Bernstein-Walsh type inequalities in unbounded regions with piecewise asymptotically conformal curve in the weighted Lebesgue space

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Abstract. In this work, we obtain pointwise Bernstein–Walsh-type estimation for algebraic polynomials in the unbounded regions with piecewise asymptotically conformal boundary, having exterior and interior zero angles, in the weighted Lebesgue space.

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1. Introduction and Definitions

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}; \ G \subset \mathbb{C}$ be a bounded region, with $0 \in \ G$ and the boundary $L := \partial G$ be a Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = exp L$. Denote by $w = \Phi(z)$ the univalent conformal mapping of $\Omega$ onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$.

For $t \geq 1$, $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set:

$L_t := \{z : |\Phi(z)| = t\}$ ($L_1 \equiv L$), $G_t := int L_t$, $\Omega_t := exp L_t$;

$d(z, M) = dist(z, M) := \inf \{|z - \zeta| : \zeta \in M\}$.

Let $\{\xi_j\}_{j=1}^m$ be a fixed system of distinct points on curve $L$ located in the positive direction. For some fixed $R_0$, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$h(z) := \prod_{j=1}^m |z - \xi_j|^{\gamma_j}$, \hspace{1cm} (1.1)
where $\gamma_j > -1$ for all $j = 1, 2, \ldots, m$.

For a rectifiable Jordan curve $L$ and for $0 < p \leq \infty$, let $L_p(h, L)$ denote the weighted Lebesgue space of complex-valued functions on $L$. Specifically, $f \in L_p(h, L)$ if $f$ is measurable and the following quasinorm (a norm for $1 \leq p \leq \infty$ and a $p$-norm for $0 < p < 1$) is finite:

$$\|f\|_p := \left( \int_{L} h(z) |f(z)|^p \, dz \right)^{1/p}, \quad 0 < p < \infty; \quad (1.2)$$

For $p = \infty$,

$$\|f\|_{\infty} := \text{ess sup}_{z \in L} |f(z)|, \quad p = \infty.$$

We denote by $\wp_n$, $n = 1, 2, \ldots$, the set of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Bernstein–Walsh Lemma [28] says that for any $P_n \in \wp_n$ and $R > 1$, the following

$$\|P_n\|_{C(\overline{G_R})} \leq R^n \|P_n\|_{C(\overline{G})} \quad (1.3)$$

holds. In [28] also was given some similar estimates for various norms on the right-hand side of (1.3). Analogously estimation with respect to the quasinorm (1.2) for $p > 0$ was obtained in [19] for $h(z) \equiv 1$ (i.e., $\gamma_j = 0$ for all $j = 1, 2, \ldots, m$). Moreover, in [6, Lemma 2.4] this estimate has been generalized for $h(z) \neq 1$, defined as in (1.1) and was proved the following:

$$\|P_n\|_{L_p(h, L_R)} \leq R^{\gamma^* + \frac{1 + \gamma^*}{p}} \|P_n\|_{L_p(h, L)}, \quad \gamma^* = \max \{0; \gamma_j : j \leq m\}. \quad (1.4)$$

For any $p > 0$ we also introduce:

$$\|P_n\|_{A_p(h, G)} := \left( \int_{G} h(z) |P_n(z)|^p \, d\sigma_z \right)^{1/p} < \infty, \quad 0 < p < \infty, \quad (1.5)$$

where $\sigma_z$ is the two-dimensional Lebesgue measure.

The Bernstein–Walsh type estimates for the quasinorm (1.5), for the regions with quasiconformal boundary (see, below) and weight function $h(z)$, defined in (1.1) with $\gamma_j > -2$, for all $p > 0$ as follows

$$\|P_n\|_{A_p(h, G_R)} \leq c_1 R^{n + \frac{1}{p}} \|P_n\|_{A_p(h, G)}, \quad (1.6)$$

was found in [3] (see, also [2]), where $R^* := 1 + c_2 (R - 1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from $n$ and $R$. In [4, Theorem 1.1], analogously estimate was studied for $A_p(1, G)$-norm, $p > 0$, for arbitrary
Jordan region and was obtained: for any \( P_n \in \varphi_n \), \( R_1 = 1 + \frac{1}{n} \) and arbitrary \( R \), \( R > R_1 \), the following estimate

\[
\|P_n\|_{Ap(G,R)} \leq c \cdot R^{n+\frac{2}{p}} \|P_n\|_{Ap(G,R_1)},
\]

is true, where \( c = \left( \frac{2^p}{p-1} \right)^{\frac{1}{p}} \left[ 1 + O\left( \frac{1}{n} \right) \right] \), \( n \to \infty \). Note that, the \( c \) is the sharp constant.

In [27] was given a new version of the Bernstein–Walsh Lemma: For quasiconformal and rectifiable curve \( L \) there exists a constant \( c = c(L) > 0 \) depending only on \( L \) such that

\[
|P_n(z)| \leq c \cdot \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,
\]

holds for every \( P_n \in \varphi_n \).

In this work, continue investigated pointwise estimations in unbounded region \( \Omega \) of the type

\[
|P_n(z)| \leq c_2 \eta_n(G,h,d(z,L)) \|P_n\|_p |\Phi(z)|^{n+1},
\]

where \( c_2 = c_2(G,p) > 0 \) is a constant independent of \( n, h \) and \( P_n \), and \( \eta_n(G,h,d(z,L)) \to \infty \), \( n \to \infty \), depending on the properties of the \( G \) and \( h \).

Analogous results of (1.8)-type for some norms and for different unbounded regions were obtained by S. N. Bernstein [28], N. A. Lebedev, P. M. Tamrazov, V. K. Dzjadyk, I. A. Shevchuk (see, for example, [14]), N. Stylianopoulos [27] and others. Recent results (1.8) for some regions and the weight function \( h(z) \) defined as in (1.1) with \( \gamma_j > -1 \) were also obtained: in [6] for \( p > 1 \) and in [22] for \( p > 0 \), for regions bounded by piecewise Dini-smooth boundary with interior and exterior zero angles; in [7] for \( p > 0 \) and for regions bounded by piecewise quasiconformal boundary with interior and exterior zero angles; in [5] for \( p > 1 \) and for regions bounded by piecewise smooth boundary with exterior zero angles (without interior zero angles); in [8] for \( p > 0 \) and for regions bounded by piecewise quasismooth boundary with interior and exterior zero angles and in others.

Now, we begin to give some definitions and notations.

Let \( z_1, z_2 \) be an arbitrary points on \( l \) and \( l(z_1,z_2) \) denotes the subarc of \( l \) of shorter diameter with endpoints \( z_1 \) and \( z_2 \). The curve \( l \) is a quasicircle if and only if the quantity

\[
\sup_{z_1,z_2 \in l; \ z \in l(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} = 1.
\]
is bounded. Following to Lesley [21], the curve \( l \) to be said “\( c-\)quasiconformal”, if the quantity (1.9) bounded by positive constant \( c \), independent from points \( z_1, z_2 \) and \( z \). At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, Def. 3.1, [23, p. 286–294], [20, p. 105], [9, p. 81], [24, p. 107]).

The Jordan curve \( l \) is called asymptotically conformal [13,24], if
\[
\sup_{z_1,z_2 \in l; \ z \in l(z_1,z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \quad |z_1 - z_2| \to 0. \tag{1.10}
\]

We will denote this class as \( AC \), and will write \( G \in AC \), if \( L := \partial G \in AC \).

The asymptotically conformal curves occupies a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems has been studied by J. M. Anderson, J. Becker and F. D. Lesley [10], E. M. Dyn’kin [15], Ch. Pommerenke, S. E. Warschawski [25], V. Ya. Gutlyanskii, V. I. Ryazanov [16–18] and others. According to the geometric criteria of quasiconformality of the curves ([9, p. 81], [24, p. 107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [12], [20, p. 104]). The same is true for asymptotically conformal curves.

We say that \( L \in \overline{AC} \), if \( L \in AC \) and \( L \) is rectifiable. A Jordan arc \( \ell \) is called asymptotically conformal arc, when \( \ell \) is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curves having interior and exterior cusps at the connecting points of boundary arcs.

Throughout this paper, \( c, c_0, c_1, c_2, \ldots \) are positive and \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \) are sufficiently small positive constants (generally, different in different relations), which depend on \( G \) in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

For any \( k \geq 0 \) and \( m > k \), notation \( i = \overline{k,m} \) means \( i = k, k+1, \ldots, m \). For any \( i = 1, 2, \ldots, k = 0, 1, 2 \) and \( \varepsilon_1 > 0 \), we denote by \( f_i : [0, \varepsilon_1] \to \mathbb{R}^+ \) and \( g_i : [0, \varepsilon_1] \to \mathbb{R}^+ \) twice differentiable functions such that
\[
f_i(0) = g_i(0) = 0, \quad f_i^{(k)}(x) > 0, \quad g_i^{(k)}(x) > 0, \quad 0 < x \leq \varepsilon_1. \tag{1.11}
\]

**Definition 1.1.** We say that a Jordan region \( G \in AC(f_i, g_i) \), for some \( f_i = f_i(x) \), \( i = \overline{1,m} \) and \( g_i = g_i(x) \), \( i = \overline{m_1+1,m} \), defined as in (1.11), if \( L = \partial G = \bigcup_{i=0}^{m} L_i \) is the union of the finite number of asymptotically conformal arcs \( L_i \), connecting at the points \( \{z_i\}_{i=0}^{m} \in L \) and such that \( L \)
is a locally asymptotically conformal arc at the $z_0 \in L \setminus \{x_i\}_{i=1}^m$ and, in the $(x,y)$ local co-ordinate system with its origin at the $z_i$, $1 \leq i \leq m$, the following conditions are satisfied:

a) for every $z_i \in L$, $i = 1, m_1$, $m_1 \leq m$,

$$
\{ z = x + iy : |z| \leq \varepsilon_1, \; c_{1i} f_i(x) \leq y \leq c_{1i} f_i(x), \; 0 \leq x \leq \varepsilon_1 \} \subset \overline{G},
$$

$$
\{ z = x + iy : |z| \leq \varepsilon_1, \; |y| \geq \varepsilon_2 x, \; 0 \leq x \leq \varepsilon_1 \} \subset \overline{\Omega};
$$

b) for every $z_i \in L$, $i = m_1 + 1, m$,

$$
\{ z = x + iy : |z| \leq \varepsilon_3, \; c_{2i} g_i(x) \leq y \leq c_{2i} g_i(x), \; 0 \leq x \leq \varepsilon_3 \} \subset \overline{\Omega},
$$

$$
\{ z = x + iy : |z| \leq \varepsilon_3, \; |y| \geq \varepsilon_4 x, \; 0 \leq x \leq \varepsilon_3 \} \subset \overline{G},
$$

for some constants $-\infty < c_{11} < c_{12} < \infty$, $-\infty < c_{21} < c_{22} < \infty$ and $\varepsilon_\alpha > 0$, $s = 1, 4$.

**Definition 1.2.** We say that a Jordan region $G \in \overline{AC}(f_i, g_i)$, $f_i = f_i(x), \; i = 1, m_1, g_i = g_i(x), \; i = m_1 + 1, m$, if $G \in AC(f_i, g_i)$ and $L := \partial G$ is rectifiable.

It is clear from Definitions 1.2 and 1.1, that each region $G \in \overline{AC}(f_i, g_i)$ may have $m_1$ interior and $m - m_1$ exterior zero angles (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^m \in L$. If a region $G$ does not have interior zero angles ($m_1 = 0$) (exterior zero angles ($m_1 = m$)), then it is written as $G \in \overline{AC}(0, g_i)$ ($G \in \overline{AC}(f_i, 0)$). If a region $G$ does not have such angles ($m = 0$), then we will assume that $G$ is bounded by a asymptotically conformal curve and in this case we set $\overline{AC}(0, 0) \equiv \overline{AC}$.

Throughout this work, we will assume that the points $\{\xi_i\}_{i=1}^m \in L$ defined in (1.1) and the points $\{z_i\}_{i=1}^m \in L$ defined in Definition 1.2 and 1.1 coincide. Without loss of generality, we also will assume that the points $\{z_i\}_{i=0}^m$ are ordered in the positive direction on the curve $L$ such that $G$ has interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$, if $m_1 \geq 1$ and exterior zero angles at the points $\{z_i\}_{i=m_1+1}^m$, if $m \geq m_1 + 1$.

### 2. Main Results

Now, we can state our new results. Our first result is related to the general case. Namely, let region $G$ has $m_1 \geq 1$ interior zero angles at the points $\{z_i\}_{i=1}^{m_1}$ and $m - m_1$ exterior zero angles at the points $\{z_i\}_{i=m_1+1}^m$. In this case, we have the following estimate, i.e. with respect to each points $\{z_i\}_{i=1}^m$. 
Theorem 2.1. Let $p > 0; G \in \overline{AC}(f_1, g_1)$, for some $f_1(x) = c_i x^{1+\alpha_i}, \alpha_i \geq 0$, $i = \overline{1,m}$, and $g_1(x) = c_i x^{1+\beta_i}, \beta_i > 0$, $i = \overline{1,m}$; $h(z)$ defined as in (1.1). Then, for any $\gamma_i > -1, i = \overline{1,m}$, and $P_n \in \wp_n, n \in \mathbb{N}$, there exists $c_1 = c_1(G, p, \epsilon, \gamma_i, \beta_i) > 0$ such that:

$$|P_n(z)| \leq c_1 \frac{\Phi(z)^{|n+1|}}{d^{2/p(z, L_R)}} \left( \sum_{i=1}^{m} B_{n,1} + \sum_{i=m+1}^{m} B_{n,2} \right) \|P_n\|_p, z \in \Omega_R,$$

where

$$B_{n,1} := \begin{cases} n^{\gamma_i - \frac{1}{p} + \varepsilon}, & \gamma_i > 2 + \beta_i - \varepsilon; \\
\left( n \ln n \right)^{\frac{1}{p}}, & \gamma_i = 2 + \beta_i - \varepsilon; \\
\frac{1}{n^p}, & 0 < \gamma_i < 2 + \beta_i - \varepsilon; \\
\frac{\varepsilon}{n^p}, & -1 < \gamma_i \leq 0; \end{cases}$$

$$\varepsilon := \left\{ \begin{array}{ll} 1, & \alpha_i \neq 0, \\
\epsilon, & \alpha_i = 0; \end{array} \right.$$ and

$$B_{n,2} := \begin{cases} n^{\gamma_i - \frac{1}{p} + \varepsilon}, & \gamma_i > 2 + \beta_i - \varepsilon; \\
\left( n \ln n \right)^{\frac{1}{p}}, & \gamma_i = 2 + \beta_i - \varepsilon; \\
\frac{1}{n^p}, & 0 < \gamma_i < 2 + \beta_i - \varepsilon; \\
\frac{\varepsilon}{n^p}, & -1 < \gamma_i \leq 0. \end{cases}$$

Now, we assume that, $i = \overline{1,2}; m_1 = 1, m = 2$.

Theorem 2.2. Let $p > 0; G \in \overline{AC}(f_1, g_2)$, for some $f_1(x) = c_1 x^{1+\alpha_1}, \alpha_1 \geq 0$, and $g_2(x) = c_2 x^{1+\beta_2}, \beta_2 > 0; h(z)$ defined as in (1.1) for $m = 2$. Then, for any $\gamma_1 > -1, i = \overline{1,2}$, and $P_n \in \wp_n, n \in \mathbb{N}$, there exists $c_2 = c_2(G, p, \epsilon, \gamma_i, \beta_i) > 0$ such that:

$$|P_n(z)| \leq c_2 \frac{\Phi(z)^{|n+1|}}{d^{2/p(z, L_R)}} B_n \|P_n\|_p, z \in \Omega_R,$$

where

$$B_n := \begin{cases} n^{2(\gamma_1 - \frac{1}{p})}, & \gamma_1 > 1 + \frac{\gamma_2 - 1}{2(1 + \beta_2)}, \gamma_2 > 2 + \beta_2 - \varepsilon; \\
\frac{\gamma_1 - 1}{n^{p(1+\beta_2)} + \varepsilon}, & 0 < \gamma_1 \leq 1 + \frac{\gamma_2 - 1}{2(1 + \beta_2)}, \gamma_2 > 2 + \beta_2 - \varepsilon; \\
\frac{2(\gamma_1 - 1)}{n^p}, & \gamma_1 > \frac{3}{2}, 0 < \gamma_2 < 2 + \beta_2 - \varepsilon; \\
\frac{1}{n^p}, & 0 < \gamma_1 < \frac{3}{2}, 0 < \gamma_2 < 2 + \beta_2 - \varepsilon; \\
\left( n \ln n \right)^{\frac{1}{p}}, & \gamma_1 = \frac{3}{2}, \gamma_2 = 2 + \beta_2 - \varepsilon; \\
\frac{\varepsilon}{n^p}, & -1 < \gamma_1 \leq 0, -1 < \gamma_2 \leq 0. \end{cases}$$
In particular, if $\alpha_1 = 0$, i.e. $G$ has only exterior zero angle at the $z_2$, then we have:

**Theorem 2.3.** Let $p > 0$; $G \in \overset{\sim}{AC} (0, g_2)$, for some $g_2(x) = c_2x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in (1.1) for $m = 2$. Then, for any $\gamma_1 > -1$, $i = 1, 2$, and $P_n \in \varphi_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(G, p, \varepsilon, \gamma_i, \beta_2) > 0$ such that:

$$|P_n(z)| \leq c_3 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_R)} B_n \|P_n\|_p, \ z \in \Omega_R,$$

(2.5)

where

$$B_n := \begin{cases} \frac{n^{-1}}{p}, & \gamma_1 > 1 + \gamma_2 - 1 \frac{1+\beta_2}{1+\beta_2}, \\
\frac{2}{p^{1+\beta_2} + \varepsilon}, & 2 \leq \gamma_1 \leq 1 + \gamma_2 - 1 \frac{1+\beta_2}{1+\beta_2}, \\
\frac{n^{-1}}{p^{1+\varepsilon}}, & \gamma_1 \geq 2, 0 < \gamma_2 < 2 + \beta_2, \\
\frac{1}{n^{\varepsilon}}, & 0 < \gamma_1 < 2, 0 < \gamma_2 < 2 + \beta_2, \\
(\ln n)^{\frac{1}{p}}, & \gamma_1 = 2 - \varepsilon, \gamma_2 = 2 + \beta_2 - \varepsilon, \\
n^{1-p}, & -1 < \gamma_1 \leq 0, -1 < \gamma_2 \leq 0. \end{cases}$$

(2.6)

**Remark 2.1.** In Theorems 2.1–2.3, in the right hand sides of estimations (2.1), (2.3), (2.5) and their corollaries there exist value $d^{2/p}(z, L_R)$. We can replace $d^{2/p}(z, L_R)$ with $d(z, L_R)$, if we consider only the values $p > 1$ instead of $p > 0$.

The sharpness of the estimations (2.1)–(2.6) for some special cases can be discussed by comparing them with the following:

**Remark 2.2.** For any $n \in \mathbb{N}$ there exist polynomials $P_n^* \in \varphi_n$, regions $G^* \subset \mathbb{C}$ and constant $c_4 = c_4(G) > 0$, such that

$$|P_n^*(z)| \geq c_4 |\Phi(z)|^{n+1} \|P_n^*\|_{L_2(\partial G^*)}, \ \forall z \in F \subset C \bar{G}^*.$$  

(2.7)

3. Some auxiliary results

For $a > 0$ and $b > 0$, we shall use the notations “$a \leq b$” (order inequality), if $a \leq cb$ and “$a \asymp b$” are equivalent to $c_1a \leq b \leq c_2a$ for some constants $c, c_1, c_2$ (independent of $a$ and $b$) respectively.

The following definitions of the $K$-quasiconformal curves are well known (see, for example, [9], [20, p. 97] and [26]):

**Definition 3.1.** The Jordan arc (or curve) $L$ is called $K$–quasiconformal ($K \geq 1$), if there is a $K$–quasiconformal mapping $f$ of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).
Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms $f$ of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let defines

$$K_L := \inf \{ K(f) : f \in F(L) \},$$

where $K(f)$ is the maximal dilatation of a such mapping $f$. $L$ is a quasiconformal curve, if $K_L < \infty$, and $L$ is a $K-$quasiconformal curve, if $K_L \leq K$.

**Lemma 3.1.** \[1\] Let $L$ be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{ z : |z - z_1| \leq d(z_1, L_{r_0}) \}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then

a) The statements $|z_1 - z_2| \leq |z_1 - z_3|$ and $|w_1 - w_2| \leq |w_1 - w_3|$ are equivalent.

So are $|z_1 - z_2| \simeq |z_1 - z_3|$ and $|w_1 - w_2| \simeq |w_1 - w_3|$.

b) If $|z_1 - z_2| \leq |z_1 - z_3|$, then

$$\frac{|w_1 - w_3|}{|w_1 - w_2|} \simeq \frac{|z_1 - z_3|}{|z_1 - z_2|} \simeq \frac{|w_1 - w_3|}{|w_1 - w_2|}^c,$$

where $\varepsilon_1 < 1$, $c > 1$, $0 < r_0 < 1$ are constants, depending on $G$ and $L_{r_0} := \{ z = \psi(w) : |w| = r_0 \}$.

**Lemma 3.2.** \[21, p. 342\] Let $L$ be an asymptotically conformal curve. Then, $\Phi$ and $\Psi$ are $\text{Lip}_\alpha$ for all $\alpha < 1$ in $\overline{\Omega}$ and $\overline{\Delta}$, correspondingly.

**Lemma 3.3.** Let $L$ be an asymptotically conformal curve. Then,

$$|\Psi(w_1) - \Psi(w_2)| \simeq |w_1 - w_2|^{1+\varepsilon},$$

for all $w_1, w_2 \in \overline{\Delta}$ and $\forall \varepsilon > 0$.

This fact follows from Lemma 3.2. We also will use the estimation for the $\Psi'$ (see, for example, [11, Th. 2.8]):

$$|\Psi'(\tau)| \simeq \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \quad (3.1)$$

Let $\{ z_j \}_{j=1}^m$ be a fixed system of the points on $L$ and the weight function $h(z)$ defined as (1.1).

**Lemma 3.4.** \[8, 19, h(z) \equiv 1\] Let $L$ be a rectifiable Jordan curve; $h(z)$ defined as in (1.1). Then, for arbitrary $P_n(z) \in \Phi_n$, any $R > 1$ and $n \in \mathbb{N}$

$$\| P_n \|_{L_p(h, L_R)} \leq R^{n+\frac{1+\gamma}{p}} \| P_n \|_{L_p(h, L)} , \quad p > 0, \quad (3.2)$$

is true, where $\tilde{\gamma} := \max \{ 0; \gamma_i : i = \overline{1, m} \}$. 
4. Proof of Theorems

4.1. Proof of Theorems 2.1–2.3

Proof. Suppose that $G \in \mathcal{AC}(f_i, g_i)$, for some $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = 1, m_1$, and $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = m_1 + 1, m$; $h(z)$ be defined as in (1.1). Let $\{\zeta_j^*\}$, $1 \leq j \leq m \leq n$, be zeros of $P_n(z)$ lying on $\Omega$ and let

$$B_m(z) := \prod_{j=1}^m \tilde{B}_j(z) = \prod_{j=1}^m \frac{\Phi(z) - \Phi(\zeta_j^*)}{1 - \Phi(\zeta_j^*)}\Phi(z)$$

denote a Blaschke function with respect to zeros $\{\zeta_j^*\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$. For any $p > 0$ and $z \in \Omega$, let us set:

$$G_n(z) := \left[\frac{P_n(z)}{B_m(z) \Phi^{n+1}(z)}\right]^{p/2}. \quad (4.1)$$

Cauchy integral representation for the unbounded region $\Omega$ gives:

$$G_n(z) = -\frac{1}{2\pi i} \int_{L_R} G_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in \Omega_R. \quad (4.2)$$

Since $|B_m(\zeta)| = 1$, for $\zeta \in L$, then, for arbitrary $\varepsilon$, $0 < \varepsilon < \varepsilon_1$, there exists a circle $|w| = 1 + \frac{\varepsilon_1}{n}$, such that for any $j = 1, m$ the following is satisfied:

$$|\tilde{B}_j(\Psi(w))| > 1 - \varepsilon.$$

Then, $|B_m(\zeta)| > (1 - \varepsilon)^m \geq 1$ for each $\varepsilon \leq \varepsilon_1$. On the other hand, $|\Phi(\zeta)| = R > 1$, for $\zeta \in L_R$. Therefore, for any $z \in \Omega_R$, we have:

$$\left|\left[\frac{P_n(z)}{B_m(z) \Phi^{n+1}(z)}\right]^{p/2}\right| \leq \frac{1}{2\pi} \int_{L_R} \left|\frac{P_n(\zeta)}{B_m(\zeta) \Phi^{n+1}(\zeta)}\right|^{p/2} \left|\frac{d\zeta}{\zeta - z}\right| \quad (4.3)$$

$$\leq \frac{1}{d(z, L_R)} \int_{L_R} |P_n(\zeta)|^{p/2} |d\zeta| =: \frac{1}{d(z, L_R)} A_n.$$

To estimate the integral $A_n$, we introduce:

$$w_j := \Phi(z_j), \ \varphi_j := \arg w_j, \ L^j_R := L_R \cap \Omega_j, \ j = 1, m,$$

where $\Omega_j := \Psi(\Delta_j')$;

$$\Delta_1' := \left\{ t = Re^{i\theta} : R > 1, \ \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$

$$\Delta_m' := \left\{ t = Re^{i\theta} : R > 1, \ \frac{\varphi_m - 1 + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2} \right\}.$$
and, for $j = 2, m - 1$

\[
\Delta_j' := \left\{ t = Re^{i\theta} : R > 1, \ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}.
\]

Then, we have:

\[
A_n = \sum_{i=1}^{m} \int_{L_R} |P_n(\zeta)|^{p/2} |d\zeta|.
\] (4.4)

Multiplying the numerator and denominator of the integrand by $h^{1/2}(\zeta)$, after applying the Hölder inequality, we obtain:

\[
A_n \leq \sum_{i=1}^{m} \left( \int_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{1/2} \times \left( \int_{L_R} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{\gamma_j}} \right)^{1/2}
\] (4.5)

\[
= \sum_{i=1}^{m} \tilde{J}_{n,1}^i \cdot \tilde{J}_{n,2}^i.
\]

According to Lemma 3.4, for the $\tilde{J}_{n,1}^i$ we get:

\[
\tilde{J}_{n,1}^i \leq \|P_n\|_p^{p/2}, \ i = 1, m.
\] (4.6)

Then, from (4.5) and (4.6) we have:

\[
A_n \leq \|P_n\|_p^{p/2} \sum_{i=1}^{m} \tilde{J}_{n,2}^i.
\]

For the integral $J_{n,2}^i$ we obtain:

\[
\left( \tilde{J}_{n,2}^i \right)^2 := \int_{L_R} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{\gamma_j}} \asymp \int_{L_R} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}, \ i = 1, 2,
\] (4.7)

since the points $\{z_j\}_{j=1}^{m}$ are distinct on $L$. Then, from (4.7), we have:

\[
A_n \leq \|P_n\|_p^{p/2} \sum_{i=1}^{m} \tilde{J}_{n,2}^i,
\] (4.8)

where

\[
\tilde{J}_{n,2}^1 = \int_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}}; \ \tilde{J}_{n,2}^2 = \int_{L_R^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}}.
\] (4.9)
It remains to estimate these integrals for each $i = 1, m$. For simplicity of our next calculations, we assume that:

\[ i = 1, 2; \ m_1 = 1, \ m = 2; \ z_1 = -1, \ z_2 = 1; \ (-1, 1) \subset G; \ R = 1 + \frac{\varepsilon_0}{n}, \]

and let local co-ordinate axis in Definitions 1.1 and 1.2 is parallel to $OX$ and $OY$ in the $OXY$ co-ordinate system; $L = L^+ \cup L^-$, where $L^+ := \{ z \in L : \text{Im} \ z \geq 0 \}$, $L^- := \{ z \in L : \text{Im} \ z < 0 \}$. Let $w^\pm := \{ w = e^{i\theta} : \theta = \frac{x^1 \pm x^2}{2} \}$, $z^\pm \in \Psi(w^\pm)$ and $L^i$ an arc, connecting the points $z_i^+, z_i^- \in L$; $L_i^{i,+} := L^i \cap L_i^+$, $i = 1, 2$. Let $z_0$ be taken as an arbitrary point on $L^+$ (or on $L^-$ subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z^+$ ($z_0 = z^-$).

Analogously to the previous notations, we introduce the following: $L_R = L_{R}^+ \cup L_{R}^-$, where $L_{R}^+ := \{ z \in L_R : \text{Im} \ z \geq 0 \}$, $L_{R}^- := \{ z \in L_R : \text{Im} \ z < 0 \}$. Let \( w_R^{\pm} := \{ w = Re^{i\theta} : \theta = \frac{x^1 \pm x^2}{2} \}, z_R^{\pm} \in \Psi(w_R^{\pm}). \) We set: $z_{i,R} \in L_R$, such that $d_{i,R} = |z_{i} - z_{i,R}|$ and $\zeta^{\pm} \in L^{\pm}$, such that $d(z_{2,R}, L^2 \cap L^{\pm}) = d(z_{2,R}, L^{\pm})$; $z_i^{\pm} := \{ \zeta \in L^i : |\zeta - z_i| = c_i d(z_{i,R}, L_R) \}$, $z_{i,R}^{\pm} := \{ \zeta \in L_R^i : |\zeta - z_{i,R}| = c_i d(z_{i,R}, L_R) \}$, $w_i^{\pm} = \Phi(z_{i,R}^{\pm})$. Let $L^i_R$, $i = 1, 2$, denote arcs, connecting the points $z_i^+, z_i^{R}, z_i^- \in L_R$, $L_i^{i,+} := L^i_R \cap L_i^+$ and $l_{i,R}(z_{i,R}^+, z_{i,R}^-)$ denote arcs, connecting the points $z_{i,R}^+$ with $z_{i,R}^-$, respectively and $l_{i,R} := \text{mes} l_{i,R}(z_{i,R}^+, z_{i,R}^-)$, $i = 1, 2$. We denote:

\[
S_{i,R}^{i,\pm} := \{ \zeta \in L_{R}^{i,\pm} : |\zeta - z_i| < c_i d_{i,R} \},
\]

\[
S_{2,R}^{i,\pm} := \{ \zeta \in L_{R}^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq l_{i,R} \}, \quad F_{i,R}^{i,\pm} := \Phi(S_{i,R}^{i,\pm});
\]

\[
S_1^{i,\pm} := \{ \zeta \in L_{R}^{i,\pm} : |\zeta - z_i| < c_i d_{i,R} \},
\]

\[
S_2^{i,\pm} := \{ \zeta \in L_{R}^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq l_{i,R} \}, \quad F_{i,R}^{i,\pm} := \Phi(S_{i,R}^{i,\pm}), \ i, j = 1, 2.
\]

Taking into consideration above notations, replacing the variable $\tau = \Phi(\zeta)$, according to (3.1), we have:

\[
\tilde{J}_{n,2} \propto \sum_{i,j=1}^{2} \int_{F_{i,R}^{i,\pm} \cup F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}}.
\]

\[
\propto \sum_{i,j=1}^{2} \int_{F_{i,R}^{i,\pm} \cup F_{j,R}^{i,\pm}} \frac{d(\Psi(\tau), L) |d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} (|\tau| - 1)}.
\]

\[
= \sum_{i,j=1}^{2} \left[ \tilde{J}(F_{i,R}^{i,\pm}) + \tilde{J}(F_{j,R}^{i,\pm}) \right]
\]
and, from (4.8), we have:

\[ A_n \leq \|P_n\|_p^{p/2} \sum_{i=1}^2 \mathcal{J}_{n,2} \]

\[ = : \|P_n\|_p^{p/2} \sum_{i=1}^2 \left[ I_{n,1}^{i,+}(S_{1,R}^{i,+}) + I_{n,2}^{i,-}(S_{2,R}^{-}) \right] \]

\[ = : \|P_n\|_p^{p/2} \sum_{i=1}^2 \left[ I_{n,1}^{i,+} + I_{n,2}^{i,-} \right], \ i = 1, 2, \]

where

\[ I_{n,k}^{i,\pm} := I_{n,k}^{i,(S_{k,R}^{i,\pm})} := \int_{\mathcal{F}_{k,R}^{i,\pm}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{\gamma_1}(|\tau| - 1)}; \ i, \ k = 1, 2. \quad (4.11) \]

According to (4.3) and (4.4), it is sufficient to estimate the integrals \( I_{n,k}^{i,\pm} \) for each \( i = 1, 2 \) and \( k = 1, 2 \).

Given the possible values of \( \gamma_i \) \((-1 < \gamma_i \leq 0, \ \gamma_i > 0, \ i = 1, 2)\), we will consider the estimates for the \( I_{n,k}^{i,\pm} \) separately.

1. Let \( i = 1 \).

1.1. For the integral \( I_{n,1}^{1,+} + I_{n,1}^{1,-} \), we get:

\[ I_{n,1}^{1,+} + I_{n,1}^{1,-} = \int \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}(|\tau| - 1)} \]

\[ \leq n \int_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1 - 1}} \leq n \int_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1 - 1)(1 + \overline{\varepsilon})}} \]

\[ \leq \begin{cases} n^{(\gamma_1 - 1)(1 + \overline{\varepsilon})}, & (\gamma_1 - 1)(1 + \overline{\varepsilon}) > 1, \\ n \ln n, & (\gamma_1 - 1)(1 + \overline{\varepsilon}) = 1, \\ n, & (\gamma_1 - 1)(1 + \overline{\varepsilon}) < 1, \end{cases} \]

for \( \gamma_1 > 0 \) and

\[ I_{n,1}^{1,+} + I_{n,1}^{1,-} = \int \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}(|\tau| - 1)} \]

\[ \leq nd_{1,R}^{(-\gamma_1)^+1} \int_{\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}} |d\tau| \leq n \left( \frac{1}{n} \right)^{[(-\gamma_1)^+1(1 - \varepsilon)]} \cdot mes(\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}) \]

\[ \leq n^{[\gamma_1 - 1](1 - \varepsilon)} < 1, \]
for $-1 < \gamma_1 \leq 0$.

1.2. Analogously to the (4.12) and (4.13), for the integral $I_{n,2}^{1,+} + I_{n,2}^{1,-}$, we get:

\[
I_{n,2}^{1,+} + I_{n,2}^{1,-} = \int_{\mathcal{F}_{2,R}^{1,+} \cup \mathcal{F}_{2,R}^{1,-}} \frac{d(\Psi(\tau), L)}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1} (|\tau| - 1)} \tag{4.14}
\]

\[
\leq n \int_{\mathcal{F}_{2,R}^{1,+} \cup \mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1 - 1}} \leq n \int_{\mathcal{F}_{2,R}^{1,+} \cup \mathcal{F}_{2,R}^{1,-}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1 - 1)(1 + \varepsilon)}}
\]

\[
\leq \begin{cases} 
  n^{(\gamma_1 - 1)(1 + \varepsilon)}, & (\gamma_1 - 1)(1 + \varepsilon) > 1, \\
  n \ln n, & (\gamma_1 - 1)(1 + \varepsilon) = 1, \\
  n, & (\gamma_1 - 1)(1 + \varepsilon) < 1,
\end{cases}
\]

for $\gamma_1 > 0$, and

\[
I_{n,2}^{1,+} + I_{n,2}^{1,-} = \int_{\mathcal{F}_{2,R}^{1,+} \cup \mathcal{F}_{2,R}^{1,-}} \frac{d(\Psi(\tau), L)}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1} (|\tau| - 1)} \tag{4.15}
\]

\[
\leq n \left( \frac{1}{n} \right)^{1 - \varepsilon} \int_{\mathcal{F}_{2,R}^{1,+} \cup \mathcal{F}_{2,R}^{1,-}} |\Psi(\tau) - \Psi(w_1)|^{(-\gamma_1)} |d\tau| \leq n^\varepsilon,
\]

for $-1 < \gamma_1 \leq 0$.

2. Let $i = 2$. Analogously to the previous case, we obtain:

2.1. (\[
I_{n,1}^{2,+} + I_{n,1}^{2,-} = \int_{\mathcal{F}_{1,R}^{2,+} \cup \mathcal{F}_{1,R}^{2,-}} \frac{d(\Psi(\tau), L)}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2} (|\tau| - 1)} \tag{4.16}
\]

\[
\leq n \int_{\mathcal{F}_{1,R}^{2,+} \cup \mathcal{F}_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\frac{\gamma_2 - 1}{1 + \beta_2}} |\tau - w_2|^{|\frac{\gamma_2 - 1}{1 + \beta_2}(1 + \varepsilon)} \tag{4.17}
\]

\[
\leq \begin{cases} 
  n^{\frac{\gamma_2 - 1}{1 + \beta_2} + \varepsilon}, & \frac{\gamma_2 - 1}{1 + \beta_2} > 1 - \varepsilon, \\
  n \ln n, & \frac{\gamma_2 - 1}{1 + \beta_2} = 1 - \varepsilon, \\
  n, & \frac{\gamma_2 - 1}{1 + \beta_2} < 1 - \varepsilon,
\end{cases}
\]
for $\gamma_2 > 0$ and
\[
I_{n,1}^{2,+} + I_{n,1}^{2,-} = \int_{\mathcal{F}_{1,R}^{2,+} \cup \mathcal{F}_{1,R}^{2,-}} \frac{d(\Psi(\tau), L) d\tau}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2}(|\tau| - 1)} \quad (4.17)
\]
\[
\leq nd_2(-\gamma_2)^{+1} \int_{\mathcal{F}_{1,R}^{2,+} \cup \mathcal{F}_{1,R}^{2,-}} |d\tau| \leq n \cdot m(s) (\mathcal{F}_{1,R}^{1,+} \cup \mathcal{F}_{1,R}^{1,-}) \leq 1,
\]
for $\gamma_2 \leq 0$.

2.2.
\[
I_{n,2}^{2,+} + I_{n,2}^{2,-} = \int_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} \frac{d(\Psi(\tau), L) d\tau}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2}(|\tau| - 1)} \quad (4.18)
\]
\[
\leq n \int_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2}(|\tau| - 1)} \leq \left\{ \begin{array}{ll}
\frac{n^{\gamma_2-1}}{4 + \beta_2}, & \gamma_2 > 1 - \varepsilon, \\
n \ln n, & \frac{n^{\gamma_2-1}}{4 + \beta_2} = 1 - \varepsilon, \\
n, & \frac{n^{\gamma_2-1}}{4 + \beta_2} < 1 - \varepsilon,
\end{array} \right.
\]

for $\gamma_2 > 0$, and
\[
I_{n,2}^{2,+} + I_{n,2}^{2,-} = \int_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} \frac{d(\Psi(\tau), L) d\tau}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2}(|\tau| - 1)} \quad (4.19)
\]
\[
\leq n \left( \frac{1}{n} \right)^{1-\varepsilon} \int_{\mathcal{F}_{2,R}^{2,+} \cup \mathcal{F}_{2,R}^{2,-}} |\Psi(\tau) - \Psi(w_1)|^{(-\gamma_2)} |d\tau| \leq n^{\varepsilon},
\]
for $-1 < \gamma_2 \leq 0$. Therefore, from (4.10)–(4.19), for any $p > 0$, we obtain
\[
A_{n}^{2/p} \leq \|P_n\|_{L_p(h,L)} \left\{ \begin{array}{ll}
n^{2(\gamma_1-1)/p}, & \gamma_1 > \frac{3}{2}, \\
(\ln n)^{\gamma_1 - 1/p}, & \gamma_1 = \frac{3}{2}, \\
n^{1/p}, & 0 < \gamma_1 < \frac{3}{2}, \\
n^{\varepsilon/p}, & -1 < \gamma_1 \leq 0
\end{array} \right.
\]
\[
+ \left\{ \begin{array}{ll}
n^{2(\gamma_1 - 1)/p} + \varepsilon, & \gamma_2 > 2 + \beta_2 - \varepsilon, \\
(\ln n)^{\gamma_2 - 1/p}, & \gamma_2 = 2 + \beta_2 - \varepsilon, \\
n^{1/p}, & 0 < \gamma_2 < 2 + \beta_2 - \varepsilon, \\
n^{\varepsilon/p}, & -1 < \gamma_2 \leq 0.
\end{array} \right.
\]
if $\alpha_1 \neq 0$, and

\[
A_n^{2/p} \leq \|P_n\|_{L_p(h,L)} \begin{cases}
\frac{n^{\gamma_1-1}}{p}, & \gamma_1 > 2 - \varepsilon, \\
\frac{1}{n^{p}}, & 0 < \gamma_1 < 2 - \varepsilon, \\
\frac{n^{\varepsilon}}{p}, & -1 < \gamma_1 \leq 0
\end{cases}
+ \begin{cases}
\frac{n^{\gamma_2-1}}{p}, & \gamma_2 > 2 + \beta_2 - \varepsilon, \\
\frac{n^{\varepsilon}}{p}, & 0 < \gamma_2 < 2 + \beta_2 - \varepsilon, \\
\frac{n^{\gamma_1-1}}{p}, & \gamma_1 > 2 + \beta_2 - \varepsilon,
\end{cases}
\]

if $\alpha_1 = 0$. So, for $A_n$ we get

\[
A_n^{2/p} \leq \|P_n\|_{L_p(h,L)} \begin{cases}
\frac{n^{2(\gamma_1-1)}}{p}, & \gamma_1 > 1 + \frac{\gamma_2-1}{2(1+\beta_2)}, \gamma_2 > 2 + \beta_2 - \varepsilon, \\
\frac{n^{\gamma_2-1}}{p}, & 0 < \gamma_1 \leq 1 + \frac{\gamma_2-1}{2(1+\beta_2)}, \gamma_2 > 2 + \beta_2 - \varepsilon, \\
\frac{n^{\gamma_1-1}}{p}, & \gamma_1 > \frac{3}{2}, 0 < \gamma_2 < 2 + \beta_2 - \varepsilon, \\
\frac{n^{\gamma_2-1}}{p}, & 0 < \gamma_1 < \frac{3}{2}, 0 < \gamma_2 < 2 + \beta_2 - \varepsilon, \\
\frac{n^{\varepsilon}}{p}, & -1 < \gamma_1 \leq 0, \gamma_2 = 2 + \beta_2 - \varepsilon,
\end{cases}
\]

if $\alpha_1 \neq 0$, and

\[
A_n^{2/p} \leq \|P_n\|_{L_p(h,L)} \begin{cases}
\frac{n^{\gamma_1-1}}{p}, & \gamma_1 > 1 + \frac{\gamma_2-1}{1+\beta_2}, \gamma_2 \geq 2 + \beta_2, \\
\frac{n^{\gamma_2-1}}{p}, & 2 \leq \gamma_1 \leq 1 + \frac{\gamma_2-1}{1+\beta_2}, \gamma_2 \geq 2 + \beta_2, \\
\frac{n^{\gamma_1-1}}{p}, & \gamma_1 \geq 2, 0 < \gamma_2 < 2 + \beta_2, \\
\frac{n^{\gamma_2-1}}{p}, & 0 < \gamma_1 < 2, 0 < \gamma_2 < 2 + \beta_2, \\
\frac{n^{\varepsilon}}{p}, & -1 < \gamma_1 \leq 0, -1 < \gamma_2 \leq 0
\end{cases}
\]

if $\alpha_1 = 0$.

Comparing (4.3) and (4.20), we get:

\[
|P_n(z)| \leq \left[ \frac{A_n}{d(z, L_R)} \right]^{2/p} |B_m(z)\Phi^{n+1}(z)|,
\]

where $A_n$ taken from (4.20). The function $B_m(z)$ is analytic in $\Omega$, continuous on $\overline{\Omega}$ and $|B_m(z)| = 1$ on $L$. Then, according to the maximum
modulus principle, we get

\[ |B_m(z)| < 1, \ z \in \Omega_R, \]

and, so the proof is complete.

References


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