x_j} dx + \int_\Omega b u_2 \varphi\, dx = \langle f, \varphi\rangle \ \ \forall \varphi \in C_0^\infty(\Omega),$$

if G is an open set with $\overline{G} \subset \Omega$, implies that

$$\left| \sum_{\substack{|r|=2\\|s|=2}} \int_{\Omega} a_{rs} D^r u_1 D^s \varphi\, dx \right| \le c \|\varphi\|_{H^{1,p}(G)} \quad \forall \varphi \in C_0^\infty(\Omega)$$

($c = \text{const} > 0$ independent on φ) from which we get statement γ_1) [10].

Let us show γ_2). Let $\bar{x} \in \operatorname{int}\Gamma_0$, S_{r_1} and S_{r_2} with $r_1 < r_2$ and $\bar{S}_{r_2} \subset \Omega - \{x^1, x^2\}$. If $\chi \in C_0^\infty(R^2)$, with $0 \le \chi \le 1$, $\chi = 1$ on $\bar{S}_{r_1}$ and $\operatorname{supp}\chi \subset S_{r_2}$, and if

$$F = -\sum_{i,j=1}^{2} b_{ij}(u_2)_{x_i}\chi_{x_j} - \sum_{i,j=1}^{2} \left(b_{ij} u_2 \chi_{x_i}\right)_{x_j},$$

$$K_{01} = \left\{ v \in H_0^1(S_{r_2}) :\ v \ge \chi u_1 \ \text{on}\ \Gamma_{0r_2} \right\},$$

we can easily see that χu_2 is the solution of the variational inequality

$$\chi u_2 \in K_{01}: \quad \sum_{i,j=1}^{2} \int_{S_{r_2}} b_{ij}(\chi u_2)_{x_i}(v - \chi u_2)_{x_j}\, dx + \int_{S_{r_2}} b\chi u_2 (v - \chi u_2)\, dx \ge$$
$$\ge \int_{S_{r_2}} F(v - \chi u_2)\, dx \quad \forall v \in K_{01}. \tag{51}$$

Introducing the functions

$$\psi_1 \in H_0^1(\Sigma_{r_2}^+):\quad B\psi_1 = F - b\chi u_2 - B(\chi u_1) \quad \text{in the sense of } D'(\Sigma_{r_2}^+), \tag{52}$$

$$\psi_2 \in H_0^1(\Sigma_{r_2}^-):\quad B\psi_2 = F - b\chi u_2 - B(\chi u_1) \quad \text{in the sense of } D'(\Sigma_{r_2}^-), \tag{53}$$

$$\psi = \begin{cases} \psi_1 & \text{on}\ \Sigma_{r_2}^+; \\ \psi_2 & \text{on}\ \Sigma_{r_2}^-, \end{cases}$$

since [11]

$$\psi_1 + \chi u_1 \in H^2(\Sigma_{r_2}^+), \qquad \psi_2 + \chi u_1 \in H^2(\Sigma_{r_2}^-), \tag{54}$$

and

$$\psi \in H_0^1(S_{r_2}) \cap C^0(\bar{S}_{r_2}), \quad \psi = 0 \quad \text{on} \quad \Gamma_{0r_2}, \tag{55}$$

let us verify that χu_2 is the solution of the variational inequality

$$\chi u_2 \in K_{02}: \quad \sum_{i,j=1}^{2} \int_{S_{r_2}} b_{ij}(\chi u_2)_{x_i}(v - \chi u_2)_{x_j}\, dx + \int_{S_{r_2}} b\chi u_2 (v - \chi u_2)\, dx \ge$$
$$\ge \int_{S_{r_2}} F(v - \chi u_2)\, dx \quad \forall v \in K_{02} \tag{56}$$

where

$$K_{02} = \left\{ v \in H_0^1(S_{r_2}) :\ v \ge \psi + \chi u_1 \ \text{a.e. on}\ S_{r_2} \right\}.$$

For this purpose it is only necessary to show that

$$\chi u_2 \ge \psi + \chi u_1 \quad \text{a.e. on } S_{r_2}. \tag{57}$$

Relation (55) and the belonging of χu_2 to K_{01} imply that $\pm(\chi u_2 - (\chi u_1 + \psi))^- + \chi u_2 \in K_{01}$; then, using (51) with $v = \pm(\chi u_2 - (\chi u_1 + \psi))^- + \chi u_2$, we get the equality

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}(\chi u_2)_{x_i}\left((\chi u_2-(\chi u_1+\psi))^-\right)_{x_j}dx =$$
$$= \int_{S_{r_2}}(F-b\chi u_2)(\chi u_2-(\chi u_1+\psi))^- dx; \tag{58}$$

on the other hand because of the (52), (53) we also obtain

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}(\chi u_1+\psi)_{x_i}\left((\chi u_2-(\chi u_1+\psi))^-\right)_{x_j}dx =$$
$$= \sum_{i,j=1}^{2}\int_{\Sigma_{r_2}^+} b_{ij}(\chi u_1+\psi_1)_{x_i}\left((\chi u_2-(\chi u_1+\psi_1))^-\right)_{x_j}dx +$$
$$+ \sum_{i,j=1}^{2}\int_{\Sigma_{r_2}^-} b_{ij}(\chi u_1+\psi_2)_{x_i}\left((\chi u_2-(\chi u_1+\psi_2))^-\right)_{x_j}dx =$$
$$= \int_{S_{r_2}}(F-b\chi u_2)(\chi u_2-(\chi u_1+\psi))^- dx. \tag{59}$$

From (58), (59) we get

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}\left((\chi u_2-(\chi u_1+\psi))^-\right)_{x_i}\left((\chi u_2-(\chi u_1+\psi))^-\right)_{x_j}dx = 0$$

from which the (57) follows.

Setting $q=\dfrac{p'}{2-p'}$ (>2), let us show that

$$\chi u_2 \in H_0^{1,q}(S_{r_2}). \tag{60}$$

Let $\{\varepsilon_n\}$ be a infinitesimal sequence of positive numbers. Let us consider the function

$$\theta_n(t) = \begin{cases} 1 & \text{if } t\le 0;\\ 1-\dfrac{t}{\varepsilon_n} & \text{if } 0<t\le\varepsilon_n;\\ 0 & \text{if } t>\varepsilon_n, \end{cases}$$

and, taking into account the continuity of the embedding from $H^1(S_{r_2})$ to $L^2(\Gamma_{0_{r_2}})$, the operator $L\colon H_0^1(S_{r_2})\to H^{-1}(S_{r_2})$ such that

$$\overline{L}u,\ v\pi = -\int_{\Gamma_{0_{r_2}}}\left[\left(-\sum_{i=1}^{2} b_{i2}\left(\overline{\chi u_1}+\psi_1\right)_{x_i}\right)^+ + \left(\sum_{i=1}^{2} b_{i2}\left(\overline{\overline{\chi u_1}}+\psi_2\right)_{x_i}\right)^+\right]\theta_n(u-\chi u_1)v\,d\sigma$$
$$\forall u,\ v\in H_0^1(S_{r_2})$$

where $\bar{\ },.\pi$ is the pairing between $H^{-1}(S_{r_2})$ and $H_0^1(S_{r_2})$, $\overline{\chi u_1}$ [resp. $\overline{\overline{\chi u_1}}$] is the restriction of χu_1, to $\Sigma_{r_2}^+$ [resp. $\Sigma_{r_2}^-$] and σ is the measure on $\Gamma_{0_{r_2}}$.

Observing that L is bounded, monotone and hemicontinuous, the variational equation

$$w_n\in H_0^1(S_{r_2})\colon\quad \sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}(w_n)_{x_i}v_{x_j}dx = \int_{S_{r_2}}(F-b\chi u_2)v\,dx +$$

$$+ \int_{\Gamma_{0_{r_2}}} \left[\left(-\sum_{i=1}^{2} b_{i2}(\overline{\chi u_1} + \psi_1)_{x_i} \right)^+ + \left(\sum_{i=1}^{2} b_{i2}(\overline{\overline{\chi u_1}} + \psi_2)_{x_i} \right)^+ \right] \theta_n (w_n - \chi u_1) v \, d\sigma \quad (61)$$

$$\forall v \in H_0^1(S_{r_2})$$

has unique solution [12] and we have:

$$\| w_n \|_{H_0^1(S_{r_2})} \le c \quad (c = \text{const} > 0 \text{ independent on } n). \quad (62)$$

Remarking the continuity of the embeddings

$$H^{2,p'}(S_{r_2}) \subseteq H^{1,q}(\Gamma_{0_{r_2}}), \quad (63)$$

$$H^{1,q'}(S_{r_2}) \subseteq L^{q'}(\Gamma_{0_{r_2}}), \quad (64)$$

$$H^{1,q'}(S_{r_2}) \subseteq L^2(S_{r_2}), \quad (65)$$

$$H^{2,p'}(S_{r_2}) \subseteq H^{1,2q}(S_{r_2}), \quad (66)$$

let us observe that relation (63), by virtue of $\gamma_1)$ and (54), gives the relations

$$\sum_{i=1}^{2} b_{i2}(\overline{\chi u_1} + \psi_1)_{x_i} \in L^q(\Gamma_{0_{r_2}}), \qquad \sum_{i=1}^{2} b_{i2}(\overline{\overline{\chi u_1}} + \psi_2)_{x_i} \in L^q(\Gamma_{0_{r_2}}) \quad (67)$$

and by virtue of $\gamma_1)$ and (66) we have

$$B(\chi u_1) \in L^{2q}(S_{r_2}). \quad (68)$$

Taking into account (64), (65), (67), from (61) we get

$$\left| \sum_{i,j=1}^{2} \int_{S_{r_2}} b_{ij}(w_n)_{x_i} v_{x_j} dx \right| \le c_1 \| v \|_{H_0^{1,q'}(S_{r_2})} \quad \forall v \in C_0^\infty(S_{r_2})$$

$$(c_1 = \text{const} > 0 \text{ independent on } v \text{ and } n)$$

then [10]

$$w_n \in H_0^{1,q}(S_{r_2}),$$

$$\| w_n \|_{H_0^{1,q}(S_{r_2})} \le c_2 \left(c_1 + \| w_n \|_{L^q(S_{r_2})} \right) \quad (c_2 = \text{const} > 0 \text{ independent on } n). \quad (69)$$

From (69), taking into account (62) and the continuity of the embeddings

$$H^1(S_{r_2}) \subseteq L^q(S_{r_2}), \qquad H^{1,q}(S_{r_2}) \subseteq C^{0,1-2/q}(\overline{S}_{r_2}),$$

we get the upper limitations

$$\| w_n \|_{H_0^{1,q}(S_{r_2})} \le c,$$

$$\| w_n \|_{H^{0,1-2/q}(\overline{S}_{r_2})} \le c \quad (70)$$

$$(c = \text{const} > 0 \text{ independent on } n).$$

The relations (70) assure the existence of $w \in H_0^{1,q}(S_{r_2})$ and of a subsequence of $\{w_n\}$, which we denote with the same symbol, such that

$$w_n \to w \quad \text{weakly in} \quad H_0^{1,q}(S_{r_2}), \quad (71)$$

$$w_n \to w \quad \text{in} \quad C^0(\overline{S}_{r_2}). \quad (72)$$

Let us control that w is the solution of (56), and this shows relation (60). First of all from (52) ÷ (55) and (61), we obtain

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}\big((w_n-(\psi+\chi u_1))^-\big)_{x_i}\big((w_n-(\psi+\chi u_1))^-\big)_{x_j}\,dx =$$

$$= \sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}(w_n)_{x_i}\big((w_n-(\psi+\chi u_1))^-\big)_{x_j}\,dx -$$

$$-\sum_{i,j=1}^{2}\int_{\Sigma_{r_2}^+} b_{ij}(\overline{\chi u_1}+\psi_1)_{x_i}\big((w_n-(\overline{\chi u_1}+\psi_1))^-\big)_{x_j}\,dx -$$

$$-\sum_{i,j=1}^{2}\int_{\Sigma_{r_2}^-} b_{ij}(\overline{\overline{\chi u_1}}+\psi_2)_{x_i}\big((w_n-(\overline{\overline{\chi u_1}}+\psi_2))^-\big)_{x_j}\,dx =$$

$$= \int_{S_{r_2}}(F-b\chi u_2)(w_n-(\psi+\chi u_1))^-\,dx +$$

$$+\int_{\Gamma_{0r_2}}\left[\left(-\sum_{i=1}^{2} b_{i2}(\overline{\chi u_1}+\psi_1)_{x_i}'\right)^+\theta_n(w_n-\chi u_1)+\sum_{i=1}^{2} b_{i2}(\overline{\chi u_1}+\psi_1)_{x_i}\right](w_n-\chi u_1)^-\,d\sigma +$$

$$+\int_{\Gamma_{0r_2}}\left[\left(\sum_{i=1}^{2} b_{i2}\big(\overline{\overline{\chi u_1}}+\psi_2\big)_{x_i}\right)^+\theta_n(w_n-\chi u_1)-\sum_{i=1}^{2} b_{i2}\big(\overline{\overline{\chi u_1}}+\psi_2\big)_{x_i}\right](w_n-\chi u_1)^-\,d\sigma -$$

$$-\int_{\Sigma_{r_2}^+}B(\overline{\chi u_1}+\psi_1)(w_n-(\overline{\chi u_1}+\psi_1))^-\,dx-\int_{\Sigma_{r_2}^-}B\big(\overline{\overline{\chi u_1}}+\psi_2\big)\big(w_n-(\overline{\overline{\chi u_1}}+\psi_2)\big)^-\,dx \le 0$$

from which we get $w_n(x)\ge\psi(x)+\chi(x)u_1(x)\quad\forall x\in S_{r_2}$ that is

$$w(x)\ \ge\ \psi(x)+\chi(x)u_1(x)\qquad \forall x\in S_{r_2} \tag{73}$$

by virtue of (72). Setting $v\in K_{02}$, from (61), (73) we obtain

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}(w_n)_{x_i}(v-w)_{x_j}\,dx \ \ge\ \int_{S_{r_2}}(F-b\chi u_2)(v-w)\,dx +$$

$$+\int_{\substack{\Gamma_{0_{r_2}}\\ w>\chi u_1}}\left[\left(-\sum_{i=1}^{2} b_{i2}(\overline{\chi u_1}+\psi_1)_{x_i}\right)^+ +\left(\sum_{i=1}^{2} b_{i2}\big(\overline{\overline{\chi u_1}}+\psi_2\big)_{x_i}\right)^+\right]\theta_n(w_n-\chi u_1)^-(v-w)\,d\sigma$$

and it is easy to verify that the second integral at the second side converges to zero when $n\to+\infty$. Then, taking into account also (71), we have

$$\sum_{i,j=1}^{2}\int_{S_{r_2}} b_{ij}w_{x_i}(v-w)_{x_j}\,dx\ \ge\ \int_{S_{r_2}}(F-b\chi u_2)(v-w)\,dx$$

and this, in according to (73), shows our purpose.

Setting $r\in\,]0,r_1[$, by virtue of (60) we have $u_2\in H^{1,q}(S_r)$ and, taking into account relations (52), (53), (68) we can conclude that [10] $\psi\in H^{2,q}(\Sigma_{r_2}^+)\cap H^{2,q}(\Sigma_{r_2}^-)$ and then from (57) we find $u_2(x)-u_1(x)\ge\psi(x)\quad\forall x\in\overline{S}_r$.

Remark 2. The hypothesis of symmetry of the coefficients b_{ij}, nonessential for

γ_1), has been necessary because, in order to get relations (54), we used a regularity theorem for the Dirichlet's problems in bounded and convex sets of R^n.

The results expressed by Theorem 7 are improved with the following theorem.

Theorem 8. *When* $m=1$ *under the hypotheses* (49),

$$a_{rs}\in C^{0,1}_{\rm loc}(\Omega)\cap L^\infty(\Omega),\quad b_{ij}\in C^1_{\rm loc}(\Omega)\cap L^\infty(\Omega),\quad f\in\left(H^1_{\rm loc}(\Omega)\right)'\cap\left(H^2(\Omega)\right)',$$

if (u_1,u_2) *is solution of* (3) *we have*

δ_1) $u_1\in H^3_{\rm loc}(\Omega)$;

δ_2) *for each* $\bar{x}\in\operatorname{int}\Gamma_0$ *there exist* S_r, *with* $\bar{S}_r\subset\Omega-\{x^1,x^2\}$, *and* $\psi\in H^{2,q}(\Sigma_r^+)\cap H^{2,q}(\Sigma_r^-)\cap C^0(\bar{S}_r)$ *such that*

$$(u_2)_{x_1^2}\in L^2(S_r),\quad (u_2)_{x_1x_2}\in L^2(S_r),\quad (u_2)_{x_2^2}\in L^2(\Sigma_r^+)\cap L^2(\Sigma_r^-)$$

$$\psi\in H^{2,q}(\Sigma_r^+)\cap H^{2,q}(\Sigma_r^-)\cap C^0(\bar{S}_r),\quad \psi(x)=0\quad\forall x\in\Gamma_{0r},$$

$$u_2(x)-u_1(x)\ \geq\ \psi(x)\quad\forall x\in\bar{S}_r,$$

$$\text{with}\quad 2<q<+\infty;$$

δ_3) *there exists at least one point* $\bar{x}\in\operatorname{int}\Gamma_0$ *such that for each* $S_r\subset\Omega$ $u_2\notin H^2(S_r)$;

δ_4) $p_0u_2\in H^2_{\rm loc}(\Omega)$.

Proof. The statement δ_1) is a consequence [13] of the relation

$$\sum_{\substack{|r|=2\\|s|=2}}\int_\Omega a_{rs}D^ru_1\,D^s\varphi\,dx+\sum_{i,j=1}^2\int_\Omega b_{ij}\,(u_2)_{x_i}\,\varphi_{x_j}\,dx+\int_\Omega b\,u_2\varphi\,dx\ =\ \langle f,\varphi\rangle\quad\forall\varphi\in C_0^\infty(R^2).$$

In regard to δ_2), let $\bar{x}$, S_{r_1}, S_{r_3}, χ, ψ_1, ψ_2, ψ the same terms used in the proof of Theorem 7. Since the validity of (51), (57), we have:

$$(\chi u_2)_{x_1x_h}\in L^2(S_{r_2}),\qquad h\ =\ 1,2,\tag{74}$$

$$(\chi u_2)_{x_2^2}\in L^2(\Sigma_{r_2}^+)\cap L^2(\Sigma_{r_2}^-).\tag{75}$$

In fact with the same notations used in Theorem 1, since for $t\in R-\{0\}$ with sufficiently small modulus we have

$$\sum_{\substack{|r|=2\\|s|=2}}\int_{S_{r_2}} a_{rs}D^r\left[D_1^t(\chi u_1)\right]D^s\left[D_1^t(\chi u_1)\right]dx\ \leq\ c\,\|u_1\|^2_{H^3(S_{r_2})}\ +$$

$$+\sum_{\substack{|r|=2\\|s|=2}}\int_{S_{r_2}} a_{rs}D^ru_1\,D^s\left[-\chi D_1^{-t}D_1^t(\chi u_1)\right]dx,$$

$$\sum_{i,j=1}^2\int_{S_{r_2}} b_{ij}\left(D_1^t(\chi u_2)\right)_{x_i}\left(D_1^t(\chi u_2)\right)_{x_j}dx\ \leq\ c\,\|u_2\|_{H^1(\Omega)}\left(\sum_{i=1}^2\left\|D_1^t\left((\chi u_2)_{x_i}\right)\right\|^2_{L^2(S_{r_2})}\right)^{1/2}\ +$$

$$+\sum_{i,j=1}^2\int_{S_{r_2}} b_{ij}(u_2)_{x_i}\left(-\chi D_1^{-t}D_1^t(\chi u_2)\right)_{x_j}dx,$$

where c is a positive constant independent on t, and for $0<\varepsilon<t^2/2$

$$\left(u_1 \cdot \varepsilon\chi D_1^{-t} D_1^t(\chi u_1),\ u_2 + \varepsilon\chi D_1^{-t} D_1^t(\chi u_2)\right) \in K_0,$$

using (3) with $v_1 = u_1 + \varepsilon\chi D_1^{-t} D_1^t(\chi u_1)$ and $v_2 = u_2 + \varepsilon\chi D_1^{-t} D_1^t(\chi u_2)$, we get easily the inequality

$$\left(\sum_{i=1}^{2} \left\| D_1^t\left((\chi u_2)_{x_i}\right)\right\|^2_{L^2(S_{r_2})}\right)^{1/2} \le c\left(\|f\|_{(H^1(S_{r_2}))'} + \|u_1\|_{H^3(S_{r_2})} + \|u_1\|_{H^1(\Omega)}\right)$$

$$(c = \text{const} > 0 \text{ independent on } t)$$

from which we have relation (74).

The relation (75) comes from (74) taking into account that because of (51) we have

$$B(\chi u_2) = F - b\chi u_2 \qquad \text{in the sense of } D'(\Sigma^+_{r_2}),$$

$$B(\chi u_2) = F - b\chi u_2 \qquad \text{in the sense of } D'(\Sigma^-_{r_2}).$$

The statement δ_1) implies that $B(\chi u_1) \in L^q(S_{r_2})$ with $2 < q < +\infty$, from (74), (75) we get $(\chi u_2)_{x_i} \in L^q(\Sigma^+_{r_2}) \cap L^q(\Sigma^-_{r_2})$ with $2 < q < +\infty$ and consequently [10]

$$\psi \in H^{2,q}(\Sigma^+_r) \cap H^{2,q}(\Sigma^-_r) \cap C^0(\overline{S}_0) \qquad \text{with} \quad 0 < r < r_1,$$

and so δ_2) is proved.

Now let we deal with δ_3). Since there exists a Radon measure μ on R^2 such that

$$\mu(R^2) = \mu(\Gamma_0) = \langle f, 1\rangle, \tag{76}$$

$$\sum_{i,j=1}^{2} \int_\Omega b_{ij}(u_2)_{x_i}\varphi_{x_j}\,dx + \int_\Omega b u_2 \varphi\,dx = \int_{\Gamma_0} \varphi\,d\mu \qquad \forall\varphi \in C_0^\infty(R^2), \tag{77}$$

if δ_3) would be false, being

$$\sum_{i,j=1}^{2} \int_\Omega b_{ij}(u_2)_{x_i}\varphi_{x_j}\,dx + \int_\Omega b u_2 \varphi\,dx = 0 \ \ \forall\varphi \in C_0^\infty(R^2) \ \text{with } \operatorname{supp}\varphi \subset R^2 - \{x^1, x^2\},$$

the relation (77) would be written as

$$\sum_{i,j=1}^{2} \int_\Omega b_{ij}(u_2)_{x_i}\varphi_{x_j}\,dx + \int_\Omega b u_2 \varphi\,dx = \mu(\{x^1\})\varphi(x^1) + \mu(\{x^2\})\varphi(x^2);$$

and this is false taking in account also (76).

Finally let us observe that $(u_1,\ u_2 \pm p_0\varphi)$ $\forall\varphi \in C_0^\infty(\Omega)$ is an element of K_0, then

$$\sum_{i,j=1}^{2} \int_\Omega b_{ij}(u_2)_{x_i}(p_0\varphi)_{x_j}\,dx + \int_\Omega b u_2 p_0 \varphi\,dx = 0$$

that is $B(p_0 u_2) = -\,b p_0 u_2 - b_{12}(u_2)_{x_1} - (b_{21}u_2)_{x_1} - (b_{22}u_2)_{x_2} - b_{22}(u_2)_{x_2}$ in the sense of $D'(\Omega)$ from which we obtain [7] the statement δ_4).

***Remark* 3.** To prove the statement δ_1) and the regularity properties of u_2 it is only sufficient that the coefficients b_{ij} are elements of $C^{0,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$. Let us observe that if Ω is $C^{2,1}$ set, cutting off in the hypotheses the symbol "{oc" we obtain $u_1 \in H^3(\Omega)$

$(u_2)_{x_1^2} \in L^2(G)$, $(u_2)_{x_1 x_2} \in L^2(G)$ for any open set G with $\overline{G} \subset \overline{\Omega} - \{x^1, x^2\}$;

$(u_2)_{x_2^2} \in L^2(G)$ for any open set G with $\overline{G} \subset \overline{\Omega} - \Gamma_0$.

About δ_4) it needs only the hypotheses (49) and $b_{ij} \in C^{0,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$. If Ω is a $C^{1,1}$ set and $b_{ij} \in C^{0,1}(\Omega)$, then $p_0 u_2 \in H^2(\Omega)$.

Let (u_1, u_2) be a solution of (3). The γ_2) [resp. δ_2)] lets obviously the existence of an open rectangle U containing $\partial\Omega_0 \cap \Gamma_0$, with $\overline{U} \subset \Omega$, and of a function $\psi \in C^{0,1}(U)$ such that

$$\psi(x) = 0 \qquad \forall x \in U \cap \Gamma_0,$$

$$u_2(x) - u_1(x) \geq \psi(x) \qquad \forall x \in U.$$

Then, taking into account statement (50), we can conclude with next theorem

Theorem 9. *For $m = 1$ under the hypotheses of Theorem 7 [resp. Theorem 8], if (u_1, u_2) is solution of (3), all the pairs*

$$(u_1 + \lambda p_0,\ u_2) \quad \text{with} \quad \lambda \leq \inf_{\Omega_0} \frac{u_2 - u_1}{p_0}$$

are solutions of (2).

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