Український математичний вісник
Том 14 **(2017)**, № 4, 575 – 594

Factorization of generalized γ-generating matrices

Olena Sukhorukova

(*Presented by V. O. Derkach*)

Abstract. The class of γ–generating matrices and its subclasses of regular and singular γ–generating matrices were introduced by D. Z. Arov in [8], where it was shown that every γ-generating matrix admits an essentially unique regular–singular factorization. The class of generalized γ-generating matrices was introduced in [14, 20]. In the present paper subclasses of singular and regular generalized γ-generating matrices are introduced and studied. As the main result of the paper a theorem of existence of regular–singular factorization for rational generalized γ-generating matrix is found.

Key words and phrases. γ-generating matrices, J-inner matrix valued function, denominator, associated pair, generalized Schur class, reproducing kernel space, Potapov–Ginzburg transform, Kreĭn–Langer factorization.

1. Introduction

The notion of a γ–generating matrix was introduced by D. Z. Arov in [8] in connection with the study of completely indeterminate Nehari problem on the unit circle $\mathbb{T}$ (see [1,2,10]), and for a real line $\mathbb{R}$ (see [10]).

Let $j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. We recall that a mvf (matrix valued function) $\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where a_{11} and a_{22} are $p \times p$ and $q \times q$ blocks, respectively, is called a γ-generating matrix of the class $\mathfrak{M}^r(j_{pq})$, if:

(1) $\mathfrak{A}$ is measurable on $\mathbb{R}$ and takes j_{pq}-unitary values for a.e. $\mu \in \mathbb{R}$;

Received 14.10.2017
This work was supported by a Volkswagen Stiftung grant and grant of the Ministry of Education and Science of Ukraine (project 0115U000556).

ISSN 1810 – 3200.

(2) $a_{22}(\mu)$ and $a_{11}^*(\mu)$ are boundary values of holomorphic mvf's $a_{22}(\lambda)$ and $a_{11}^{\#}(\lambda)$, such that a_{22}^{-1} and $(a_{11}^{\#})^{-1}$ are outer mvf's from the Schur classes $\mathcal{S}^{p\times p}$ and $\mathcal{S}^{q\times q}$, respectively;

(3^r) $s_{21} := -a_{22}^{-1}a_{21}$ belongs to the Schur class $\mathcal{S}^{q\times p}$ of holomorphic in $\mathbb{C}_+$ with values in the set of contractive mvf's, i.e. $I_p - s(\lambda)^*s(\lambda) \geq 0$ for every point $\lambda \in \mathbb{C}_+$.

The class $\mathfrak{M}^\ell(j_{pq})$ of left γ-generating matrices was introduced in [8] as the set of mvf's $\mathfrak{A}(\mu)$ which satisfies (1), (2) and

(3^ℓ) $s_{12} := a_{12}a_{22}^{-1} \in \mathcal{S}^{p\times q}$.

As was shown in [1,2], any solution of a completely indeterminate matrix Nehari problem can be represented in the form

$$f(\mu) = T_{\mathfrak{A}}[s] = (a_{11}(\mu)s(\mu) + a_{12}(\mu))(a_{21}(\mu)s(\mu) + a_{22}(\mu))^{-1}, \qquad (1.1)$$

where $\mathfrak{A} \in \mathfrak{M}^r(j_{pq})$, and s is a mvf of the Schur class $\mathcal{S}^{p\times q}$.

A mvf $\mathfrak{A} \in \mathfrak{M}^r(j_{pq})$ is said to be right singular γ-generating matrix if $T_{\mathfrak{A}}[\mathcal{S}^{p\times q}] \subset \mathcal{S}^{p\times q}$. A mvf $\mathfrak{A} \in \mathfrak{M}^r(j_{pq})$ is said to be right regular γ-generating matrix if the factorization $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$ with a factor $\mathfrak{A}_1 \in \mathfrak{M}_r(j_{pq})$ and a right singular factor $\mathfrak{A}_2$ implies that $\mathfrak{A}_2$ is constant. These two subclasses of $\mathfrak{M}^r(j_{pq})$ will be designated $\mathfrak{M}^{r,S}(j_{pq})$ and $\mathfrak{M}^{r,R}(j_{pq})$, respectively.

Similarly, the classes $\mathfrak{M}^{\ell,S}(j_{pq})$ and $\mathfrak{M}^{\ell,R}(j_{pq})$ were introduced in [8, 10] and in fact the classes $\mathfrak{M}^{r,S}(j_{pq})$ and $\mathfrak{M}^{\ell,S}(j_{pq})$ coincide:

$$\mathfrak{M}^S(j_{pq}) := \mathfrak{M}^{r,S}(j_{pq}) = \mathfrak{M}^{\ell,S}(j_{pq}).$$

As was shown in [8] a resolvent matrix $\mathfrak{A}$ which describes solutions of the Nehari problem is a right regular γ-generating matrix.

In [8] it was shown that any γ-generating matrix admits a factorization

$$\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2, \quad \text{where} \quad \mathfrak{A}_1 \in \mathfrak{M}^{r,R}(j_{pq}),\, \mathfrak{A}_2 \in \mathfrak{M}^S(j_{pq}).$$

Classes $\mathfrak{M}_\kappa^r(j_{pq})$ and $\mathfrak{M}_\kappa^\ell(j_{pq})$ of generalized γ-generating matrices were introduced in [14,20], where also connections between generalized γ-generating matrices of the class $\mathfrak{M}_\kappa^r(j_{pq})$ (resp. $\mathfrak{M}_\kappa^\ell(j_{pq})$) and generalized j_{pq}-inner mvf's of the class $\mathcal{U}_\kappa^r(j_{pq})$ (resp. $\mathcal{U}_\kappa^\ell(j_{pq})$) were established.

Sufficient conditions for regular–singular factorization of generalized j_{pq}-inner mvf were found in [15]. In the present paper the notions of singular and regular right and left generalized γ-generating mvf's are introduced and studied.

Sufficient conditions for existance of regular-singular factorization for right and left generalized γ-generating mvf's are also found.

1.1. The generalized Schur class

Let Ω_+ be equal to either $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ or $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : -i(\lambda - \bar{\lambda}) > 0\}$. Let us set

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\overline{\omega}, & \text{if } \Omega_+ = \mathbb{D}; \\ -2\pi i(\lambda - \overline{\omega}), & \text{if } \Omega_+ = \mathbb{C}_+. \end{cases}$$

and let $\Omega_- := \{\omega \in \mathbb{C} : \rho_\omega(\omega) < 0\}$. Then $\Omega_0 := \partial\Omega_+$ is either the unit circle $\mathbb{T}$, if $\Omega_+ = \mathbb{D}$, or the real line $\mathbb{R}$, if $\Omega_+ = \mathbb{C}_+$.

Let $\kappa \in \mathbb{Z}_+$. Recall [5], that a Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m\times m}$ is said to have κ negative squares, if for every positive integer n and every choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ $(j = 1, \dots, n)$ the matrix

$$(u_k^* \mathsf{K}_{\omega_j}(\omega_k) u_j)_{j,k=1}^n$$

has at most κ, and for some choice of $n \in \mathbb{N}$, $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ exactly κ negative eigenvalues.

Denote by $\mathfrak{h}_s$ the domain of holomorphy of the mvf $s(\lambda)$ and let us set $\mathfrak{h}_s^\pm := \mathfrak{h}_s \cap \Omega_\pm$.

Let $\mathcal{S}_\kappa^{q\times p}$ denote *the generalized Schur class* of $q \times p$ mvf's s that are meromorphic in Ω_+ and for which the kernel

$$\Lambda_\omega^s(\lambda) = \frac{I_p - s(\lambda)s(\omega)^*}{\rho_\omega(\lambda)} \tag{1.2}$$

has κ negative squares on $\mathfrak{h}_s^+ \times \mathfrak{h}_s^+$ (see [17]). In the case where $\kappa = 0$ the class $\mathcal{S}_0^{q\times p}$ coincides with the Schur class $\mathcal{S}^{q\times p}$. A mvf $s \in \mathcal{S}^{q\times p}$ is said to be inner ($s \in \mathcal{S}_{in}^{q\times p}$), if $I_p - s(\mu)^* s(\mu) = 0$ for a.e. point $\mu \in \Omega_0$. Mvf $s \in \mathcal{S}^{q\times p}$ is said to be outer ($s \in \mathcal{S}_{out}^{q\times p}$), if $\overline{sH_2^p} = H_2^q$.

As was shown in [17] every mvf $s \in \mathcal{S}_\kappa^{q\times p}$ admits a factorization of the form

$$s(\lambda) = b_\ell(\lambda)^{-1} s_\ell(\lambda), \quad \lambda \in \mathfrak{h}_s^+, \tag{1.3}$$

where $b_\ell \in \mathcal{S}_{in}^{q\times q}$ is a $q \times q$ BP (Blaschke–Potapov) product of degree κ (see. [10]), $s_\ell \in \mathcal{S}^{q\times q}$ and

$$\operatorname{rank} \begin{bmatrix} b_\ell(\lambda) & s_\ell(\lambda) \end{bmatrix} = q \quad (\lambda \in \Omega_+). \tag{1.4}$$

The representation (1.3) is called a *left KL (Krein–Langer) factorization.* Similarly, every generalized Schur function $s \in \mathcal{S}_\kappa^{q\times p}$ admits a *right KL-factorization*

$$s(\lambda) = s_r(\lambda) b_r(\lambda)^{-1} \quad \text{for } \lambda \in \mathfrak{h}_s^+, \tag{1.5}$$

where $b_r \in \mathcal{S}^{p\times p}$ is a BP-product of degree κ, $s_r \in \mathcal{S}^{q\times p}$ and

$$\operatorname{rank}\begin{bmatrix} b_r(\lambda)^* & s_r(\lambda)^* \end{bmatrix} = p \quad (\lambda \in \Omega_+). \tag{1.6}$$

Recall the notations (see [10]): $\mathcal{R}^{p\times q}$ – the class of rational $p \times q$ mvf's,

$$\mathcal{N}_{\pm}^{p\times q} = \{f = h^{-1}g : g \in H_{\infty}^{p\times q}(\Omega_{\pm}), h \in \mathcal{S}_{out}^{1\times 1}(\Omega_{\pm})\};$$

$$\mathcal{N}_{out}^{p\times q} = \{f = h^{-1}g : g \in \mathcal{S}_{out}^{p\times q}, h \in \mathcal{S}_{out}^{1\times 1}\}.$$

The limit values $f(\mu)$ of mvf $f(\lambda) \in \mathcal{N}^{p\times q}(\mathbb{C}_+)$ $(\mathcal{N}^{p\times q}(\mathbb{D}))$ are defined a.e. on $\mathbb{R}$ $(\mathbb{T})$

$$f(\mu) = \lim_{\nu\downarrow 0} f(\mu + i\nu) \quad (f(\mu) = \lim_{r\uparrow 1} f(r\mu)). \tag{1.7}$$

Similarly, the limit values of $f \in \mathcal{N}^{p\times q}(\Omega_-)$ are defined a.e. on Ω_0.

Definition 1.1. *A $p\times q$ mvf f_- in Ω_- is said to be a pseudocontinuation of a mvf $f \in \mathcal{N}^{p\times q}$, if*

(1) $f_-^{\#} \in \mathcal{N}^{p\times q}$;

(2) $f_-(\mu) = f(\mu)$ a.e. on Ω_0.

The subclass of all mvf's $f \in \mathcal{N}^{p\times q}$ that admit pseudocontinuations f_- into Ω_- will be denoted $\Pi^{p\times q}$. Sometimes the superindex $p\times q$ is dropped and we denote this class by Π if it does not lead to confusion.

1.2. Generalized j_{pq}-inner mvf's

Definition 1.2. [4,13] *An $m \times m$ mvf $W(\lambda)$ that is meromorphic in Ω_+ is said to belong to the class $\mathcal{U}_\kappa(j_{pq})$ of generalized j_{pq}-inner mvf's, if:*

(i) the kernel

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)} \tag{1.8}$$

has κ negative squares in $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$, where $\mathfrak{h}_W^+$ denotes the domain of holomorphy of W in Ω_+ and

(ii) $j_{pq} - W(\mu)j_{pq}W(\mu)^* = 0$ a.e. on the boundary Ω_0 of Ω_+.

Let us recall some facts concerning the PG (Potapov–Ginzburg) transform of generalized j_{pq}-inner mvf's. As is known [4, Theorem 6.8], for every $W \in \mathcal{U}_\kappa(j_{pq})$ the matrix $w_{22}(\lambda)$ is invertible for all $\lambda \in \mathfrak{h}_W^+$ except for at most κ point in Ω_+. The PG-transform $S = PG(W)$ of W (see [3])

$$S(\lambda) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1} \tag{1.9}$$

is well defined for those $\lambda \in \mathfrak{h}_W^+$, for which $w_{22}(\lambda)$ is invertible, $S(\lambda)$ belongs to the class $\mathcal{S}_\kappa^{m\times m}$ and $S(\mu)$ is unitary for a.e. $\mu \in \Omega_0$ (see [4,13]).

The formula (1.9) can be rewritten as

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} w_{11} - w_{12}w_{22}^{-1}w_{21} & w_{12}w_{22}^{-1} \\ -w_{22}^{-1}w_{21} & w_{22}^{-1} \end{bmatrix}. \tag{1.10}$$

Since the mvf $S(\lambda)$ has unitary nontangential boundary limits a.e. on Ω_0, the pseudocontinuation of S to Ω_- can be defined by the formula $S(\lambda) = (S^\#(\lambda))^{-1}$, where the reflection function $S^\#(\lambda)$ is defined by

$$S^\#(\lambda) = S(\lambda^\circ)^*, \quad \lambda^\circ = \begin{cases} 1/\overline{\lambda} & : \text{ if } \Omega_+ = \mathbb{D},\ \lambda \neq 0; \\ \overline{\lambda} & : \text{ if } \Omega_+ = \mathbb{C}_+. \end{cases} \tag{1.11}$$

1.3. The class $\mathcal{U}_\kappa^r(j_{pq})$

Definition 1.3. [13] *An $m \times m$ mvf $W(\lambda) \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^r(j_{pq})$, if*

$$s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q\times p}. \tag{1.12}$$

Theorem 1.4. [13] *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let the BP-factors b_ℓ and b_r be defined by the KL-factorizations of s_{21}:*

$$s_{21}(\lambda) := b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1}, \quad \lambda \in \mathfrak{h}_{s_{21}}^+, \tag{1.13}$$

where $b_\ell \in \mathcal{S}_{in}^{q\times q}$, $b_r \in \mathcal{S}_{in}^{p\times p}$, $s_\ell, s_r \in \mathcal{S}^{q\times p}$. Then the mvf's $b_\ell s_{22}$ and $s_{11}b_r$ are holomorphic in Ω_+, and hence they admit the following inner-outer and outer-inner factorizations

$$s_{11}b_r = b_1a_1, \qquad b_\ell s_{22} = a_2b_2, \tag{1.14}$$

where $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$, $a_1 \in \mathcal{S}_{out}^{p\times p}$, $a_2 \in \mathcal{S}_{out}^{q\times q}$.

The pair $\{b_1, b_2\}$ is called the *right associated pair* of the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ and is written as $\{b_1, b_2\} \in \mathrm{ap}^r(W)$. In the case $\kappa = 0$ this notion was introduced in [6].

Proposition 1.5. [13, 16] *If $s \in \mathcal{S}^{q\times p}$, then there exists a set of mvf's $c_\ell \in H_\infty^{q\times q}$, $d_\ell \in H_\infty^{p\times q}$, $c_r \in H_\infty^{p\times p}$ and $d_r \in H_\infty^{p\times q}$, such that*

$$\begin{bmatrix} c_r & d_r \\ -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} b_r & -d_\ell \\ s_r & c_\ell \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}. \tag{1.15}$$

If, in addition, $s \in \Pi$, then c_ℓ, d_ℓ, c_r, d_r can be chosen from Π.

Proof. The first statement was proved in [13, Theorem 4.9] (the rational case was treated in [16]).

Assume now that $s \in \Pi$ and hence also $s_\ell \in \Pi$. Let d_ℓ be a rational mvf's such that

$$b_\ell^{-1}(I_q - s_\ell d_\ell) \in H_\infty^{q\times q}.$$

Such a mvf can be chosen via matrix Lagrange–Silvester interpolation. Then by setting

$$c_\ell := b_\ell^{-1}(I_p - s_\ell d_\ell)$$

one obtains $c_\ell \in H_\infty^{q\times q} \cap \Pi^{q\times q}$, since $b_\ell, s_\ell, d_\ell \in \Pi$.

The inclusions $c_r, d_r \in \Pi$ are implied by (1.15). □

By [13, Theorem 4.11] for every $W \in \mathcal{U}_\kappa^r(j_{pq})$ and c_ℓ and d_ℓ as in (1.15) the mvf

$$K = (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1}, \tag{1.16}$$

belongs to $H_\infty^{p\times q}$ and admits the representations

$$K = (-w_{11}d_\ell + w_{12}c_\ell)a_2 b_2, \tag{1.17}$$

where $\{b_1, b_2\} \in ap^r(W)$ and $a_2 \in \mathcal{S}_{out}^{q\times q}$ is determined by (1.14).

1.4. The class $\mathcal{U}_\kappa^\ell(j_{pq})$

The following definitions and statements concerning the dual class $\mathcal{U}_\kappa^\ell(j_{pq})$ are taken from [19].

Definition 1.6. *An $m \times m$ mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^\ell(j_{pq})$, if*

$$s_{12} := w_{12}w_{22}^{-1} \in \mathcal{S}_\kappa^{p\times q}. \tag{1.18}$$

If $W \in \mathcal{U}_\kappa(j_{pq})$ and the mvf $\widetilde{W}$ is defined by

$$\widetilde{W}(\lambda) = \begin{cases} W(\overline{\lambda})^*, & \text{if} \quad \Omega_+ = \mathbb{D}, \\ W(-\overline{\lambda})^* & \text{if} \quad \Omega_+ = \mathbb{C}_+. \end{cases} \tag{1.19}$$

then, as was shown in [19], the following equivalence holds:

$$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \Longleftrightarrow \widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq}) \tag{1.20}$$

and as a corollary of Theorem 1.4 one can get the following statement.

Theorem 1.7. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and let the BP-factors $\mathfrak{b}_\ell$ and $\mathfrak{b}_r$ be defined by the KL-factorizations of s_{12}:*

$$s_{12}(\lambda) = \mathfrak{b}_\ell(\lambda)^{-1}\mathfrak{s}_\ell(\lambda) = \mathfrak{s}_r(\lambda)\mathfrak{b}_r(\lambda)^{-1}, \quad (\lambda \in \mathfrak{h}_{s_{12}}^+), \tag{1.21}$$

where $\mathfrak{b}_\ell \in \mathcal{S}_{in}^{p\times p}$, $\mathfrak{b}_r \in \mathcal{S}_{in}^{q\times q}$, $\mathfrak{s}_\ell, \mathfrak{s}_r \in \mathcal{S}^{p\times q}$. Then

$$s_{22}\mathfrak{b}_r \in \mathcal{S}^{q\times q} \quad \text{and} \quad \mathfrak{b}_\ell s_{11} \in \mathcal{S}^{p\times p}. \tag{1.22}$$

Definition 1.8. *Consider inner-outer and outer-inner factorizations of $\mathfrak{b}_\ell s_{11}$ and $s_{22}\mathfrak{b}_r$*

$$\mathfrak{b}_\ell s_{11} = \mathfrak{a}_1\mathfrak{b}_1, \qquad s_{22}\mathfrak{b}_r = \mathfrak{b}_2\mathfrak{a}_2, \tag{1.23}$$

where $\mathfrak{b}_1 \in \mathcal{S}_{in}^{p\times p}$, $\mathfrak{b}_2 \in \mathcal{S}_{in}^{q\times q}$, $\mathfrak{a}_1 \in \mathcal{S}_{out}^{p\times p}$, $\mathfrak{a}_2 \in \mathcal{S}_{out}^{q\times q}$. The pair $\mathfrak{b}_1, \mathfrak{b}_2$ of inner factors in the factorizations (1.23) *is called the left associated pair of the mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and is written as $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$, for short.*

Remark 1.9. As was shown in [19] (3.25) if $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$, then $\widetilde{s}_{11}\widetilde{\mathfrak{b}}_\ell = \widetilde{\mathfrak{b}}_1\widetilde{\mathfrak{a}}_1$, $\quad \widetilde{\mathfrak{b}}_r\widetilde{s}_{22} = \widetilde{\mathfrak{a}}_2\widetilde{\mathfrak{b}}_2$, and, therefore, $\{\widetilde{\mathfrak{b}}_1, \widetilde{\mathfrak{b}}_2\} \in ap^r(\widetilde{W})$.

As was shown in [19], there exists a set of mvf's $\mathfrak{c}_\ell \in H_\infty^{p\times p}$, $\mathfrak{d}_\ell \in H_\infty^{q\times p}$, $\mathfrak{c}_r \in H_\infty^{q\times q}$ and $\mathfrak{d}_r \in H_\infty^{q\times p}$, such that

$$\begin{bmatrix} \mathfrak{c}_\ell & \mathfrak{s}_r \\ \mathfrak{d}_\ell & \mathfrak{b}_r \end{bmatrix} \begin{bmatrix} \mathfrak{b}_\ell & -\mathfrak{s}_\ell \\ -\mathfrak{d}_r & \mathfrak{c}_r \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}. \tag{1.24}$$

1.5. Reproducing kernel Pontryagin spaces

In this subsection we review some facts and notation from [11–13] on the theory of indefinite inner product spaces for the convenience of the reader. A linear space $\mathcal{K}$ equipped with a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ on $\mathcal{K}\times\mathcal{K}$ is called an indefinite inner product space. A subspace $\mathcal{F}$ of $\mathcal{K}$ is called positive (resp. negative) if $\langle f, f\rangle_{\mathcal{K}} > 0$, (resp. < 0) for all $f \in \mathcal{F}$, $f \neq 0$.

An indefinite inner product space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ is called a Pontryagin space, if it can be decomposed as the orthogonal sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \tag{1.25}$$

of a positive subspace $\mathcal{K}_+$ which is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and a negative subspace $\mathcal{K}_-$ of finite dimension. The number $\operatorname{ind}_-\mathcal{K} := \dim\mathcal{K}_-$ is referred to as the negative index of $\mathcal{K}$.

The isotropic part of $\mathcal{L} \subset \mathcal{K}$ is defined by $\mathcal{L}_0 := \{x \in \mathcal{L} : \langle x, y\rangle_{\mathcal{L}} = 0, y \in \mathcal{L}\}$. The subspace $\mathcal{L}$ is called nondegenerate iff $\mathcal{L}_0 = \{0\}$.

A Pontryagin space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ of $\mathbb{C}^m$-valued functions defined on a subset Ω of $\mathbb{C}$ is called *a RKPS (reproducing kernel Pontryagin space)*, if there exists a Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega\times\Omega \to \mathbb{C}^{m\times m}$, such that:

(1) for every $\omega \in \Omega$ and every $u \in \mathbb{C}^m$ the vvf $\mathsf{K}_\omega(\lambda)u$ belongs to $\mathcal{K}$;

(2) for every $f \in \mathcal{K}$, $\omega \in \Omega$ and $u \in \mathbb{C}^m$ the following identity holds

$$\langle f, \mathsf{K}_\omega u\rangle_{\mathcal{K}} = u^* f(\omega). \tag{1.26}$$

It is known (see [18]) that for every Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega\times\Omega \to \mathbb{C}^{m\times m}$ with a finite number κ of negative squares on $\Omega\times\Omega$ there is a unique Pontryagin space $\mathcal{K}$ with reproducing kernel $\mathsf{K}_\omega(\lambda)$, and that $\operatorname{ind}_-\mathcal{K} = \operatorname{sq}_-\mathsf{K} = \kappa$. In the case $\kappa = 0$ this fact is due to Aronszajn [5].

For $W \in \mathcal{U}_\kappa(j_{pq})$ we denote by $\mathcal{K}(W)$ the RKPS associated with the kernel $\mathsf{K}^W_\omega(\lambda)$ defined by (1.8).

2. A-regular–A-singular factorization of generalized J-inner mvf's

A mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is called *A-singular*, if it is an outer mvf (see [6, 19]). The set of A-singular mvf's in $\mathcal{U}_\kappa(j_{pq})$ is denoted by $\mathcal{U}^S_\kappa(j_{pq})$.

We will be also using the following subclasses of the class $\mathcal{U}^S_\kappa(j_{pq})$:

$$\mathcal{U}^{r,S}_\kappa(j_{pq}) := \mathcal{U}^r_\kappa(j_{pq}) \cap \mathcal{N}^{m\times m}_{out}, \quad \mathcal{U}^{\ell,S}_\kappa(j_{pq}) := \mathcal{U}^\ell_\kappa(j_{pq}) \cap \mathcal{N}^{m\times m}_{out}.$$

In the case $\kappa = 0$ the class $\mathcal{U}^S(j_{pq}) := \mathcal{U}^S_0(j_{pq})$ was introduced and characterized in terms of associated pairs by D. Arov in [9]. For $\kappa \neq 0$ a definition of A-singular generalized j_{pq}-inner mvf and its characterization in terms of associated pairs was given in [19].

Theorem 2.1. [19] *Let $W \in \mathcal{U}^r_\kappa(j_{pq})$ and let $\{b_1, b_2\} \in ap^r(W)$. Then:*

$$W \in \mathcal{U}^{r,S}_\kappa(j_{pq}) \Longleftrightarrow b_1 \equiv \text{const}, \quad b_2 \equiv \text{const}.$$

Theorem 2.2. [19] *Let $W \in \mathcal{U}^\ell_\kappa(j_{pq})$ and let $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$. Then:*

$$W \in \mathcal{U}^{\ell,S}_\kappa(j_{pq}) \Longleftrightarrow \mathfrak{b}_1 \equiv \text{const}, \quad \mathfrak{b}_2 \equiv \text{const}.$$

Lemma 2.3. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let $\{b_1, b_2\} \in ap^r(W)$. Then:*

$$W \in \mathcal{U}_\kappa^{r,S}(j_{pq}) \Longleftrightarrow \widetilde{W} \in \mathcal{U}_\kappa^{\ell,S}(j_{pq}).$$

Proof. Let $W \in \mathcal{U}_\kappa^{r,S}(j_{pq})$. Then by Theorem 2.1,

$$b_1 \equiv \text{const}, \quad b_2 \equiv \text{const}.$$

Due to Remark 1.9 one obtains $\widetilde{b}_1 \equiv \text{const}$, $\widetilde{b}_2 \equiv \text{const}$ and hence $\widetilde{W} \in \mathcal{U}_\kappa^{\ell,S}(j_{pq})$ by Theorem 2.2. The proof of the converse is similar. □

Lemma 2.4. [15] *Let a mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ admits a factorization*

$$W = W^{(1)}W^{(2)}, \quad W^{(1)} \in \mathcal{U}_{\kappa_1}(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq}), \tag{2.1}$$

where $\kappa_1 + \kappa_2 = \kappa$. Then:

(i) $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$;

(ii) For $\{b_1, b_2\} \in \text{ap}^r(W)$ and $\{b_1^{(1)}, b_2^{(1)}\} \in \text{ap}^r(W^{(1)})$ one has

$$\theta_1 := (b_1^{(1)})^{-1} b_1 \in S_{in}^{p\times p}, \quad \theta_2 := b_2 (b_2^{(1)})^{-1} \in S_{in}^{q\times q}. \tag{2.2}$$

Definition 2.5. [15] *A mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ is called right A-regular, if for any factorization*

$$W = W^{(1)}W^{(2)}, \quad W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq}), \tag{2.3}$$

with $\kappa_1 + \kappa_2 = \kappa$ the assumption $W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq})$ implies $W^{(2)}(\lambda) \equiv$ const.

Similarly, a mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ is called left A-regular, if for any factorization (2.3) with $\kappa_1 + \kappa_2 = \kappa$ the assumption $W^{(1)} \in \mathcal{U}_{\kappa_1}^S(j_{pq})$ implies $W^{(1)}(\lambda) \equiv$ const.

In the case $\kappa = 0$ Definition 2.5 is simplified since $\mathcal{U}_0^r(j_{pq}) = \mathcal{U}_0^\ell(j_{pq}) = \mathcal{U}(j_{pq})$ (see [7]).

In the next lemma we present one necessary and one sufficient condition for a mvf $W(\lambda) \in \mathcal{U}_\kappa^r(j_{pq})$ to be regular. Let us set

$$\mathcal{L}_W := \mathcal{K}(W) \cap L_2^m. \tag{2.4}$$

Lemma 2.6. [15] *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$, let $\mathcal{K}(W)$ be the RKPS with the kernel $\mathsf{K}_\omega^W(\lambda)$, defined by (1.8), let* $\text{ind}_-\mathcal{L}_W = \kappa$ *and let* $\kappa_1 = \text{ind}_-(\mathcal{L}_W)$, $\kappa_2 = \kappa - \kappa_1$. *Assume that:*

(A1) $\mathfrak{h}_W \cap \Omega_0 \neq \emptyset$;

(A2) The closure $\overline{\mathcal{L}_W}$ *of* $\mathcal{L}_W$ *is nondegenerate in* $\mathcal{K}(W)$.

Then the following implications hold:

(1) $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longrightarrow \overline{\mathcal{L}_W} = \mathcal{K}(W)$;

(2) $\mathcal{K}(\widetilde{W}) \subset L_2^{m\times m} \Longrightarrow W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$.

Denote by $\mathcal{R}^{m\times m}$ the set of rational $m \times m$-mvf's. The following criterion for a rational mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ to be right A-regular is given in [15]. We will present here a simpler proof of this result.

Theorem 2.7. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$ *be a rational mvf. Then*

(1) $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longleftrightarrow \mathcal{L}_W = \mathcal{K}(W)$.

(2) $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longleftrightarrow W \in \widetilde{L}_2^{m\times m}$.

Proof. Let $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \cap \mathcal{R}^{m\times m}$. Then by Lemma 2.6 $\overline{\mathcal{L}_W} = \mathcal{K}(W)$, and since W is rational, $\mathcal{L}_W = \overline{\mathcal{L}_W} = \mathcal{K}(W)$. Therefore, $\mathcal{K}(W) \subset L_2^{m\times m}$. Hence $W \in \widetilde{L}_2^{m\times m}$. The converse is immediate from Lemma 3.19(3) in [15]. □

Lemma 2.8. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$. *Then:*

$$W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longleftrightarrow \widetilde{W} \in \mathcal{U}_\kappa^{\ell,R}(j_{pq}).$$

Proof. Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and assume that $\widetilde{W} = \widetilde{W}^{(1)}\widetilde{W}^{(2)}$, where $\widetilde{W}^{(1)} \in \mathcal{U}_{\kappa_1}^{r,S}(j_{pq})$, $\widetilde{W}^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell}(j_{pq})$. Then

$$W = W^{(2)}W^{(1)}, \quad \text{where} \quad W^{(1)} \in \mathcal{U}_{\kappa_1}^{\ell,S}(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^{r}(j_{pq}).$$

By the regularity of W, $W^{(1)} \equiv const$. Hence $\widetilde{W}^{(1)} \equiv const$ and thus $\widetilde{W} \in \mathcal{U}_\kappa^{\ell,R}(j_{pq})$. The converse implication is obtained similarly. □

The following theorem was proved in [15].

Theorem 2.9. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{R}^{m\times m}$. *Then the following statements are equivalent:*

(1) W *admits the factorization*

$$W = W^{(1)}W^{(2)}, \quad W^{(1)} \in \mathcal{U}_{\kappa_1}^{r,R}(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}) \tag{2.5}$$

with $\kappa = \kappa_1 + \kappa_2$;

(2) $\mathcal{L}_W$ *is a nondegenerate subspace of* $\mathcal{K}(W)$, $ind_-\mathcal{L}_W = \kappa_1$.

Moreover, if (2) is the case then the factors $W^{(1)}$ *and* $W^{(2)}$ *in* (2.5) *are uniquely determined up to* j_{pq}*-unitary factors.*

In the classical case ($\kappa = 0$) this result coincides with the factorization Theorem in [10].

Let us present now an analog of Theorem 2.9 for A-singular-A-regular factorizations.

Corollary 2.10. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{R}^{m\times m}$. *Then the following statements are equivalent:*

(1) W *admits the factorization*

$$W = W^{(2)}W^{(1)}, \quad W^{(1)} \in \mathcal{U}_{\kappa_1}^{\ell,R}(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^{r,S}(j_{pq}) \tag{2.6}$$

with $\kappa = \kappa_1 + \kappa_2$;

(2) $\mathcal{L}_{\widetilde{W}}$ *is a nondegenerate subspace of* $\mathcal{K}(\widetilde{W})$, $ind_-\mathcal{L}_{\widetilde{W}} = \kappa_1$.

Moreover, if (2) is the case then the factors $W^{(1)}$ *and* $W^{(2)}$ *in* (2.5) *are uniquely determined up to* j_{pq}*-unitary factors.*

Proof. Assume that (2) holds and consider the mvf $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{R}^{m\times m}$ see (1.20). By Theorem 2.9

$$\widetilde{W} = \widetilde{W}^{(1)}\widetilde{W}^{(2)}, \quad \text{where} \quad \widetilde{W}^{(1)} \in \mathcal{U}_{\kappa_1}^{r,R}(j_{pq}),\ \widetilde{W}^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}), \tag{2.7}$$

with $\kappa_1 + \kappa_2 = \kappa$. Hence by Lemma 2.3 and 2.8 W admits the factorization (2.6). Conversely, let (1) holds. Then by (1.20), Lemma 2.3 and 2.8 the mvf $\widetilde{W}$ admits the factorization (2.7) and hence by Theorem 2.9 the statement (2) holds. □

The following example illustrates importance of the condition (2) of Theorem 2.9.

Example 2.11. Let

$$W_1(\lambda) = \frac{1}{2\lambda - 2}\begin{bmatrix} \lambda^2 - 3\lambda + 1 & \lambda^2 - \lambda + 1 \\ \lambda^2 - \lambda + 1 & \lambda^2 - 3\lambda + 1 \end{bmatrix}.$$

As was shown in [15], this mvf W_1 belongs to $\mathcal{U}_1^r(j_{11}) \cap \mathcal{U}_1^\ell(j_{11})$ and it does not admit the A-regular $-A$-singular factorization.

The RKPS $\mathcal{K}(W_1)$ and the subspace $\mathcal{L}_{W_1}$ take the form

$$\mathcal{K}(W_1) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\lambda - 1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{L}_{W_1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

and $\mathcal{L}_{W_1}$ is a degenerate subspace of $\mathcal{K}(W_1)$ see [15]. Therefore, condition (2) of Theorem 2.9 does not holds. By Corollary 2.10 W_1 does not admit an A-singular–A-regular factorization.

3. Generalized γ-generating matrices

Definition 3.1. *Let $\mathfrak{M}_\kappa^r(j_{pq})$ denote the class of $m \times m$ mvf's*

$$\mathfrak{A}(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{bmatrix}, \tag{3.1}$$

with blocks a_{11} of size $p \times p$ and a_{22} of size $q \times q$ such that:

(1) $\mathfrak{A}(\mu)$ is a measurable on Ω_0 mvf that is j_{pq}-unitary a.e. on Ω_0;

(2) $s_{21} = -a_{22}^{-1}a_{21} \in \mathcal{S}_\kappa^{q\times p}$;

(3) $(a_{11}^\#)^{-1}b_r = a_1 \in \mathcal{S}_{out}^{p\times p}$, $b_\ell a_{22}^{-1} = a_2 \in \mathcal{S}_{out}^{q\times q}$, where b_ℓ, b_r are BP-products of degree κ which are determined by KL-factorizations of s_{21}.

The mvf's in the class $\mathfrak{M}_\kappa^r(j_{pq})$ are called generalized right γ-generating matrices.

Definition 3.2. *Let $\mathfrak{M}_\kappa^\ell(j_{pq})$ denote the class of $m \times m$ mvf's $\mathfrak{A}(\mu)$ of the form* (3.1)*, such that:*

(1) $\mathfrak{A}(\mu)$ is a measurable on Ω_0 mvf that is j_{pq}-unitary a.e. on Ω_0;

(2) $s_{12} = a_{12}a_{22}^{-1} \in \mathcal{S}_\kappa^{p\times q}$;

(3) $\mathfrak{b}_\ell(a_{11}^\#)^{-1} = \mathfrak{a}_1 \in \mathcal{S}_{out}^{p\times p}$, $a_{22}^{-1}\mathfrak{b}_r = \mathfrak{a}_2 \in \mathcal{S}_{out}^{q\times q}$, where $\mathfrak{b}_\ell$, $\mathfrak{b}_r$ are BP-product of degree κ which are determined by KL-factorizations of s_{12}.

The mvf's in the class $\mathfrak{M}_\kappa^\ell(j_{pq})$ are called generalized left γ-generating matrices.

Definition 3.3. *An ordered pair $\{b_1, b_2\}$ of inner mvf's $b_1 \in \mathcal{N}^{p\times p}$, $b_2 \in \mathcal{N}^{q\times q}$ is called a denominator of the mvf $f \in \mathcal{N}^{p\times q}$, if $b_1 f b_2 \in \mathcal{N}_+^{p\times q}$. The set of denominators of f will be denoted by $den(f)$.*

Theorem 3.4. *Let $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq})$, let b_ℓ, s_ℓ, b_r, s_r be defined by KL-factorization of $s_{21} \in \mathcal{S}_\kappa^{q\times p}$. Let c_ℓ, d_ℓ, c_r, d_r be defined by (1.15) and let*

$$f_0^r := (-a_{11}d_\ell + a_{12}c_\ell)(-a_{21}d_\ell + a_{22}c_\ell)^{-1} = (-a_{11}d_\ell + a_{12}c_\ell)a_2. \quad (3.2)$$

Then:

(i) if $den(f_0^r) \neq \emptyset$ and $\{b_1, b_2\} \in den(f_0^r)$ then

$$W(z) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{bmatrix} \mathfrak{A}(z) \in \mathcal{U}_\kappa^r(j_{pq}), \quad \{b_1, b_2\} \in ap^r(W) \quad (3.3)$$

and hence $\mathfrak{A} \in \Pi^{m\times m}$. Conversely, if

$$W \in \mathcal{U}_\kappa^r(j_{pq}) \quad \text{and} \quad \{b_1, b_2\} \in ap^r(W). \quad (3.4)$$

then

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} W(z) \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^r(j_{pq}) \text{ and } \{b_1, b_2\} \in den(f_0^r).$$

(ii) if $\mathfrak{A} \in \Pi^{m\times m}$ then $den(f_0^r) \neq \emptyset$ and, moreover, for some choice of mvf's c_ℓ, d_ℓ, c_r, d_r in (1.15) one gets $f_0^r \in \Pi$.

(iii) if $\{c_\ell^{(i)}, d_\ell^{(i)}, c_r^{(i)}, d_r^{(i)}\}$ $(i = 1, 2)$ are two sets of mvf's satisfying (1.15) and

$$f_0^{r,i} = (-a_{11}d_\ell^{(i)} + a_{12}c_\ell^{(i)})a_2, \quad i \in \{1, 2\} \quad (3.5)$$

then $den(f_0^{r,1}) = den(f_0^{r,2})$.

Proof. (i) The first implication holds by Theorem 4.3 from [14]. The converse implication follows from Theorem 4.3 and from the fact that $W \in \Pi^{m\times m}$ since W is j_{pq}-unitary. By virtue of $\begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} \in \Pi^{m\times m}$, this implies $\mathfrak{A} \in \Pi^{m\times m}$.
(ii) Since $\mathfrak{A} \in \Pi^{m\times m}$ one has $a_{11}, a_{12}, a_2 \in \Pi$. By Proposition 1.5 the mvf's c_ℓ and d_ℓ can be chosen from Π. Therefore, $f_0^r \in \Pi$.
(iii) Let $\{b_1, b_2\} \in den(f_0^{r,1})$ and let $W(z)$ be given by (3.3). Then by item (i) $W \in \mathcal{U}_\kappa^r(j_{pq})$ and $\{b_1, b_2\} \in ap^r(W)$. Let us set

$$K^{(i)} = (-w_{11}d_\ell^{(i)} + w_{12}c_\ell^{(i)})a_2b_2, \quad i = \{1, 2\}.$$

Then by [13, Theorem 4.11]

$$(b_1 a_1)^{-1}(K^{(1)} - K^{(2)})(a_2 b_2)^{(-1)} \in H_\infty^{p\times q}. \tag{3.6}$$

Since $K^{(i)} = b_1 f_0^{r,i} b_2$ $(i = 1, 2)$ one gets from (3.6)

$$f_0^{r,1} - f_0^{r,2} \in H_\infty^{p\times q}.$$

Therefore, $\{b_1, b_2\} \in den(f_0^{r,2})$. Clearly, the converse implication is also true. □

Remark 3.5. A similar assertion also holds for the class of generalized left γ-generating matrices. Let $\mathfrak{A} \in \mathfrak{M}_\kappa^\ell(j_{pq})$, $\mathfrak{b}_\ell, \mathfrak{s}_\ell, \mathfrak{b}_r, \mathfrak{s}_r$ be defined by KL-factorization of $s_{12} \in \mathcal{S}_\kappa^{q\times p}$. Let $\mathfrak{c}_\ell, \mathfrak{d}_\ell, \mathfrak{c}_r, \mathfrak{d}_r$ defined by (1.24) and let

$$f_0^\ell := \mathfrak{a}_2(-\mathfrak{d}_r a_{11} + \mathfrak{c}_r a_{21}) = (-\mathfrak{d}_r a_{12} + \mathfrak{c}_r a_{22})^{-1}(-\mathfrak{d}_r a_{11} + \mathfrak{c}_r a_{21}). \tag{3.7}$$

Then:

(i) if $den(f_0^\ell) \neq \emptyset$ and $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in den(f_0^\ell)$ then

$$W(z) = \mathfrak{A}(z)\begin{bmatrix} \mathfrak{b}_1 & 0 \\ 0 & \mathfrak{b}_2^{-1} \end{bmatrix} \in \mathcal{U}_\kappa^\ell(j_{pq}) \tag{3.8}$$

and $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$. Conversely, if

$$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \quad \text{and} \quad \{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W), \tag{3.9}$$

then

$$\mathfrak{A}(z) = W(z)\begin{bmatrix} \mathfrak{b}_1^{-1} & 0 \\ 0 & \mathfrak{b}_2 \end{bmatrix} \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^\ell(j_{pq}) \text{ and } \{\mathfrak{b}_1, \mathfrak{b}_2\} \in den(f_0^\ell).$$

(ii) if $\mathfrak{A} \in \Pi^{m\times m}$ then $den f_0^\ell \neq \emptyset$ and, moreover, the mvf's $\mathfrak{c}_\ell, \mathfrak{d}_\ell, \mathfrak{c}_r, \mathfrak{d}_r$ in (1.24) can be chosen from Π and then $f_0^\ell \in \Pi$.

(iii) $\{\mathfrak{c}_\ell^{(i)}, \mathfrak{d}_\ell^{(i)}, \mathfrak{c}_r^{(i)}, \mathfrak{d}_r^{(i)}\}$ $(i = 1, 2)$ two sets of mvf's defined by (1.24)

$$f_0^{\ell,i} = \mathfrak{a}_2(-\mathfrak{d}_r a_{11} + \mathfrak{c}_r a_{21}), \quad \{\mathfrak{b}_1, \mathfrak{b}_2\}\alpha_2(-\mathfrak{d}_r^{(i)} a_{11} + \mathfrak{c}_r^{(i)} a_{21}),$$

$$\{\mathfrak{b}_1, \mathfrak{b}_2\} \in den f_0^{\ell,i}, \quad i = \{1, 2\},$$

then $den f_0^{\ell,1} = den f_0^{\ell,2}$.

Definition 3.6. *We define the denominator of generalized right γ-generating mvf $\mathfrak{A} \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^r(j_{pq})$ as*

$$den^r(\mathfrak{A}) := den f_0^r,$$

and the denominator of left generalized γ-generating mvf $\mathfrak{A} \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^\ell(j_{pq})$ as

$$den^\ell(\mathfrak{A}) := den f_0^\ell.$$

Definition 3.7. *Let a mvf $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq})$ is said to be*

(1) right singular and is written as $\mathfrak{A} \in \mathfrak{M}_\kappa^{r,S}$ if $f_0^r = (-a_{11}d_\ell + a_{12}c_\ell)a_2 \in H_\infty^{p\times q}$,

(2) right regular and is written as $\mathfrak{A} \in \mathfrak{M}_\kappa^{r,R}$ if the factorization $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$, with $\mathfrak{A}_1 \in \mathfrak{M}_{\kappa_1}^r(j_{pq})$ and $\mathfrak{A}_2 \in \mathfrak{M}_{\kappa_2}^{\ell,S}(j_{pq})$, $\kappa_1+\kappa_2=\kappa$ implies that $\mathfrak{A}_2 \equiv const$.

Definition 3.8. *Let a mvf $\mathfrak{A} \in \mathfrak{M}_\kappa^\ell(j_{pq})$ is said to be*

(1) left singular and is written as $\mathfrak{A} \in \mathfrak{M}_\kappa^{\ell,S}$ if $f_0^\ell = \mathfrak{a}_2(-\mathfrak{d}_r a_{11}+\mathfrak{c}_r a_{21}) \in H_\infty^{p\times q}$,

(2) left regular and is written as $\mathfrak{A} \in \mathfrak{M}_\kappa^{\ell,R}$ if the factorization $\mathfrak{A} = \mathfrak{A}_2\mathfrak{A}_1$, with $\mathfrak{A}_1 \in \mathfrak{M}_{\kappa_1}^\ell(j_{pq})$ and $\mathfrak{A}_2 \in \mathfrak{M}_{\kappa_2}^{rS}(j_{pq})$, $\kappa_1+\kappa_2=\kappa$ implies that $\mathfrak{A}_2 \equiv const$.

In the case $\kappa = 0$, the left singularity coincides with the right singularity, therefore our definition coincides with the definition in [8].

4. Fatorization of γ-generating matrices

Lemma 4.1. *Let $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq}) \cap \Pi^{m\times m}$. Then:*

$$\mathfrak{A} \in \mathfrak{M}_\kappa^{r,S}(j_{pq}) \Longleftrightarrow \mathfrak{A} \in \mathcal{U}_\kappa^{r,S}(j_{pq}).$$

Proof. Let $\mathfrak{A} \in \mathfrak{M}_\kappa^{r,S}(j_{pq})$, then $f_0 = (-a_{11}d_\ell + a_{12}c_\ell)a_2 \in H_\infty^{p\times p}$, therefore $\{I_p, I_q\} \in den f_0^r$. In view of Theorem 3.4 this implies $\mathfrak{A} \in \mathcal{U}_\kappa^r(j_{pq})$ and $\{I_p, I_q\} \in ap^r(\mathfrak{A})$. Hence by Theorem 2.1 $\mathfrak{A} \in \mathcal{U}_\kappa^{r,S}(j_{pq})$.

Conversely, if $\mathfrak{A} \in \mathcal{U}_\kappa^{r,S}(j_{pq})$ and $\{b_1, b_2\} \in ap^r(\mathfrak{A})$, then $b_i \equiv const$, $i = \{1,2\}$. Hence by Theorem 3.4 $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq})$, $\{b_1, b_2\} \in den f_0^r$, i.e. $f_0^r \in H_\infty^{p\times q}$, and thus $\mathfrak{A} \in \mathfrak{M}_\kappa^{r,S}(j_{pq})$ □

Corollary 4.2. *Let $\mathfrak{A} \in \mathfrak{M}_\kappa^\ell(j_{pq}) \cap \Pi^{m\times m}$. Then:*

$$\mathfrak{A} \in \mathfrak{M}_\kappa^{\ell,S}(j_{pq}) \Longleftrightarrow \mathfrak{A} \in \mathcal{U}_\kappa^{\ell,S}(j_{pq}).$$

Proof. Let $\mathfrak{A} \in \mathfrak{M}_\kappa^{\ell,S}(j_{pq})$, then $\widetilde{\mathfrak{A}} \in \mathfrak{M}_\kappa^{r,S}(j_{pq})$, hence by Lemma 4.1 $\widetilde{\mathfrak{A}} \in \mathcal{U}_\kappa^{r,S}(j_{pq})$, and thus $\mathfrak{A} \in \mathcal{U}_\kappa^{\ell,S}(j_{pq})$. Analogously, the assumption $\mathfrak{A} \in \mathcal{U}_\kappa^{\ell,S}$ implies $\mathfrak{A} \in \mathfrak{M}_\kappa^{\ell,S}$. □

Lemma 4.3. *Let* $\mathfrak{A}', \mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq})$ *and* $\mathfrak{A}' = \begin{bmatrix} \theta_1^{-1} & 0 \\ 0 & \theta_2 \end{bmatrix} \mathfrak{A}$, $\theta_1 \in \mathcal{S}_{in}^{p\times p}$, $\theta_2 \in \mathcal{S}_{in}^{q\times q}$. *Then* $\theta_1 \equiv const$, $\theta_2 \equiv const$.

Proof. Let $\mathfrak{A}'(\mu) = \begin{bmatrix} a'_{11}(\mu) & a'_{12}(\mu) \\ a'_{21}(\mu) & a'_{22}(\mu) \end{bmatrix}$ and let the mvf $\mathfrak{A}(\mu)$ has block representation (3.1). Then

$$\mathfrak{A}' = \begin{bmatrix} \theta_1^{-1} & 0 \\ 0 & \theta_2 \end{bmatrix} \mathfrak{A} = \begin{bmatrix} \theta_1^{-1} a_{11} & \theta_1^{-1} a_{12} \\ \theta_2 a_{21} & \theta_2 a_{22} \end{bmatrix},$$

and hence by Definition 3.1

$$s'_{21} := -(a'_{22})^{-1} a'_{21} = -a_{22}^{-1}\theta_2^{-1}\theta_2 a_{21} = -a_{22}^{-1} a_{21} = s_{21} \in \mathcal{S}_\kappa^{q\times p}.$$

This means that the Krein-Langer factorizations of s_{21} and s'_{21} coincide

$$s'_{21} = s_{21} = b_\ell^{-1} s_\ell = s_r b_r^{-1},$$

where $b_\ell \in \mathcal{S}_{in}^{q\times q}$, $b_r \in \mathcal{S}_{in}^{q\times q}$, $s_\ell, s_r \in \mathcal{S}^{q\times p}$. Hence

$$a'_1 = (a'_{11})^{-\#} b_r = \theta_1^{\#}(a_{11})^{-\#} b_r \in \mathcal{S}_{out}^{p\times p}, \quad \text{and} \quad a_1 = a_{11}^{-\#} b_r \in \mathcal{S}_{out}^{q\times p}.$$

This is possible only when $\theta_1 \equiv const$. Analogously,

$$a'_2 = b_\ell (a'_{22})^{-1} = b_\ell a_{22}^{-1}\theta_2^{-1} \in \mathcal{S}_{out}^{q\times q} \quad \text{and} \quad a_2 = b_\ell a_{22}^{-1} \in \mathcal{S}_{out}^{q\times q}$$

consequently $\theta_2 \equiv const$. □

Lemma 4.4. *Let a mvf* $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq}) \cap \Pi^{m\times m}$ *admits the factorization*

$$\mathfrak{A} = \mathfrak{A}^{(1)}\mathfrak{A}^{(2)}, \quad \textit{where} \quad \mathfrak{A}^{(1)} \in \mathfrak{M}_{\kappa_1}^r(j_{pq}), \quad \mathfrak{A}^{(2)} \in \mathfrak{M}_{\kappa_2}^{\ell,S}(j_{pq}), \tag{4.1}$$

with $\kappa_1 + \kappa_2 = \kappa$. *Then* $den^r(\mathfrak{A}^{(1)}) \subset den^r(\mathfrak{A})$.

Proof. Let a mvf $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq}) \cap \Pi^{m\times m}$ admits the factorization (4.1). Since $\mathfrak{A}^{(2)} \in \mathfrak{M}_{\kappa_2}^{\ell,S}$, then $f_0 \in H_\infty$ and then $\mathfrak{A}^{(2)} \in \Pi^{m\times m}$. Therefore $\mathfrak{A}^{(1)} = \mathfrak{A}(\mathfrak{A}^{(2)})^{-1}$ and thus $\mathfrak{A}^{(1)} \in \Pi$. Let $\{b_1^{(1)}, b_2^{(1)}\} \in den^r(\mathfrak{A}^{(1)})$ and $\kappa_1 + \kappa_2 = \kappa$. By Theorem 3.4

$$W^{(1)} = \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A}^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq}), \quad \{b_1^{(1)}, b_2^{(2)}\} \in ap^r W^{(1)},$$

$$W^{(2)} = \mathfrak{A}^{(2)} \in \mathcal{U}^{\ell,S}_{\kappa_2}(j_{pq}).$$

Let us set

$$W' := \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A} = \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A}^{(1)}\mathfrak{A}^{(2)} = W^{(1)}W^{(2)}.$$

Then $W' \in \mathcal{U}_{\kappa'}$, $\kappa' \le \kappa_1 + \kappa_2 = \kappa$ (see [4] or [13]).

On the other hand, $s_{21} = -(w'_{22})^{-1}w'_{21} = -a_{22}^{-1}a_{21} \in \mathcal{S}_\kappa^{p\times q}$, hence $\kappa' \ge \kappa$ and therefore $\kappa' = \kappa$. Then $W' \in \mathcal{U}^r_\kappa(j_{pq})$.

Let $\{b'_1, b'_2\} \in ap^r(W')$, hence, in view of Lemma 2.4 $b'_1 = b_1^{(1)}\theta_1$, $b'_2 = \theta_2 b_2^{(1)}$. By Theorem 3.4

$$\mathfrak{A}' = \begin{bmatrix} {b'}_1^{-1} & 0 \\ 0 & b'_2 \end{bmatrix} W' = \begin{bmatrix} {b'}_1^{-1} & 0 \\ 0 & b'_2 \end{bmatrix} W^{(1)}W^{(2)}$$

$$= \begin{bmatrix} {b'}_1^{-1} & 0 \\ 0 & b'_2 \end{bmatrix} \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(2)})^{-1} \end{bmatrix} \mathfrak{A}^{(1)}\mathfrak{A}^{(2)} = \begin{bmatrix} \theta_1^{-1} & 0 \\ 0 & \theta_2 \end{bmatrix} \mathfrak{A} \in \mathfrak{M}^r_\kappa(j_{pq}),$$

Hence, by Lemma 4.3 $\theta_1 \equiv const$, $\theta_2 \equiv const$. Consequently $\{b_1^{(1)}, b_2^{(1)}\} \in ap^r(W')$. Thus by Theorem 3.4 $\{b_1^{(1)}, b_2^{(1)}\} \in den^r(\mathfrak{A})$.

□

Lemma 4.5. *Let a mvf* $\mathfrak{A} \in \mathfrak{M}^r_\kappa(j_{pq}) \cap \widetilde{L}_2^{p\times q} \cap \mathcal{R}^{m\times m}$. *Then* $\mathfrak{A} \in \mathfrak{M}^{r,R}_\kappa(j_{pq})$.

Proof. Let $\mathfrak{A} = \mathfrak{A}^{(1)}\mathfrak{A}^{(2)}$, where $\mathfrak{A}^{(1)} \in \mathfrak{M}^r_{\kappa_1}(j_{pq})$, $\mathfrak{A}^{(2)} \in \mathfrak{M}^{\ell,S}_{\kappa_2}(j_{pq})$, $\kappa_1 + \kappa_2 = \kappa$. Let $\{b_1^{(1)}, b_2^{(1)}\} \in den^r(\mathfrak{A}^{(1)})$, then by Lemma 4.4 the pair $\{b_1^{(1)}, b_2^{(2)}\} \in den^r(\mathfrak{A})$. By Theorem 3.4

$$W^{(1)} = \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A}^{(1)} \in \mathcal{U}^r_{\kappa_1}(j_{pq}) \quad W^{(2)} = \mathfrak{A}^{(2)} \in \mathcal{U}^{\ell,S}_{\kappa_2}(j_{pq}),$$

and

$$W = \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A} = \begin{bmatrix} b_1^{(1)} & 0 \\ 0 & (b_2^{(1)})^{-1} \end{bmatrix} \mathfrak{A}^{(1)}\mathfrak{A}^{(2)}$$
$$= W^{(1)}W^{(2)} \in \mathcal{U}_\kappa(j_{pq}).$$

Since $W \in \mathcal{U}_\kappa(j_{pq}) \cap \widetilde{L}_2^{m\times m}$, then by Theorem 2.7(2) $W \in \mathcal{U}^{r,R}_\kappa(j_{pq})$.

By this condition $\mathfrak{A}^{(2)} = W^{(2)} \equiv const$. This implies $\mathfrak{A} \in \mathfrak{M}^{r,R}_\kappa(j_{pq})$.

□

An analogous statement for the left class $\mathfrak{M}^{\ell}_{\kappa}(j_{pq})$ can be easily obtained with the help of the transformation (1.19).

Lemma 4.6. *Let a mvf* $\mathfrak{A} \in \mathfrak{M}^{\ell}_{\kappa}(j_{pq}) \cap \widetilde{L}_2^{p\times q} \cap \mathcal{R}^{m\times m}$. *Then* $\mathfrak{A} \in \mathfrak{M}^{\ell,R}_{\kappa}(j_{pq})$.

Theorem 4.7. *Let a mvf* $\mathfrak{A} \in \mathfrak{M}^{r}_{\kappa}(j_{pq}) \cap \mathcal{R}^{m\times m}$, $\{b_1, b_2\} \in den^r(\mathfrak{A})$, *let* W *be given by* (3.3) *and let:*

(1) $W(z) \in \mathcal{U}^{\ell}_{\kappa}(j_{pq})$,

(2) $\mathcal{L}_W$ *be a nondegenerate subspace of* $\mathcal{K}(W)$.

Then the mvf $\mathfrak{A}$ *admits regular–singular factorization*

$$\mathfrak{A} = \mathfrak{A}^{(1)}\mathfrak{A}^{(2)}, \quad \mathfrak{A}^{(1)} \in \mathfrak{M}^{r,R}_{\kappa_1}(j_{pq}), \quad \mathfrak{A}^{(2)} \in \mathfrak{M}^{\ell,S}_{\kappa_2}(j_{pq}), \tag{4.2}$$

where $\kappa_1 = ind_{-}\mathcal{L}_W$ *and* $\kappa_2 = \kappa - \kappa_1$.

Proof. Let condition (1) holds and let $\{b_1, b_2\} \in ap(W)$. Then by Theorem 2.9, W admits the factorization $W = W^{(1)}W^{(2)}$, where $W^{(1)} \in \mathcal{U}^{r,R}_{\kappa_1}(j_{pq})$ and $W^{(2)} \in \mathcal{U}^{\ell,S}_{\kappa_2}(j_{pq})$, $\kappa = \kappa_1 + \kappa_2$. By Theorem 3.4 $W \in \mathcal{U}^{r}_{\kappa}(j_{pq})$ and $\{b_1, b_2\} \in ap^r(W)$. By Theorem 2.7 (2) and hence by Lemma 3.12 [15] $ap^r(W) = ap^r(W^{(1)})$.

Hence, upon applying $\begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix}$ to the mvf W and by Theorem 3.4 and Lemma 4.1 we obtain

$$\mathfrak{A} = \mathfrak{A}^{(1)}\mathfrak{A}^{(2)}, \text{ where } \mathfrak{A}^{(1)} \in \mathfrak{M}^{r}_{\kappa_1}(j_{pq}),\ \mathfrak{A}^{(2)} \in \mathfrak{M}^{\ell,S}_{\kappa_2}(j_{pq}),\ \kappa_1 + \kappa_2 = \kappa.$$

Since $W^{(1)} \in \widetilde{L}_2^{m\times m}$, then $\mathfrak{A}^{(1)} \in \widetilde{L}_2^{m\times m}$, and thus by Lemma 4.5 $\mathfrak{A}^{(1)} \in \mathfrak{M}^{r,R}_{\kappa}(j_{pq})$. □

Theorem 4.8. *Let a mvf* $\mathfrak{A} \in \mathfrak{M}^{\ell}_{\kappa}(j_{pq}) \cap \mathcal{R}$, $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in den^{\ell}(\mathfrak{A})$, *let* W *be given by* (3.8) *and let:*

(1) $W(z) \in \mathcal{U}^{r}_{\kappa}(j_{pq})$,

(2) $\mathcal{L}_{\widetilde{W}}$ *be a nondegenerate of* $\mathcal{K}(\widetilde{W})$, *with negative index* $ind_{-}\mathcal{L}_{\widetilde{W}} = \kappa_1$.

Then $\mathfrak{A}$ *admits regular–singular factirization*

$$\mathfrak{A} = \mathfrak{A}^{(2)}\mathfrak{A}^{(1)}, \quad \mathfrak{A}^{(1)} \in \mathfrak{M}^{\ell,R}_{\kappa_1}(j_{pq}), \quad \mathfrak{A}^{(2)} \in \mathfrak{M}^{r,S}_{\kappa_2}(j_{pq}), \tag{4.3}$$

where $\kappa_1 = ind_{-}\mathcal{L}_{\widetilde{W}}$ *and* $\kappa_2 = \kappa - \kappa_1$.

Proof. Let $\mathfrak{A} \in \mathfrak{M}^{\ell}_{\kappa}(j_{pq})$, then $\widetilde{\mathfrak{A}} \in \mathfrak{M}^{r}_{\kappa}(j_{pq})$ and the mvf's $\widetilde{W}(z)$ and $\widetilde{\mathfrak{A}}(z)$ satisfy the assumptions of Theorem 4.7. By Theorem 4.7 $\widetilde{\mathfrak{A}}$ admits a factorization

$$\widetilde{\mathfrak{A}} = \widetilde{\mathfrak{A}}^{(1)}\widetilde{\mathfrak{A}}^{(2)}, \quad \widetilde{\mathfrak{A}}^{(1)} \in \mathfrak{M}^{r,R}_{\kappa_1}(j_{pq}), \quad \widetilde{\mathfrak{A}}^{(2)} \in \mathfrak{M}^{\ell,S}_{\kappa_2}(j_{pq}), \tag{4.4}$$

where $\kappa_1 + \kappa_2 = \kappa$.

Using the transformation (1.19) again, we obtain (4.3). □

References

[1] V. M. Adamjan, D. Z. Arov, M. G. Krein, *Infinite Hankel matrices and generalized problems of Carathodory-Fejer and I. Schur problems* // Funktsional. Anal. i Prilozhen, **2** (1968), No. 4, 1–17.

[2] V. M. Adamjan, *Nondegenerate unitary couplings of semiunitary operators* // Funktsional. Anal. i Prilozhen., **7** (1973), No. 4, 1–16.

[3] D. Alpay, A. Dijksma, J. Rovnyak, H. S. V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces* // Oper. Theory: Adv. Appl., **96**, (1997), 229.

[4] D. Alpay, H. Dym, *On applications of reproducing kernel spaces to the Schur algorithm and rational J unitary factorization. I. Schur methods in operator theory and signal processing* // Oper. Theory Adv. Appl., **18** (1986), 89–159.

[5] N. Aronszajn, *Theory of reproducing kernels* // Trans. Amer. Math. Soc., **68** (1950), 337–404.

[6] D.Z. Arov, *Realization of matrix-valued functions according to Darlington* // Izv. Akad. Nauk SSSR Ser. Mat., **37** (1973), 1299–1331.

[7] D. Z. Arov, *Regular and singular J-inner matrix functions and corresponding extrapolation problems* // Funktsional. Anal. i Prilozhen., **22** (1988), No. 1, 57–59 [in Russian]; transl. in Funct. Anal. Appl., **22** (1988), No. 1, 46–48.

[8] D. Z. Arov,*γ-generating matrices, j-inner matrix-functions and related extrapolation problems* // Teor. Funktsii Funktsional. Anal. i Prilozhen, I, **51** (1989), 61–67; II, **52** (1989), 103–109 [in Russian]; transl. in J. Soviet Math. I, **52** (1990), 3487–3491; II, **52** (1990), 3421–3425.

[9] D. Z. Arov, H. Dym, *J-inner matrix function, interpolation and inverse problems for canonical system, I: foundation, Integral Equations* // Operator Theory, **28** (1997), 1–16.

[10] D. Z. Arov, H. Dym,*J-Contractive Matrix Valued Functions and Related Topics*, Cambridge University Press, Cambridge, 2008.

[11] T. Ya. Azizov, I. S. Iokhvidov, *Foundations of the theory of linear operators in spaces with an indefinite metric*, Nauka, Moscow, 1986.

[12] J. Bognar, *Indefinite inner product spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, New York-Heidelberg, **78**, 1974.

[13] V. A. Derkach, H. Dym, *On linear fractional transformations associated with generalized J-inner matrix functions* // Integ. Eq. Oper. Th., **65** (2009), 1–50.

[14] V. A. Derkach, O. Sukhorukova, *Generalized γ-generating matrices and Nehari-Takagi problem* // Oper. Matrices, **10** (2016), No. 4, 1073–1091.

[15] V. Derkach, O. Sukhorukova, *A-regular–A-singular factorizations of generalized J-inner matrix functions* //Journal of Mathematical Sciences, **23** (2017), No. 3, 231–251.

[16] B. Francis, *A Cource in H_∞ Control Theory*, Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, **88**, 1987.

[17] M. G. Krein, H. Langer, *Über die verallgemeinerten Resolventen und die characteristische Function eines isometrischen Operators im Raume* Π_κ // Hilbert space Operators and Operator Algebras (Proc. Intern. Conf.,Tihany, 1970); Colloq. Math. Soc. Janos Bolyai, **5**, (1972), 353–399.

[18] L. Schwartz, *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associes* // J. Analyse Math., **13** (1964), 115–256.

[19] O. Sukhorukova, *Factorization formulas for some classes of generalized J-inner matrix valued functions* // Methods Funct. Anal. Topology, **20** (2014), No. 4, 365–378.

[20] O. Sukhorukova, *Generalized γ-generating matrices* // Ukr. Mat. Visn, **12** (2015), No.4, 539–558 [in Russian]; transl. in J. of Math. Sci., **218**, (2016) No. 1, 89–104.

CONTACT INFORMATION

Olena Sukhorukova National Pedagogical Dragomanov University, Kiev, Ukraine
E-Mail: sukhorukova.elena.ua@gmail.com