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On the Cauchy theorem for hyperholomorphic functions of spatial variable

Oleg F. Herus

(*Presented by V. Ya. Gutlyanskii*)

Abstract. We proved a theorem about integral of quaternionic-differentiable functions of spatial variable over the closed surface. It is an analog of the Cauchy theorem from complex analysis.

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1. Introduction

Several researchers (see, e.g., [1, 2]) tried to generalize methods of complex analysis onto analysis of functions acting in several-dimensional algebras. At that, generalizations of different but mutually equivalent definitions of holomorphy in complex analysis generate diverse classes of hyperholomorphic functions in several-dimensional algebras.

Hypercomplex analysis in the space $\mathbb{R}^3$ was launched in the work of G. Moisil and N. Theodoresco [3], where a three-dimensional analog of the Cauchy–Riemann system was posed for the first time. R. Fueter [4] first introduced a class of "regular" quaternion functions by means of a four-dimensional generalization of the G. Moisil and N. Theodoresco system. He proved quaternion analogues of the Cauchy theorem, the integral Cauchy formula, the Liouville theorem and constructed an analog of the Laurent series.

Now quaternion analysis gained wide evolution (more detailed see [1, 5–7]) previously thanks to its physical applications. At that in most works it was usual to consider functions having continuous partial derivatives in a domain and satisfying the above Cauchy–Riemann-type system. In particular in the book [1] a spatial analog of the Cauchy theorem was

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proved by using the quaternion Stokes formula for bounded domains with a piecewise-smooth boundary and for functions having continuous partial derivatives in the closure of the domain.

In the survey paper [6] the continuity of partial derivatives was replaced by the weaker condition of real-differentiability for components of the quaternion function. In the work [8] we consider the same class of functions defined in a three-dimensional domain with the piecewise-smooth boundary and requiring only the component-wise real-differentiability and satisfying Cauchy–Riemann-type conditions like the class of holomorphic functions in complex analysis (see e.g. [9]).

In the present work we extend the result of the paper [8] onto more wide class of surfaces by using methods of the work [10], where a similar theorem was proved for functions taking values in finite-dimensional commutative associative algebras.

2. Quaternion hyperholomorphic functions

Let $\mathbb{H}(\mathbb{C})$ be the associative algebra of *complex quaternions*

$$a = \sum_{k=0}^{3} a_k \boldsymbol{i}_k,$$

where $\{a_k\}_{k=0}^3 \subset \mathbb{C}$, $\boldsymbol{i}_0 = 1$ and $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ be the imaginary quaternion units with the multiplication rule $\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$. The module of quaternion is defined by the formula

$$|a| := \sqrt{\sum_{k=0}^{3} |a_k|^2}.$$

Lemma 2.1 ([11]). *$|ab| \leqslant \sqrt{2}\,|a|\,|b|$ for all $\{a; b\} \subset \mathbb{H}(\mathbb{C})$.*

For $\{z_k\}_{k=1}^3 \subset \mathbb{R}$ consider vector quaternions $z := z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3$ as points of the Euclidean space $\mathbb{R}^3$ with the basis $\{\boldsymbol{i_k}\}_{k=1}^3$. Let Ω be a domain of $\mathbb{R}^3$. For functions $f : \Omega \to \mathbb{H}(\mathbb{C})$ having first order partial derivatives consider differential operators

$$D_l[f] := \sum_{k=1}^{3} \boldsymbol{i}_k \frac{\partial f}{\partial z_k},$$

$$D_r[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial z_k} \boldsymbol{i}_k.$$

Definition 2.1. *Function* $f := f_0 + f_1\boldsymbol{i}_1 + f_2\boldsymbol{i}_2 + f_3\boldsymbol{i}_3$ *is called* left- *or* right-$\mathbb{H}$-differentiable *in a point* $z^{(0)} \in \mathbb{R}^3$ *if its components* f_0, f_1, f_2, f_3 *are* $\mathbb{R}^3$*-differentiable functions in* $z^{(0)}$ *and the next condition*

$$D_l[f](z^{(0)}) = 0 \tag{2.1}$$

or

$$D_r[f](z^{(0)}) = 0$$

holds true respectively.

There is the notion of $\mathbb{C}$-differentiability of a function $f(\zeta) = u(x,y) + v(x,y)\boldsymbol{i}$, $\zeta = x + y\boldsymbol{i}$, in complex analysis (see [9, p. 33–34]). It is equivalent to $\mathbb{R}^2$-differentiability in the point (x_0, y_0) of the components $u(x,y)$ and $v(x,y)$ and satisfaction the condition

$$\frac{\partial f(\zeta_0)}{\partial x} + \frac{\partial f(\zeta_0)}{\partial y}\boldsymbol{i} = 0.$$

Thus the defined above notion of $\mathbb{H}$-differentiability is the exact analog of $\mathbb{C}$-differentiability from complex analysis.

It is well known (see [9, p. 35]) that $\mathbb{C}$-differentiability of a complex function is equivalent to existence of its derivative. But in quaternion analysis only the linear functions of special form have a derivative (see [12]).

The operator D_l is called the Dirac operator (see [13]) or the Moisil–Theodoresco operator (see [14]) and equality (2.1) is equivalent to the Moisil–Theodoresco system [3].

Definition 2.2. *A function* f *is called* left- *or* right-hyperholomorphic *in a domain* Ω *if it is left- or right-*$\mathbb{H}$*-differentiable in every point of the domain.*

3. Quaternion surface integral

Consider notions of surface and closed surface like to defined in the work [10].

Definition 3.1. *A* surface $\Gamma \subset \mathbb{R}^3$ *is an image of a closed set* $G \subset \mathbb{R}^2$ *under a homeomorphic mapping* $\varphi : G \to \mathbb{R}^3$

$$\varphi(u,v) := (z_1(u,v), z_2(u,v), z_3(u,v)),\ (u,v) \in G,$$

such that Jacobians

$$A := \frac{\partial z_2}{\partial u}\frac{\partial z_3}{\partial v} - \frac{\partial z_2}{\partial v}\frac{\partial z_3}{\partial u},\ B := \frac{\partial z_3}{\partial u}\frac{\partial z_1}{\partial v} - \frac{\partial z_3}{\partial v}\frac{\partial z_1}{\partial u},\ C := \frac{\partial z_1}{\partial u}\frac{\partial z_2}{\partial v} - \frac{\partial z_1}{\partial v}\frac{\partial z_2}{\partial u}$$

exist almost everywhere on the set G *and summable on* G.

The area of the surface Γ is calculated by the formula

$$\mathcal{L}(\Gamma) = \iint\limits_G \sqrt{A^2 + B^2 + C^2} du dv,$$

where the integral is understood in the Lebesgue sense.

A surface Γ is called *quadrable* (see [10]) if $\mathcal{L}(\Gamma) < +\infty$.

Let $\Gamma \subset \mathbb{R}^3$ be an image of a sphere $S \subset \mathbb{R}^3$ in a such homeomorphic mapping $\psi : S \to \mathbb{R}^3$ that the image of a great circle γ on the sphere S is a closed Jordan rectifiable curve $\widetilde{\gamma}$ on the set Γ. The sphere S is the union of two half-spheres S_1, S_2 with common edge γ. It is ease to see that there exist continuously differentiable mappings $\varphi_1 : K \to S_1$, $\varphi_2 : K \to S_2$ of the disk $K := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leqslant 1\}$. So the set Γ is the union of two sets $\Gamma_1 = \psi(\varphi_1(K))$, $\Gamma_2 = \psi(\varphi_2(K))$ with the intersection $\widetilde{\gamma} = \psi(\varphi_1(\partial K)) = \psi(\varphi_2(\partial K))$.

Definition 3.2. *A set Γ is called a* closed surface *if there exist a such homeomorphic mapping $\psi : S \to \mathbb{R}^3$ that the sets Γ_1, Γ_2 are surfaces in the sense of Definition 3.1 and orientation of the circle ∂K induces two mutually opposite orientations of the curve $\widetilde{\gamma}$ under mappings $\psi \circ \varphi_1$ and $\psi \circ \varphi_2$ respectively.*

Let $\Gamma^\varepsilon := \left\{z \in \mathbb{R}^3 : \rho(z, \Gamma) \leqslant \varepsilon\right\}$ (ρ denotes the Euclidean distance) be the closed ε-neighborhood of the surface Γ, $V(\Gamma^\varepsilon)$ be the space Lebesgue measure of the set Γ^ε and $\mathcal{M}^*(\Gamma) := \overline{\lim\limits_{\varepsilon \to 0}} \frac{V(\Gamma^\varepsilon)}{2\varepsilon}$ be the two-dimensional upper Minkowski content (see [15, p. 79]) of the surface Γ. For functions $f : \Gamma \to \mathbb{H}(\mathbb{C})$, $g : \Gamma \to \mathbb{H}(\mathbb{C})$ in the case of non-closed quadrable surface Γ the *quaternion surface integral* is defined by the formula

$$\iint\limits_\Gamma f(z)\, \sigma\, g(z) := \iint\limits_G f(\varphi(u, v))(A\boldsymbol{i}_1 + B\boldsymbol{i}_2 + C\boldsymbol{i}_3) g(\varphi(u, v)) du\, dv,$$

where $\sigma := dz_2 dz_3 \boldsymbol{i}_1 + dz_3 dz_1 \boldsymbol{i}_2 + dz_1 dz_2 \boldsymbol{i}_3$, and in the case of a closed surface — by the formula

$$\iint\limits_\Gamma f(z)\, \sigma\, g(z) := \iint\limits_{\Gamma_1} f(z)\, \sigma\, g(z) + \iint\limits_{\Gamma_2} f(z)\, \sigma\, g(z).$$

In particular, $\iint\limits_\Gamma |\sigma| = \mathcal{L}(\Gamma)$.

Theorem 3.1 ([8]). *Let P be the surface of a closed cube contained in a simply connected domain $\Omega \subset \mathbb{R}^3$, let a function $f : \Omega \to \mathbb{H}(\mathbb{C})$ be right-hyperholomorphic and a function $g : \Omega \to \mathbb{H}(\mathbb{C})$ be left-hyperholomorphic.*

Then

$$\iint\limits_{P} f(z)\,\sigma\, g(z) = 0.$$

Let $\delta > 0$, let $\omega_\Gamma(f,\delta) := \sup\limits_{\substack{|z_1-z_2|\leqslant\delta \\ z_1,z_2\in\Gamma}} |f(z_1) - f(z_2)|$ be the module of continuity of a function f on Γ, and let $d(\Gamma)$ be the diameter of Γ.

Lemma 3.1 ([10])**.** *Let Γ be a quadrable closed surface. Then*

$$\iint\limits_{\Gamma} \sigma = 0. \tag{3.1}$$

Lemma 3.2. *Let Γ be a quadrable closed surface and let $f : \Omega \to \mathbb{H}(\mathbb{C})$ and $g : \Omega \to \mathbb{H}(\mathbb{C})$ be continuous functions. Then*

$$\left|\iint\limits_{\Gamma} f(z)\,\sigma\, g(z)\right|$$
$$\leqslant 2\,\mathcal{L}(\Gamma)\left(\omega_\Gamma(f, d(\Gamma))\,\max_{z\in\Gamma}|g(z)| + \omega_\Gamma(g, d(\Gamma))\,\max_{z\in\Gamma}|f(z)|\right). \tag{3.2}$$

Proof. Thanks to the formula (3.1) we have

$$\iint\limits_{\Gamma} f(z_0)\,\sigma\, g(z_0) = 0$$

for any point $z_0 \in \Gamma$. Therefore

$$\iint\limits_{\Gamma} f(z)\,\sigma\, g(z) = \iint\limits_{\Gamma} (f(z) - f(z_0))\,\sigma\, g(z_0)$$
$$+ \iint\limits_{\Gamma} f(z)\,\sigma\,(g(z) - g(z_0)),$$

from which follows the estimate (3.2), taking into account Lemma 2.1. □

Theorem 3.2. *Let $\mathbb{R}^3 \supset \Omega$ be a bounded simply connected domain with the quadrable closed boundary Γ, for which*

$$\mathcal{M}^*(\Gamma) < +\infty, \tag{3.3}$$

let Ω have Jordan measurable intersections with planes perpendicular to coordinate axes, let a function $f : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be right-hyperholomorphic in Ω and continuous in the closure $\overline{\Omega}$ and let a function $g : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be left-hyperholomorphic in Ω and continuous in $\overline{\Omega}$. Then

$$\iint\limits_{\Gamma} f(z)\,\sigma\, g(z) = 0. \tag{3.4}$$

Proof. Let us use the method proposed in the work [10] under the proving of Theorem 6.1. Thanks to the condition (3.3) there exists such a constant $c > 0$ that for all sufficiently small $\varepsilon > 0$ the following inequality holds

$$V(\Gamma^{\varepsilon}) \leqslant c\varepsilon. \tag{3.5}$$

Decompose the space by planes perpendicular to coordinate axes onto closed cubes with the edge of length $\frac{\varepsilon}{\sqrt{3}}$. Let $\{K_j\}$, $j \in J$, be the finite set of formed cubes having nonempty intersection with the surface Γ.

The integral (3.4) is representable in the form

$$\iint\limits_{\Gamma} f(z)\,\sigma\, g(z) = \sum_{j\in J} \iint\limits_{\partial(\Omega\cap K_j)} f(z)\,\sigma\, g(z) + \sum_{K_j\subset\Omega} \iint\limits_{\partial K_j} f(z)\,\sigma\, g(z). \tag{3.6}$$

By the Theorem 3.1 the second sum in the equality (3.6) is equal zero.

Every set $\Omega \cap K_j$ consists of finite or infinite totality of connected components. Applying the estimate (3.2) to the boundary of the every component, we obtain

$$\begin{aligned} &\left| \iint\limits_{\partial(\Omega\cap K_j)} f(z)\,\sigma\, g(z) \right| \\ &\leqslant 2(\mathcal{L}(\Gamma\cap K_j) + 2\varepsilon^2) \Big(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| \\ &+ \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \Big). \end{aligned} \tag{3.7}$$

Substituting the inequality (3.7) into the equality (3.6), we obtain

$$\begin{aligned} &\left| \iint\limits_{\Gamma} f(z)\,\sigma\, g(z) \right| \\ &\leqslant 2 \left(\mathcal{L}(\Gamma) + 2\sum_{j\in J} \varepsilon^2 \right) \left(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| + \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \right). \end{aligned}$$

Since $\bigcup\limits_{j\in J} K_j \subset \Gamma^\varepsilon$, we obtain from the inequality (3.5) that

$$\frac{1}{3\sqrt{3}} \sum_{j\in J} \varepsilon^3 \leqslant V(\Gamma^\varepsilon) \leqslant c\varepsilon.$$

Therefore

$$\left| \iint\limits_{\Gamma} f(z)\,\sigma\, g(z) \right|$$
$$\leqslant 2(\mathcal{L}(\Gamma) + 6\sqrt{3}c) \left(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| + \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \right)$$

and the equality (3.4) be obtained from here by passaging to the limit when $\varepsilon \to 0$. □

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Contact information

Oleg F. Herus
Zhytomyr Ivan Franko State University,
Zhytomyr, Ukraine
E-Mail: ogerus@zu.edu.ua