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On Fredholm parameter-dependent boundary-value problems in Sobolev spaces

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We consider the most general class of linear inhomogeneous boundary-value problems for systems of r-th order ordinary differential equations whose solutions and right-hand sides belong to appropriate Sobolev spaces. For parameter-dependent problems from this class, we prove a constructive criterion under which their solutions are continuous in the Sobolev space with respect to the parameter. We also prove a two-sided estimate for the degree of convergence of these solutions to the solution of the nonperturbed problem.

Keywords: *differential system, boundary-value problem, Sobolev space, continuity in parameter.*

Introduction. The investigation of solutions of the systems of ordinary differential equations is an important part of numerous problems of contemporary analysis and its applications. Unlike Cauchy problems, the solutions of such problems may not exist or may not be unique. Thus, it is interesting to investigate the nature of the solvability of inhomogeneous boundary-value problems in Sobolev spaces and the dependence of their solutions on the parameter [1-3]. For first-order differential systems, the most general results on this topic were obtained in [4, 5]. The purpose of the present paper is to generalize these results to differential systems of an arbitrary order.

Statement of the problem. We arbitrarily choose a compact interval $[a,b]\subset\mathbb{R}$ and parameters

$$\{m, n+1, r\}\subset\mathbb{N} \quad \text{and} \quad 1\leqslant p\leqslant\infty.$$

We use the complex Sobolev space $W_p^n := W_p^n([a,b];C)$ and set $W_p^0 := L_p$. By $(W_p^n)^m := W_p^n([a,b]; C^m)$ and $(W_p^n)^{m\times m} := W_p^n([a,b];C^{m\times m})$, we denote the Sobolev spaces of vector-valued and matrix-valued functions, respectively, whose entries belong to the Sobolev space W_p^n of scalar functions on $[a,b]$, with the vectors having m entries and with the matrices being of the $m\times m$ type.

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By $\|\cdot\|_{n,p}$ we denote the norms in these spaces. They are the sums of the corresponding norms in W_p^n of all components of vector-valued or matrix-valued functions from these spaces. It will be always clear from context in which Sobolev space (scalar or vector-valued or matrix-valued functions) these norms are considered. For $m=1$, all these spaces coincide. It is known that the spaces W_p^n are Banach spaces. They are separable, if and only if $p<\infty$.

Let the parameter ε run through the set $[0,\varepsilon_0)$, with the number $\varepsilon_0>0$ being fixed. We consider the following parameter-dependent inhomogeneous boundary-value problem for a system of m linear differential equations of order r:

$$L(\varepsilon)y(t,\varepsilon):=y^{(r)}(t,\varepsilon)+\sum_{j=1}^{r}A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon)=f(t,\varepsilon)\,,\ t\in(a,b)\,, \tag{1}$$

$$B(\varepsilon)y(\cdot,\varepsilon)=c(\varepsilon)\,. \tag{2}$$

Here, for every $\varepsilon\in[0,\varepsilon_0)$, the unknown vector-valued function $y(\cdot,\varepsilon)$ belongs to the space $(W_p^{n+r})^m$, and we arbitrarily choose the matrix-valued functions $A_{r-j}(\cdot,\varepsilon)\in(W_p^n)^{m\times m}$ with $j\in\{1,\ldots,r\}$, vector-valued function $f(\cdot,\varepsilon)\in(W_p^n)^m$, vector $c(\varepsilon)\in\mathbb{C}^{rm}$, and linear continuous operator

$$B(\varepsilon):(W_p^{n+r})^m\to\mathbb{C}^{rm}\,.$$

We interpret vectors and vector-valued functions as columns. Note that the functions $A_{r-j}(t,\varepsilon)$ are not assumed to have any regularity in ε.

Let us indicate the sense in which Eq. (1) is understood. A solution of the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot,\varepsilon)\in(W_p^{n+r})^m$ that satisfies Eq. (1) (everywhere for $n\geqslant 1$ and almost everywhere for $n=0$) on (a,b) and equality (2) (which means rm scalar boundary conditions). The boundary condition (2) with the arbitrary continuous operator $B(\varepsilon)$ is the most general for the differential system (1). Indeed, if the right-hand side $f(\cdot,\varepsilon)$ of the system runs through the whole space $(W_p^n)^m$, then the solution $y(\cdot,\varepsilon)$ of the system runs through the whole space $(W_p^{n+r})^m$. This condition covers all the classical types of boundary conditions such as initial conditions in the Cauchy problem, various multipoint conditions, integral conditions, conditions used in mixed boundary-value problems, and also nonclassical conditions containing the derivatives (generally fractional) $y^{(k)}(\cdot,\varepsilon)$ with $0<k\leqslant n+r-1$. Therefore, the boundary-value problem (1), (2) is generic with respect to the Sobolev space W_p^{n+r}.

The main aim of the present paper is to give a constructive criterion under which the solutions of parameter-dependent problems are continuous in the Sobolev space in a parameter.

Main Results. With the boundary-value problem (1), (2), we associate the linear continuous operator

$$(L(\varepsilon),B(\varepsilon)):(W_p^{n+r})^m\to(W_p^n)^m\times\mathbb{C}^{rm}\,. \tag{3}$$

According to [5, Theorem 1], operator (3) is a Fredholm one with zero index for every $\varepsilon\in[0,\varepsilon_0)$.

Let us consider the following condition.

Condition (0). The homogeneous boundary-value problem

$$L(0)y(t,0)=0, \quad t\in(a,b), \quad B(0)y(\cdot,0)=0$$

has only the trivial solution.

Let us now give our basic concepts.

Definition. The solution of the boundary-value problem (1), (2) depends continuously on the parameter ε at $\varepsilon=0$, if the following two conditions are satisfied:

there exists a positive number $\varepsilon_1<\varepsilon_0$ such that, for any $\varepsilon\in[0,\varepsilon_0)$ and arbitrarily chosen right-hand sides $f(\cdot,\varepsilon)\in(W_p^n)^m$ and $c(\varepsilon)\in\mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot,\varepsilon)$ from the space $(W_p^{n+r})^m$;

the convergence of the right-hand sides $f(\cdot,\varepsilon)\to f(\cdot,0)$ in $(W_p^n)^m$ and $c(\varepsilon)\to c(0)$ in $\mathbb{C}^{rm}$ as $\varepsilon\to 0+$ implies the convergence of the solutions $y(\cdot,\varepsilon)\to y(\cdot,0)$ in $(W_p^{n+r})^m$ as $\varepsilon\to 0+$.

We also consider the next two conditions on the left-hand sides of this problem.

Limit Conditions as $\varepsilon\to 0+$:

(I) $A_{r-j}(\cdot,\varepsilon)\to A_{r-j}(\cdot,0)$ in the space $(W_p^n)^{m\times m}$ for each number $j\in\{1,\dots,r\}$;

(II) $B(\varepsilon)y\to B(0)y$ in the space $\mathbb{C}^{rm}$ for every $y\in(W_p^{n+r})^m$.

Let us formulate the main result of the paper.

Theorem 1. *The solution of the boundary-value problem* (1), (2) *depends continuously on the parameter* ε *at* $\varepsilon=0$, *if and only if this problem satisfies Condition* (0) *and Limit Conditions* (I) *and* (II).

Remark. In the case of $r=1$, Theorem 1 is proved in [4, Theorem 1].

Paper [3] gives a constructive criterion under which the solutions of parameter-dependent problems are continuous in a parameter in the Sobolev spaces W_p^n, where $1\leqslant p<\infty$. The proof of the criterion is based on the fact that the continuous linear operator B, for every $\varepsilon\in[0,\varepsilon_0)$ and $1\leqslant p<\infty$, admits the following unique analytic representation:

$$By=\sum_{s=0}^{n+r-1}\alpha_s y^{(s)}(a)+\int_a^b \Phi(t)y^{(n+r)}(t)\,dt, \quad y(\cdot)\in(W_p^{n+r})^m.$$

Here, the matrices α_s belong to the space $\mathbb{C}^{rm\times m}$, and the matrix-valued function $\Phi(\cdot)$ belongs to the space $L_{p'}([a,b];\mathbb{C}^{rm\times m})$, with $1/p+1/p'=1$. For $p=\infty$, this formula also defines a continuous operator $B:(W_\infty^{n+r})^m\to\mathbb{C}^{rm}$. However, there exist continuous operators from $(W_\infty^{n+r})^m$ to $\mathbb{C}^{rm}$ specified by the integrals over finitely additive measures [6].

Our method of proof allows one to investigate such problems in the Sobolev spaces W_p^n, where $1\leqslant p\leqslant\infty$, and some others function spaces (see [7, 8]).

We supplement our result with a two-sided estimate of the error $\|y(\cdot,0)-y(\cdot,\varepsilon)\|_{n+r,p}$ of the solution $y(\cdot,\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon):=\|L(\varepsilon)y(\cdot,0)-f(\cdot,\varepsilon)\|_{n,p}+\|B(\varepsilon)y(\cdot,0)-c(\varepsilon)\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot,0)$ as an approximate solution of problem (1), (2).

Theorem 2. *Let the boundary-value problem* (1), (2) *satisfy Conditions* (0) *and Limit Conditions* (I) *and* (II). *Then there exist positive numbers* $\varepsilon_2<\varepsilon_1$, γ_1, *and* γ_2 *such that*

$$\gamma_1\widetilde{d}_{n,p}(\varepsilon)\leqslant\|y(\cdot,0)-y(\cdot,\varepsilon)\|_{n+r,p}\leqslant\gamma_2\widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in [0, \varepsilon_2)$ *. Here, the numbers* ε_2 *,* γ_1 *, and* γ_2 *are independent of* $y(\cdot, 0)$ *and* $y(\cdot, \varepsilon)$ *.*

Thus, the error and discrepancy of the solution of problem (1), (2) are of the same degree.

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ПРО ФРЕДГОЛЬМОВІ КРАЙОВІ ЗАДАЧІ
З ПАРАМЕТРОМ У ПРОСТОРАХ СОБОЛЄВА

Досліджено найбільш загальний клас лінійних неоднорідних крайових задач для систем звичайних диференціальних рівнянь довільного порядку, розв'язки і права частина яких належать до відповідних просторів Соболєва. Для залежних від параметрів задач цього класу встановлено конструктивний критерій неперервності за параметром розв'язків у просторі Соболєва. Знайдено двосторонню оцінку швидкості збіжності цих розв'язків до розв'язку незбуреної задачі.

Ключові слова: *диференціальна система, крайова задача, простір Соболєва, неперервність за параметром.*

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О ФРЕДГОЛЬМОВЫХ КРАЕВЫХ ЗАДАЧАХ
С ПАРАМЕТРОМ В ПРОСТРАНСТВАХ СОБОЛЕВА

Исследуется наиболее общий класс линейных неоднородных краевых задач для систем обыкновенных дифференциальных уравнений произвольного порядка, решения и правые части которых принадлежат соответствующим пространствам Соболева. Для зависящих от параметров задач из этого класса установлен конструктивный критерий того, что решения задач непрерывны по параметру в пространстве Соболева. Найдена двусторонняя оценка скорости сходимости этих решений к решению невозмущенной задачи.

Ключевые слова: *дифференциальная система, краевая задача, пространство Соболева, непрерывность по параметру.*