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Prime ends in the mapping theory on the Riemann surfaces

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Abstract. It is proved criteria for continuous and homeomorphic extension to the boundary of mappings with finite distortion between domains on the Riemann surfaces by prime ends of Caratheodory.

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1. Introduction

The theory of the boundary behavior in the prime ends for the mappings with finite distortion has been developed in [12] for the plane domains and in [15] for the spatial domains. The pointwise boundary behavior of the mappings with finite distortion in regular domains on Riemann surfaces was recently studied by us in [30] and [31]. Moreover, the problem was investigated in regular domains on the Riemann manifolds for $n \geq 3$ as well as in metric spaces, see e.g. [1] and [34]. It is necessary to mention also that the theory of the boundary behavior of Sobolev's mappings has significant applications to the boundary value problems for the Beltrami equations and for analogs of the Laplace equation in anisotropic and inhomogeneous media, see e.g. [3, 8, 10, 11, 13, 14, 20, 23, 26] and relevant references therein.

For basic definitions and notations, discussions and historic comments in the mapping theory on the Riemann surfaces, see our previous papers [29–32].

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2. Definition of the prime ends and preliminary remarks

We act similarly to Caratheodory [5] under the definition of the prime ends of domains on a Riemann surface $\mathbb{S}$, see Chapter 9 in [6]. First of all, recall that a continuous mapping $\sigma : \mathbb{I} \to \mathbb{S}$, $\mathbb{I} = (0, 1)$, is called a **Jordan arc** in $\mathbb{S}$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations σ, $\overline{\sigma}$ and $\partial\sigma$ for $\sigma(\mathbb{I})$, $\overline{\sigma(\mathbb{I})}$ and $\overline{\sigma(\mathbb{I})} \setminus \sigma(\mathbb{I})$, correspondingly. A Jordan arc σ in a domain $D \subset \mathbb{S}$ is called a **cross–cut** of the domain D if σ splits D, i.e. $D \setminus \sigma$ has more than one (connected) component, $\partial\sigma \subseteq \partial D$ and $\overline{\sigma}$ is a compact set in $\mathbb{S}$.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of D is called a **chain** in D if:

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \varnothing$ for every $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) σ_m splits D into 2 domains one of which contains σ_{m+1} and another one σ_{m-1} for every $m > 1$;

(iii) $\delta(\sigma_m) \to 0$ as $m \to \infty$.

Here $\delta(E) = \sup\limits_{p_1, p_2 \in \mathbb{S}} \delta(p_1, p_2)$ denotes the diameter of a set E in $\mathbb{S}$ with respect to an arbitrary metric δ in $\mathbb{S}$ agreed with its topology, see [29]–[31].

Correspondingly to the definition, a chain of cross-cuts σ_m generates a sequence of domains $d_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial\, d_m = \sigma_m$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called **equivalent** if, for every $m = 1, 2, \ldots$, the domain d_m contains all domains d'_k except a finite number and, for every $k = 1, 2, \ldots$, the domain d'_k contains all domains d_m except a finite number, too. A **prime end** P of the domain D is an equivalence class of chains of cross-cuts of D. Later on, E_D denote the collection of all prime ends of a domain D and $\overline{D}_P = D \cup E_D$ is its completion by prime ends.

Next, we say that a sequence of points $p_l \in D$ is **convergent to a prime end** P of D if, for a chain of cross–cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains all points p_l except their finite collection. Further, we say that a sequence of prime ends P_l converge to a prime end P if, for a chain of cross–cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains chains of cross–cuts $\{\sigma'_k\}$ in all prime ends P_l except their finite collection.

Now, let D be a domain in the compactification $\overline{\mathbb{S}}$ of a Riemann surface $\mathbb{S}$ by Kerekjarto–Stoilow, see a discussion in [29]– [31]. Denote by E_D the union of D and all its prime ends. Open neighborhoods of points in D is induced by the topology of $\overline{\mathbb{S}}$. A basis of neighborhoods of a prime end P of D can be defined in the following way. Let d be an arbitrary domain from a chain in P. Denote by d^* the union of d and all prime ends of D having some chains in d. Just all such d^* form a basis of open neighborhoods of the prime end P. The corresponding topology on $\overline{D}_P$ is called the **topology of prime ends**.

Let P be a prime end of D on a Riemann surface $\mathbb{S}$, $\{\sigma_m\}$ and $\{\sigma'_m\}$ be two chains in P, d_m and d'_m be domains corresponding to σ_m and σ'_m. Then

$$\bigcap_{m=1}^{\infty}\overline{d_m}\subseteq\bigcap_{m=1}^{\infty}\overline{d'_m}\subset\bigcap_{m=1}^{\infty}\overline{d_m}\ ,$$

and, thus,

$$\bigcap_{m=1}^{\infty}\overline{d_m}=\bigcap_{m=1}^{\infty}\overline{d'_m}\ ,$$

i.e. the set named by a **body of the prime end** P

$$I(P):=\bigcap_{m=1}^{\infty}\overline{d_m} \tag{2.1}$$

depends only on P but not on a choice of a chain of cross–cuts $\{\sigma_m\}$ in P.

It is necessary to note also that, for any chain $\{\sigma_m\}$ in the prime end P,

$$\Omega\ :=\ \bigcap_{m=1}^{\infty}d_m\ =\ \varnothing\ . \tag{2.2}$$

Indeed, every point p in Ω belongs to D. Moreover, some open neighborhood of p in D should belong to Ω. In the contrary case each neighborhood of p should have a point in some σ_m. However, in view of condition (iii) then $p\in\partial D$ that should contradict the inclusion $p\in D$. Thus, Ω is an open set and if Ω would be not empty, then the connectedness of D would be broken because $D=\Omega\cup\Omega^*$ with the open set $\Omega^*:=D\setminus I(P)$.

In view of conditions (i) and (ii), we have by (2.2) that

$$I(P)=\bigcap_{m=1}^{\infty}(\partial d_m\cap\partial D)=\partial D\ \cap\ \bigcap_{m=1}^{\infty}\partial d_m\ .$$

Thus, we obtain the following statement.

Proposition 2.1. *For each prime end P of a domain D on a Riemann surface,*

$$I(P)\subseteq\partial D. \tag{2.3}$$

Remark 2.1. If ∂D is a compact set in $\mathbb{S}$, then $I(P)$ is a continuum, i.e. it is a connected compact set, see e.g. I(9.12) in [37], see also I.9.3 in [4], and $I(P)$ belongs to only one (connected) component Γ of ∂D. In the case, we say that the component Γ is **associated with the prime end** P.

Moreover, in the case of a compact boundary of D, every prime end of D contains a **convergent chain** $\{\sigma_m\}$, i.e., that is contracted to a point $p_0 \in \partial D$. Furthermore, each prime end P contains a **spherical chain** $\{\sigma_m\}$ lying on circles $S(p_0, r_m) = \{p \in \mathbb{S} : \ \delta(p, p_0) = r_m\}$ with $p_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$. The proof is perfectly similar to Lemma 1 in [15] after the replacement of metrics, see also Theorem 7.1 in [22], and hence we omit it. Note by the way that condition (iii) does not depend on the choice of the metric δ agreed with the topology of $\mathbb{S}$ because ∂D has a compact neighborhood.

3. The main lemma

Lemma 3.1. *Let D be a domain in a Riemann surface $\mathbb{S}$ and let Γ be a compact isolated component of ∂D in $\mathbb{S}$ that is not degenerated to a point. Then Γ has a neighborhood U with a conformal mapping h of $U^* := U \cap D$ onto a ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ where one may assume that $\gamma := \partial U^* \cap D$ is a closed Jordan curve and*

$$C(\gamma, h) = \{z \in \mathbb{C} : |z| = 1\}, \ C(\Gamma, h) = \{z \in \mathbb{C} : |z| = r\} \ .$$

Furthermore, the map h can be extended to a homeomorphism of E_{U^*} onto $\overline{R}$.

Here we use the notation of the **cluster set** of the mapping h for $B \subseteq \partial D$,

$$C(B, h) := \left\{ \ z \in \mathbb{C} : z = \lim_{k \to \infty} h(p_k), \ p_k \to p \in B, \ p_k \in D \ \right\}$$

Note that the first statement is obvious in the case of isolated boundary points of ∂D with $r = 0$ and the punctured unit disk $R = \mathbb{D}_0 := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Proof. By the Kerekjarto–Stoilow representation of $\mathbb{S}$, see a discussion in [29]– [31], Γ has an open neighborhood V in $\mathbb{S}$ of a finite genus. Without loss of generality, we may assume that V is connected and does not intersect $\partial D \setminus \Gamma$ because Γ is an isolated component of ∂D. Thus, $V \cap D$ is a Riemann surface of finite genus with an isolated boundary element

g corresponding to Γ. However, a Riemann surface of finite genus has only boundary elements of the first kind, see, e.g., IV.II.6 in [35]. Consequently, Γ has a neighborhood U^* from the side of D of genus zero with a closed Jordan curve $\gamma = \partial U^* \cap D$. Set $U = U^* \cup (V \setminus D)$. Correspondingly to the Kerekjarto–Stoilow representation, the latter means that U^* is homeomorphic to a plane domain and, consequently, by the general principle of Koebe, see e.g. Section II.3 in [17], U^* is conformally equivalent to a plane domain D^*. Note that by the construction U^* had two boundary components. Hence there is a conformal mapping h of U^* onto a ring $D^* = R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ with $C(\gamma, h) = \{z \in \mathbb{C} : |z| = 1\}$ and $C(\Gamma, h) = \{z \in \mathbb{C} : |z| = r\}$, see e.g. Proposition 2.5 in [25] or Proposition 13.5 in [20].

Now, U^* and R are Riemann surfaces of hyperbolic type and the modulus M of curve families are invariant under the conformal mapping h, see a discussion in [29]– [31]. By condition (i) we have, for a chain $\{\sigma_m\}$ in a prime end P associated with the component Γ and localized in U^*, that

$$M(\Delta(\sigma_m, \sigma_{m+1}, U^*)) \ < \ \infty \qquad \forall\, m = 1, 2, \ldots \tag{3.1}$$

where $\Delta(E, F, G)$ denotes a family of all curves joining the sets E and F through the set G. Moreover, by Remark 2.1 the prime end P contains a convergent chain $\{\sigma_m\}$ for which and any continuum C in U^*

$$\lim_{m\to\infty} \ M(\Delta(\sigma_m, C, U^*)) \ = \ 0\ . \tag{3.2}$$

Similarly, prime ends associated with γ satisfy conditions (3.1) and (3.2). Thus, the prime ends of U^* in the sense (i)–(iii) and their images in R are the prime ends in the sense of Section 4 in [21]. The Näkki prime ends in R has a natural one-to-one correspondence with the points of ∂R whose extension to the correspondence between $\overline{R}$ and $\overline{R}_P$ by the identity in R is a homeomorphism with respect to the topologies of $\overline{R}$ and $\overline{R}_P$ or with respect to convergence of points and prime ends, correspondingly, see Theorems 4.1 and 4.2 in [21]. □

Remark 3.1. So, the space of $\overline{U^*}_P$ with the topology of prime ends is metrizable by $\rho(p_1, p_2) := |\tilde{h}(p_1) - \tilde{h}(p_2)|$, where $\tilde{h}$ is the extension of $h : U^* \to R$ to the homeomorphism $\tilde{h} : \overline{U^*}_P \to \overline{R}$ from Lemma 3.1, and the space $(\overline{U^*}_P, \rho)$ is compact.

Furthermore, if D be a domain in the Kerekjarto–Stoilow compactification $\overline{\mathbb{S}}$ of a Riemann surface $\mathbb{S}$ and ∂D is a set in $\mathbb{S}$ with a finite collection of components, then the whole space $\overline{D}_P$ can be metrized through the theory of pseudometric spaces, see e.g. Section 2.21.XV in [18], and it is compact. Namely, let ρ_0 be one of the metrics on $\overline{\mathbb{S}}$ and let $\rho_1, \ldots, \rho_n$ be the above metrics on $\overline{U^*_1}_P, \ldots, \overline{U^*_n}_P$ for the corresponding components $\Gamma_1, \ldots, \Gamma_n$ of ∂D. Then $\rho^*_j := \rho_j/(1+\rho_j) \le 1$, $j = 0, 1, \ldots, n$, be also metrics generated the same topologies on $\overline{\mathbb{S}}$, $\overline{U^*_1}_P, \ldots, \overline{U^*_n}_P$, correspondingly, see e.g. Section 2.21.V in [18]. Then the topology of prime ends on $\overline{D}_P$ is generated by the metric $\rho = \sum\limits_{j=0}^{n} 2^{-(j+1)}\tilde{\rho}_j < 1$ where the pseudometrics $\tilde{\rho}_j$ are extensions of ρ^*_j onto $\overline{D}_P$ by 1, see e.g. Remark 2 in point 2.21.XV of [18].

4. Some general topological lemmas

Let us give definitions of topological notions and facts of a general character that will be useful in what follows. Let T be an arbitrary topological space. Then a **path in** T is a continuous map $\gamma : [a, b] \to T$. Given $A, B, C \subseteq T$, $\Delta(A, B, C)$ denotes a collection of all paths γ joining A and B in C, i.e., $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$. In what follows, $|\gamma|$ denotes the locus of γ, i.e. the image $\gamma([a, b])$.

Proposition 4.1. *Let T be a topological space. Suppose that E_1 and E_2 are sets in T with $\overline{E_1} \cap \overline{E_2} = \varnothing$. Then*

$$\Delta\,(\,E_1\,,\,E_2\,,\,T\,)\;>\;\Delta\,(\,\partial E_1\,,\,\partial E_2\,,\,T \setminus (\overline{E}_1 \cup \overline{E}_2)\,)\;. \tag{4.1}$$

Proof. Indeed, let $\gamma \in \Delta(E_1, E_2, T)$, i.e. the path $\gamma : [a, b] \to T$ is such that $\gamma(a) \in E_1$ and $\gamma(b) \in E_2$. Note that the set $\alpha := \gamma^{-1}(\overline{E_1})$ is a closed subset of the segment $[a, b]$ because γ is continuous, see e.g. Theorem 1 in Section I.2.1 of [4]. Consequently, α is compact because $[a, b]$ is a compact space, see e.g. I.9.3 in [4]. Then there is $a_* := \max_{t\in\alpha} t < b$ because $\gamma(b) \in E_2$ and by the hypothesis of the proposition $\overline{E_1} \cap \overline{E_2} = \varnothing$. Thus, $\gamma' := \gamma|_{[a_*, b]}$ belongs to $\Delta(\partial E_1, E_2, T \setminus \overline{E}_1)$ because γ is continuous and hence $\gamma'(a_*)$ cannot be an inner point of E_1.

Arguing similarly in the space $T' = T \setminus E_1$ with $E'_1 := E_2$ and $E'_2 := \partial E_1$, we obtain that there is $b_* := \min_{\gamma'(t)\in\overline{E_2}} t > a_*$. Thus, by the given construction $\gamma_* := \gamma|_{[a_*, b_*]}$ just belongs to $\Delta(\partial E_1, \partial E_2, T \setminus (\overline{E}_1 \cup \overline{E}_2))$. $\square$

Lemma 4.1. *In addition to the hypothesis of Proposition 4.1, let T be a subspace of a metric space (M, ρ). Suppose that*

$$\partial E_1 \subseteq C_1 := \{p \in M : \rho(p, p_0) = R_1\},$$

$$\partial E_2 \subseteq C_2 := \{p \in M : \rho(p, p_0) = R_2\}$$

with $p_0 \in M \setminus T$ *and* $R_1 < R_2$. *Then*

$$\Delta\,(\,E_1\,,\,E_2\,,\,T\,)\ >\ \Delta\,(\,C_1\,,\,C_2\,,\,A\,) \tag{4.2}$$

where

$$A\ =\ A(p_0, R_1, R_2)\ :=\ \{p \in M:\ R_1 < \rho(p, p_0) < R_2\}\ .$$

Note that here, generally speaking, $C_1 \cap T \neq E_1$ and $C_2 \cap T \neq E_2$ as well as γ_* in the proof of Proposition 4.1 is not in R.

Proof. First of all, note that by the continuity of γ_* the set $\omega := \gamma_*^{-1}(R)$ is open in $[a_*, b_*]$ and ω is the union of a countable collection of disjoint intervals (a_1, b_1), (a_2, b_2), $\ldots$ with ends in $\Gamma := \gamma_*^{-1}(\partial R)$. If there is a pair a_k and b_k in the different sets $\Gamma_i := \gamma_*^{-1}(C_i)$, $i = 1, 2$, $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, then the proof is complete.

Let us assume that such a pair is absent. Then the given collection is split into 2 collections of disjoint intervals (a_l', b_l') and (a_l'', b_l'') with ends $a_l', b_l' \in \Gamma_1$ and $a_l'', b_l'' \in \Gamma_2$, $l = 1, 2, \ldots$. Set $\alpha_1 = \bigcup\limits_l (a_l', b_l')$ and $\alpha_2 = \bigcup\limits_l (a_l'', b_l'')$.

Arguing by contradiction, it is easy to show that $\gamma_* : [a_*, b_*] \to (M, \rho)$ is uniformly continuous because $[a_*, b_*]$ is a compact space. Indeed, let us assume that there is $\varepsilon > 0$ and a sequence of pairs a_n^* and $b_n^* \in [a_*, b_*]$, $n = 1, 2, \ldots$, such that $|b_n^* - a_n^*| \to 0$ as $n \to \infty$ and simultaneously $\rho(\gamma_*(a_n^*), \gamma_*(b_n^*)) \geq \varepsilon$. However, by compactness of $[a_*, b_*]$ there is a subsequence $a_{n_k}^* \to a_0 \in [a_*, b_*]$ and then also $b_{n_k}^* \to a_0$ as $k \to \infty$. Hence by the continuity of γ_* it should be $\rho(\gamma_*(a_{n_k}^*), \gamma_*(a_0)) \to 0$ as well as $\rho(\gamma_*(b_{n_k}^*), \gamma_*(a_0)) \to 0$ and then by the triangle inequality also $\rho(\gamma_*(a_{n_k}^*), \gamma_*(b_{n_k}^*)) \to 0$ as $k \to \infty$. The contradiction disproves the assumption.

Note that $b_l' - a_l' \to 0$ as $l \to \infty$ and by the uniform continuity of γ_* on $[a_*, b_*]$ we have that $|\gamma_l'| \to C_1$ in the sense that

$$\sup_{p \in |\gamma_l'|}\ \inf_{q \in C_1} \rho(p, q)\ \to\ 0 \qquad \text{as} \quad l \to \infty$$

where $\gamma_l' := \gamma_*|_{[a_l', b_l']}$, $l = 1, 2, \ldots$. Thus, there is $R_2' \in (R_1, R_2)$ such that the set $L_1 := \bigcup\limits_l |\gamma_l'|$ lies outside of $B_2 := \{p \in M : \rho(p, p_0) > R_2'\}$.

Arguing similarly, we obtain that there is $R_1' \in (R_1, R_2')$ such that the set $L_2 := \bigcup\limits_l |\gamma_l''|$ lies outside of $B_1 := \{p \in M : \rho(p, p_0) < R_1'\}$. Remark that the sets $\beta_1 := \gamma_*^{-1}(B_1)$ and $\beta_2 := \gamma_*^{-1}(B_2)$ are open in $[a_*, b_*]$ because γ_* is continuous and by the construction $\delta_1 := \alpha_1 \cup \beta_1$

and $\delta_2 := \alpha_2 \cup \beta_2$ are open, mutually disjoint and together cover the segment $[a_*, b_*]$. The latter contradicts to connectedness of the segment and, thus, disproves the above assumption. $\square$

5. On boundary behavior in prime ends of inverse maps

The main base for extending inverse mappings is the following fact.

Lemma 5.1. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of components, and let $f : D \to D^*$ be a homeomorphism of finite distortion with $K_f \in L^1_{\mathrm{loc}}$. Then*

$$C(P_1, f) \cap C(P_2, f) = \varnothing \tag{5.1}$$

for all prime ends $P_1 \neq P_2$ in the domain D.

Here we use the notation of the **cluster set** of the mapping f at $P \in E_D$,

$$C(P, f) := \left\{ P' \in E_{D'} : P' = \lim_{k\to\infty} f(p_k),\ p_k \to P,\ p_k \in D \right\}.$$

As usual, we also assume here that the dilatation K_f of the mapping f is extended by zero outside of the domain D.

Proof. First of all note that $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$ are metrizable spaces. Hence their compactness is equivalent to their sequential compactness, see e.g. Remark 41.I.3 in [19], and, consequently, ∂D and $\partial D'$ are compact subsets of $\mathbb{S}$ and $\mathbb{S}'$, correspondingly, see e.g. Proposition I.9.3 in [4]. Thus, in view of Remarks 2.1 and 3.1 and Lemma 3.1, we may assume that $\mathbb{S}$ is hyperbolic, $\overline{D}$ is a compact set in $\mathbb{S}$, $K_f \in L^1(D)$, P_1 and P_2 are associated with the same component Γ of ∂D and D' is a ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ and

$$A_k \ := \ C(P_k, f)\ , \qquad k \ = \ 1,\ 2$$

are sets of points in the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$, ∂D consists of 2 components: Γ and a closed Jordan curve γ, $C(\gamma, f) = C_* := \{z \in \mathbb{C} : |z| = 1\}$, $C(C_*, f^{-1}) = \gamma$, $C(C_r, f^{-1}) = \Gamma$, see also Proposition 2.5 in [25] or Proposition 13.5 in [20]. Furthermore, then the sets A_k are continua, i.e. closed arcs of the circle C_r, because

$$A_k \ = \ \bigcap_{m=1}^{\infty} \overline{f\left(d_m^{(k)}\right)}\ , \qquad k \ = \ 1,\ 2\ ,$$

where $d_m^{(k)}$ are domains corresponding to chains of cross–cuts $\{\sigma_m^{(k)}\}$ in the prime ends P_k, $k=1,2$, see e.g. I(9.12) in [37] and also I.9.3 in [4]. In addition, by Remark 2.1 we may assume also that $\sigma_m^{(k)}$ are open arcs of the hyperbolic circles $C_m^{(k)} := \{p \in \mathbb{S} : h(p,p_k) = r_m^{(k)}\}$ on $\mathbb{S}$ with $p_k \in \partial D$ and $r_m^{(k)} \to 0$ as $m \to \infty$, $k=1,2$.

Set $p_0 = p_1$. By the definition of the topology of the prime ends in the space $\overline{D}_P$, we have that $d_m^{(1)} \cap d_m^{(2)} = \varnothing$ for all large enough m because $P_1 \neq P_2$. For a such m, set $R_1 = r_{m+1}^{(1)} < R_2 = r_m^{(1)}$ and

$$U_k = d_m^{(k)}\ , \quad \Sigma_k = \sigma_m^{(k)}\ , \quad C_k = \{p \in \mathbb{S} : h(p,p_0) = R_k\}, \quad k = 1,2\ .$$

Let K_1 and K_2 be arbitrary continua in U_1 and U_2, correspondingly. Applying Proposition 4.1 and Lemma 4.1 with $T = D$, $E_1 = d_{m+1}^{(1)}$ and $E_2 = D \setminus d_m^{(1)}$, and taking into account the inclusion $\Delta(K_1,K_2,D) \subset \Delta(E_1,E_2,D)$, we obtain that

$$\Delta(K_1,K_2,D)\ >\ \Delta(C_1,C_2,A)\ , \quad A := \{p \in \mathbb{S} : R_1 < h(p,p_0) < R_2\}\ , \tag{5.2}$$

which means that any path $\alpha : [a,b] \to \mathbb{S}$ joining K_1 and K_2 in D, $\alpha(a) \in K_1$, $\alpha(b) \in K_2$ and $\alpha(t) \in D$, $t \in (a,b)$, has a subpath joining C_1 and C_2 in A. Thus, since f is a homeomorphism, we have also that

$$\Delta(fK_1,fK_2,fD)\ >\ \Delta(fC_1,fC_2,fA) \tag{5.3}$$

and by the minorization principle, see e.g. [7, p. 178], we obtain that

$$M(\Delta(fK_1,fK_2,fD))\ \leq\ M(\Delta(fC_1,fC_2,fA))\ . \tag{5.4}$$

So, by Lemma 3.1 in [30] and [31] we conclude that

$$M(\Delta(fK_1,fK_2,fD))\ \leqslant\ \int\limits_A K_f(p)\cdot\ \xi^2(h(p,p_0))\ dh(p) \tag{5.5}$$

for all measurable functions $\xi : (R_1,R_2) \to [0,\infty]$ such that

$$\int\limits_{R_1}^{R_2} \xi(R)\ dR \geqslant\ 1\ . \tag{5.6}$$

In particular, for $\xi(R) \equiv 1/\delta$, $\delta = R_2 - R_1 > 0$, we get from here that

$$M(\Delta(fK_1,fK_2,fD))\ \leqslant\ M_0\ :=\ \frac{1}{\delta}\int\limits_D K_f(p)\ dh(p)\ <\ \infty\ . \tag{5.7}$$

Since f is a homeomorphism, (5.7) means that

$$M(\Delta(\mathcal{K}_1, \mathcal{K}_2, D')) \leqslant M_0 < \infty \tag{5.8}$$

for all continua $\mathcal{K}_1$ and $\mathcal{K}_2$ in the domains $V_1 = fU_1$ and $V_2 = fU_2$, correspondingly.

Let us assume that $A_1 \cap A_2 \neq \varnothing$. Then by the construction there is $p_0 \in \partial R \cap \partial V_1 \cap \partial V_2$. However, the latter contradicts (5.8) because the ring R is a QED (quasiextremal distance) domains, see e.g. Theorem 3.2 in [20], see also Theorem 10.12 in [36]. $\square$

Theorem 5.1. *Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}'}$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of nondegenerate components, and let $f : D \to D'$ be a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$. Then the inverse mapping $g = f^{-1} : D' \to D$ can be extended to a continuous mapping $\tilde{g}$ of $\overline{D'}_P$ onto $\overline{D}_P$.*

Proof. Recall that by Remark 3.1 the spaces $\overline{D}_P$ and $\overline{D'}_P$ are compact and metrizable with metrics ρ and ρ'. Let a sequence $p_n \in D'$ converges as $n \to \infty$ to a prime end $P' \in E_{D'}$. Then any subsequence of $p_n^* := g(p_n)$ has a convergent subsequence by compactness of $\overline{D}_P$. By Lemma 5.1 any such convergent subsequence should have the same limit. Thus, the sequence p_n^* is convergent, see e.g. Theorem 2 of Section 2.20.II in [18]. Note that p_n^* cannot converge to an inner point of D because $I(P) \subseteq \partial D$ by Proposition 2.1 and, consequently, p_n is convergent to $\partial D'$, see e.g. Proposition 2.5 in [25] or Proposition 13.5 in [20]. Thus, $E_{D'}$ is mapped into E_D under this extension $\tilde{g}$ of g. In fact, $\tilde{g}$ maps $E_{D'}$ onto E_D because $p_n = f(p_n^*)$ has a convergent subsequence for every sequence $p_n^* \in D$ that is convergent to a prime end P of the domain D because $\overline{D'}_P$ is compact. The map $\tilde{g}$ is continuous. Indeed, let a sequence $P_n' \in \overline{D'}_P$ be convergent to $P' \in \overline{D'}_P$. Then there is a sequence $p_n \in D'$ such that $\rho'(P_n', p_n) < 2^{-n}$ and $\rho(p_n^*, P_n^*) < 2^{-n}$ where $p_n^* := g(p_n)$, $P_n^* := \tilde{g}(P_n)$ and $P^* = \tilde{g}(P')$. Then $p_n \to P'$ and by the above $p_n^* \to P^*$ as well as $P_n^* \to P^*$ as $n \to \infty$. $\square$

6. Lemma on extension to boundary of direct mappings

In contrast with the case of the inverse mappings, as it was already established in the plane, no degree of integrability of the dilatation leads to the extension to the boundary of direct mappings with finite distortion, see the example in the proof of Proposition 6.3 in [20]. The nature of the corresponding conditions has a much more refined character as the following lemma demonstrates.

Lemma 6.1. *Under the hypothesis of Theorem 5.1, let in addition*

$$\int\limits_{R(p_0,\varepsilon,\varepsilon_0)} K_f(p)\cdot\psi^2_{p_0,\varepsilon,\varepsilon_0}(h(p,p_0))\ dh(p) = o\left(I^2_{p_0,\varepsilon_0}(\varepsilon)\right) \qquad \forall\ p_0\in\partial D \tag{6.1}$$

as $\varepsilon\to 0$ for all $\varepsilon_0<\delta(p_0)$ where $R(p_0,\varepsilon,\varepsilon_0)=\{p\in\mathbb{S}:\varepsilon<h(p,p_0)<\varepsilon_0\}$ and $\psi_{p_0,\varepsilon,\varepsilon_0}(t):(0,\infty)\to[0,\infty]$, $\varepsilon\in(0,\varepsilon_0)$, is a family of measurable functions such that

$$0\ <\ I_{p_0,\varepsilon_0}(\varepsilon)\ :=\ \int\limits_{\varepsilon}^{\varepsilon_0}\psi_{p_0,\varepsilon,\varepsilon_0}(t)\ dt\ <\ \infty \qquad\qquad \forall\ \varepsilon\in(0,\varepsilon_0)\ .$$

Then f can be extended to a continuous mapping $\tilde f$ of $\overline{D}_P$ onto $\overline{D'}_P$.

We assume here that the function K_f is extended by zero outside of D.

Proof. By Remarks 2.1 and 3.1 and Lemma 3.1, arguing as in the beginning of the proof of Lemma 5.1, we may assume that $\overline{D}$ is a compact set in $\mathbb{S}$, ∂D consists of 2 components: a closed Jordan curve γ and one more nondegenerate component Γ, D' is a ring $R=\{z\in\mathbb{C}:0<r<|z|<1\}$, $\overline{D'}_P=\overline{R}$,

$$C(\Gamma,f)=C_r:=\{z\in\mathbb{C}:|z|=r\},\quad C(\gamma,f)=C_*:=\{z\in\mathbb{C}:|z|=1\}$$

and that f is extended to a homeomorphism of $D\cup\gamma$ onto $D'\cup C_*$.

Let us first prove that the set $L:=C(P,f)$ consists of a single point of C_r for a prime end P of the domain D associated with Γ. Note that $L\neq\varnothing$ by compactness of the set $\overline{R}$ and, moreover, $L\subseteq C_r$ by Proposition 2.1.

Let us assume that there is at least two points ζ_0 and $\zeta_*\in L$. Set $U=\{\zeta\in\mathbb{C}:|\zeta-\zeta_0|<\rho_0\}$ where $0<\rho_0<|\zeta_*-\zeta_0|$.

Let σ_k, $k=1,2,\ldots$, be a chain in the prime end P from Remark 2.1 lying on the circles $S_k:=\{p\in\mathbb{S}:h(p,p_0)=r_k\}$ where $p_0\in\Gamma$ and $r_k\to 0$ as $k\to\infty$. Let d_k be the domains associated with σ_k. Then there exist points ζ_k and ζ_k^* in the domains $d_k'=f(d_k)\subset R$ such that $|\zeta_0-\zeta_k|<\rho_0$ and $|\zeta_0-\zeta_k^*|>\rho_0$ and, moreover, $\zeta_k\to\zeta_0$ and $\zeta_k^*\to\zeta_*$ as $k\to\infty$. Let γ_k be paths joining ζ_k and ζ_k^* in d_k'. Note that by the construction $\partial U\cap\gamma_k\neq\varnothing$, $k=1,2,\ldots$.

By the condition of strong accessibility of the point ζ_0 in the ring R, there is a continuum $E\subset R$ and a number $\delta>0$ such that

$$M(\Delta(E,\gamma_k;R))\ \geqslant\ \delta \tag{6.2}$$

for all large enough k. Note that $C = f^{-1}(E)$ is a compact subset of D and hence $h(p_0, C)) > 0$. Let $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 := \min(\delta(p_0), h(p_0, C))$. Without loss of generality, we may assume that $r_k < \varepsilon_0$ and that (6.2) holds for all $k = 1, 2, \ldots$.

Let Γ_m be the family of paths joining the circle $S_0 := \{p \in \mathbb{S} : h(p, p_0) = \varepsilon_0\}$ and σ_m, $m = 1, 2, \ldots$, in the intersection of $D \setminus d_m$ and the ring $R_m := \{p \in \mathbb{S} : r_m < h(p, p_0) < \varepsilon_0\}$. Applying Proposition 4.1 and Lemma 4.1 with $T = D$, $E_1 = d_m$ and $E_2 = B_0 := \{p \in \mathbb{S} : h(p, p_0) > \varepsilon_0\}$, and taking into account the inclusion $\Delta(C, C_k, D) \subset \Delta(E_1, E_2, D) = \Delta(B_0, d_m, D)$ where $C_k = f^{-1}(\gamma_k)$, we have that $\Delta(C, C_k, D) > \Gamma_m$ for all $k \geqslant m$ because by the construction $C_k \subset d_k \subset d_m$. Thus, since f is a homeomorphism, we have also that $\Delta(E, \gamma_k, D) > f\Gamma_m$ for all $k \geqslant m$, and by the principle of minorization, see e.g. [7], p. 178, we obtain that $M(f(\Gamma_m)) \geqslant \delta$ for all $m = 1, 2, \ldots$.

On the other hand, every function

$$\xi(t) = \xi_m(t) := \psi_{p_0, r_m, \varepsilon_0}(t)/I_{p_0, \varepsilon_0}(r_m), \qquad m = 1, 2, \ldots,$$

satisfies the condition (5.6) and by Lemma 3.1 in [30] and [31]

$$M(f\Gamma_m) \ \leqslant \ \int\limits_{R_m} K_f(p) \cdot \xi_m^2(h(p, p_0))\ dh(p)\ ,$$

i.e., $M(f\Gamma_m) \to 0$ as $m \to \infty$ in view of (6.1).

The obtained contradiction disproves the assumption that the cluster set $C(P, f)$ consists of more than one point.

Thus, we have the extension $\tilde{f}$ of f to $\overline{D}_P$ such that $\tilde{f}(E_D) \subseteq E_{D'}$. In fact, $\tilde{f}(E_D) = E_{D'}$. Indeed, if $\zeta_0 \in D'$, then there is a sequence ζ_n in D' that is convergent to ζ_0. We may assume with no loss of generality that $f^{-1}(\zeta_n) \to P_0 \in \overline{D}_P$ because $\overline{D}_P$ is compact, see Remark 3.1. Hence $\zeta_0 \in E_D$ because $\zeta_0 \notin D$, see e.g. Proposition 2.5 in [25] or Proposition 13.5 in [20].

Finally, let us show that the extended mapping $\tilde{f} : \overline{D}_P \to \overline{D'}_P$ is continuous. Indeed, let $P_n \to P_0$ in $\overline{D}_P$. The statement is obvious for $P_0 \in D$. If $P_0 \in E_D$, then by the last item we are able to choose $P_n^* \in D$ such that $\rho(P_n, P_n^*) < 2^{-n}$ and $\rho'(\tilde{f}(P_n), \tilde{f}(P_n^*)) < 2^{-n}$ where ρ and ρ' are some metrics on $\overline{D}_P$ and $\overline{D'}_P$, correspondingly, see Remark 3.1. Note that by the first part of the proof $f(P_n^*) \to f(P_0)$ because $P_n^* \to P_0$. Consequently, $\tilde{f}(P_n) \to \tilde{f}(P_0)$, too. $\square$

Remark 6.1. Note that condition (6.1) holds, in particular, if

$$\int\limits_{D(p_0, \varepsilon_0)} K_f(p) \cdot \psi^2(h(p, p_0))\ dh(p) \ < \ \infty \qquad\qquad \forall\ p_0 \in \partial D \qquad (6.3)$$

where $D(p_0,\varepsilon_0)=\{p\in\mathbb{S}:h(p,p_0)<\varepsilon_0\}$ and where $\psi(t):(0,\infty)\to[0,\infty]$ is a locally integrable function such that $I_{p_0,\varepsilon_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. In other words, for the extendability of f to a continuous mapping of $\overline{D}_P$ onto $\overline{D'}_P$, it suffices for the integrals in (6.3) to be convergent for some nonnegative function $\psi(t)$ that is locally integrable on $(0,\infty)$ but that has a non-integrable singularity at zero.

7. On the homeomorphic extension to the boundary

Combining Lemma 6.1 and Theorem 5.1, we obtain the significant conclusion:

Lemma 7.1. *Under the hypothesis of Lemma 6.1, the homeomorphism $f:D\to D'$ can be extended to a homeomorphism $\tilde{f}:\overline{D}_P\to\overline{D'}_P$.*

Proof. Indeed, by Lemma 5.1 the mapping $\tilde{f}:\overline{D}_P\to\overline{D'}_P$ from Lemma 6.1 is injective and hence it has the well defined inverse mapping $\tilde{f}^{-1}:\overline{D'}_P\to\overline{D}_P$ and the latter coincides with the mapping $\tilde{g}:\overline{D'}_P\to\overline{D}_P$ from Theorem 5.1 because a limit under a metric convergence is unique. The continuity of the mappings $\tilde{g}$ and $\tilde{f}$ follows from Theorem 5.1 and Lemma 6.1, respectively. □

We assume everywhere in this section that the function K_f is extended by zero outside of D.

Theorem 7.1. *Under the hypothesis of Theorem 5.1, let in addition*

$$\int\limits_0^{\varepsilon_0}\frac{dr}{||K_f||(p_0,r)}\ =\ \infty\qquad\qquad \forall\ p_0\in\partial D,\quad \varepsilon_0<\delta(p_0)\tag{7.1}$$

where

$$||K_f||(p_0,r)\ :=\ \int\limits_{S(p_0,r)}K_f(p)\ ds_h(p)\ .\tag{7.2}$$

Then f can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Here $S(p_0,r)$ denotes the circle $\{p\in\mathbb{S}:h(p,p_0)=r\}$.

Proof. Indeed, for the functions

$$\psi_{p_0,\varepsilon_0}(t)\ :=\ \begin{cases}1/||K_f||(p_0,t), & t\in(0,\varepsilon_0),\\ 0, & t\in[\varepsilon_0,\infty),\end{cases}\tag{7.3}$$

we have by the Fubini theorem that

$$\int\limits_{R(p_0,\varepsilon,\varepsilon_0)} K_f(p)\cdot\psi^2_{p_0,\varepsilon_0}(h(p,p_0)\ dh(p)\ =\ \int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{||K_f||(p_0,r)} \tag{7.4}$$

where $R(p_0,\varepsilon,\varepsilon_0)$ denotes the ring $\{p\in\mathbb{S}:\varepsilon<h(p,p_0)<\varepsilon_0\}$ and, consequently, condition (6.1) holds by (7.1) for all $p_0\in\partial D$ and $\varepsilon_0\in(0,\varepsilon(p_0))$.

Here we have used the standard conventions in the integral theory that $a/\infty=0$ for $a\neq\infty$ and $0\cdot\infty=0$, see, e.g., Section I.3 in [33].

Thus, Theorem 7.1 follows immediately from Lemma 7.1. $\Box$

Corollary 7.1. *In particular, the conclusion of Theorem 7.1 holds if*

$$k_{p_0}(r)=O\left(\log\frac{1}{r}\right)\qquad\qquad \forall\ p_0\in\partial D \tag{7.5}$$

as $r\to 0$ where $k_{p_0}(r)$ is the average of K_f over the infinitesimal circle $S(p_0,r)$.

Choosing in (6.1) $\psi(t):=\frac{1}{t\log 1/t}$, we obtain by Lemma 7.1 the next result, see also Lemma 4.1 in [25] or Lemma 13.2 in [20].

Theorem 7.2. *Under the hypothesis of Theorem 5.1, let K_f have a dominant Q_{p_0} in a neighborhood of each point $p_0\in\partial D$ with finite mean oscillation at p_0. Then f can be extended to a homeomorphism $\tilde{f}:\overline{D}_P\to\overline{D'}_P$.*

By Corollary 4.1 in [25] or Corollary 13.3 in [20] we obtain the following.

Corollary 7.2. *In particular, the conclusion of Theorem 7.2 holds if*

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{D(p_0,\varepsilon)} K_f(p)\ dh(p)\ <\ \infty\qquad\qquad \forall\ p_0\in\partial D \tag{7.6}$$

where $D(p_0,\varepsilon)$ is the infinitesimal disk $\{p\in\mathbb{S}:\ h(p,p_0)\ <\ \varepsilon\}$.

Corollary 7.3. *The conslusion of Theorem 7.2 holds if every point $p_0\in\partial D$ is a Lebesgue point of the function K_f or its dominant Q_{p_0}.*

The next statement also follows from Lemma 7.1 under the choice $\psi(t)=1/t$.

Theorem 7.3. *Under the hypothesis of Theorem 5.1, let, for some $\varepsilon_0 > 0$,*

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} K_f(p)\ \frac{dh(p)}{h^2(p,p_0)} \ = \ o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad as \ \ \varepsilon\to 0 \quad \forall\ p_0\in\partial D \tag{7.7}$$

Then f can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Remark 7.1. Choosing in Lemma 7.1 the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$, (7.7) can be replaced by the more weak condition

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} \frac{K_f(p)\ dh(p)}{\left(h(p,p_0)\ \log\frac{1}{h(p,p_0)}\right)^2} \ = \ o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{7.8}$$

and (7.5) by the condition

$$k_{p_0}(r) \ = \ o\left(\log\frac{1}{\varepsilon}\log\,\log\frac{1}{\varepsilon}\right). \tag{7.9}$$

Of course, we could give here the whole scale of the corresponding condition of the logarithmic type using suitable functions $\psi(t)$.

8. On interconnections between integral conditions

For every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the **inverse function** Φ^{-1} can be well defined by setting

$$\Phi^{-1}(\tau) \ = \ \inf_{\Phi(t)\geq\tau} t\ . \tag{8.1}$$

As usual, here inf is equal to ∞ if the set of $t\in[0,\infty]$ such that $\Phi(t)\geq\tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 8.1. Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \ \leq \ t \qquad \forall\ t\in[0,\infty] \tag{8.2}$$

with the equality in (8.2) except intervals of constancy of the function $\Phi(t)$.

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called convex if

$$\Phi(\lambda t_1 + (1-\lambda)t_2) \ \leq \ \lambda\ \Phi(t_1) \ + \ (1-\lambda)\ \Phi(t_2)$$

for all t_1, $t_2 \in [0,\infty]$ and $\lambda\in[0,1]$.

In what follows, $\mathbb{H}(R)$ denotes the hyperbolic disk centered at the origin with the hyperbolic radius $R = \log{(1+r)/(1-r)},\ r \in (0,1)$ is its Euclidean radius:

$$\mathbb{H}(R) \ = \ \{ \ z \in \mathbb{C} : \ h(z,0) \ < \ R \ \} \, , \qquad R \in (0, \infty) \ . \tag{8.3}$$

Further we also use the notation of the **hyperbolic sine**: $\sinh t \ := (e^t - e^{-t})/2$.

The following statement is an analog of Lemma 3.1 in [28] adopted to the hyperbolic geometry in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Lemma 8.1. *Let $Q : \mathbb{H}(\varepsilon) \to [0, \infty]$, $\varepsilon \in (0,1)$, be a measurable function and $\Phi : [0,\infty] \to (0,\infty]$ be a non-decreasing convex function with a finite mean integral value $M(\varepsilon)$ of the function $\Phi \circ Q$ on $\mathbb{H}(\varepsilon)$. Then*

$$\int\limits_0^{\varepsilon} \frac{d\rho}{\rho q(\rho)} \ \geq \ \frac{1}{2} \int\limits_{\delta(\varepsilon)}^{\infty} \frac{d\tau}{\tau\, [\Phi^{-1}(\tau)]} \tag{8.4}$$

where $q(\rho)$ is the average of Q on the circle $\mathbb{S}(\rho) = \{z \in \mathbb{D} : h(z,0) = \rho\}$ and

$$\delta(\varepsilon) \ = \ \exp\left(4\sinh^2 \frac{\varepsilon}{2}\right) \cdot \frac{M(\varepsilon)}{\varepsilon^2} \ > \ \tau_0 \ := \ \Phi(0) \ > \ 0 \ . \tag{8.5}$$

Proof. Since $M(\varepsilon) < \infty$ we may assume with no loss of generality that $\Phi(t) < \infty$ for all $t \in [0, \infty)$ because in the contrary case $Q \in L^{\infty}$ and then the left-hand side in (8.4) is equal to ∞. Moreover, we may assume that $\Phi(t)$ is not constant because in the contrary case $\Phi^{-1}(\tau) \equiv \infty$ for all $\tau > \tau_0$ and hence the right-hand side in (8.4) is equal to 0. Note also that $\Phi(\tau)$ is (strictly) increasing, convex and continuous in the segment $[t_*, \infty]$ and

$$\Phi(t) \ \equiv \ \tau_0 \qquad \forall \ t \ \in [0, t_*] \quad \text{where} \quad t_* \ := \ \sup_{\Phi(t) = \tau_0} \ t \ . \tag{8.6}$$

Setting $H(t) := \log \Phi(t)$, we see that $H^{-1}(\eta) = \Phi^{-1}(e^{\eta})$, $\Phi^{-1}(\tau) = H^{-1}(\log\ \tau)$. Thus, we obtain that

$$q(\rho) = H^{-1}\left(\log \frac{h(\rho)}{\rho^2}\right) = H^{-1}\left(2\log\frac{1}{\rho} + \log\ h(\rho)\right) \qquad \forall \ \rho \ \in R_* \tag{8.7}$$

where $h(\rho) := \ \rho^2 \Phi(q(\rho))$ and $R_* = \{ \ \rho \in (0, \varepsilon) : \ q(\rho) \ > \ t_*\}$. Then also

$$q(e^{-s}) \ = \ H^{-1}\left(2s \ + \ \log\ h(e^{-s})\right) \qquad \forall \ s \ \in S_* \tag{8.8}$$

where $S_* = \{s \in (\log \frac{1}{\varepsilon} , \infty) : q(e^{-s}) > t_*\}$.

Now, by the Jensen inequality, see e.g. Theorem 2.6.2 in [24], we have that

$$\int\limits_{\log \frac{1}{\varepsilon}}^{\infty} h(e^{-s})\, ds = \int\limits_0^{\varepsilon} h(\rho)\, \frac{d\rho}{\rho} = \int\limits_0^{\varepsilon} \Phi(q(\rho))\, \rho\, d\rho \tag{8.9}$$

$$\leq \int\limits_0^{\varepsilon} \left(\fint_{S(\rho)} \Phi(Q(z))\, ds_h(z) \right) \rho\, d\rho \leq 2\sinh^2 \frac{\varepsilon}{2} \cdot M(\varepsilon)$$

because $\mathbb{H}(\varepsilon)$ has the hyperbolic area $A(\varepsilon) = 4\pi \sinh^2 \frac{\varepsilon}{2}$ and $\mathbb{S}(\rho)$ has the hyperbolic length $L(\rho) = 2\pi \sinh \rho$, see e.g. Theorem 7.2.2 in [2], and, moreover, $\sinh \rho \geq \rho$ by the Taylor expansion. Then arguing by contradiction it is easy to see for the set $T := \{ s \in (\log \frac{1}{\varepsilon} , \infty) : h(e^{-s}) > M(\varepsilon) \}$ that its length

$$|T| = \int\limits_T ds \leq 2\sinh^2 \frac{\varepsilon}{2} . \tag{8.10}$$

Next, let us show for $T_* := T \cap S_*$ that

$$q\left(e^{-s}\right) \leq H^{-1}\left(2s + \log M(\varepsilon)\right) \qquad \forall\, s \in \left(\log \frac{1}{\varepsilon}, \infty\right) \setminus T_* . \tag{8.11}$$

Indeed, note that $\left(\log \frac{1}{\varepsilon}, \infty\right) \setminus T_* = \left[\left(\log \frac{1}{\varepsilon}, \infty\right) \setminus S_*\right] \cup \left[\left(\log \frac{1}{\varepsilon}, \infty\right) \setminus T\right] = \left[\left(\log \frac{1}{\varepsilon}, \infty\right) \setminus S_*\right] \cup [S_* \setminus T]$. The inequality (8.11) holds for $s \in S_* \setminus T$ by (8.8) because H^{-1} is a non-decreasing function. Note also that

$$e^{2s} M(\varepsilon) > \Phi(0) = \tau_0 \qquad \forall\, s \in \left(\log \frac{1}{\varepsilon} , \infty\right) \tag{8.12}$$

and then

$$t_* < \Phi^{-1}\left(e^{2s} M(\varepsilon)\right) = H^{-1}\left(2s + \log M(\varepsilon)\right) \qquad \forall\, s \in \left(\log \frac{1}{\varepsilon} , \infty\right) \tag{8.13}$$

Consequently, (8.11) holds for all $s \in (\log \frac{1}{\varepsilon} , \infty) \setminus S_*$, too.

Since H^{-1} is non-decreasing, we have by (8.10)–(8.11) that, for $\Delta := \log M(\varepsilon)$,

$$\int\limits_0^{\varepsilon} \frac{d\rho}{\rho q(\rho)} = \int\limits_{\log \frac{1}{\varepsilon}}^{\infty} \frac{ds}{q(e^{-s})} \geq \int\limits_{\left(\log \frac{1}{\varepsilon}, \infty\right) \setminus T_*} \frac{ds}{H^{-1}(2s + \Delta)} \tag{8.14}$$

$$\geq \int\limits_{|T_*|+\log\frac{1}{\varepsilon}}^{\infty} \frac{ds}{H^{-1}(2s+\Delta)} \geq \int\limits_{2\sinh^2\frac{\varepsilon}{2}+\log\frac{1}{\varepsilon}}^{\infty} \frac{ds}{H^{-1}(2s+\Delta)}$$

$$= \frac{1}{2} \int\limits_{4\sinh^2\frac{\varepsilon}{2}+\log\frac{M(\varepsilon)}{\varepsilon^2}}^{\infty} \frac{d\eta}{H^{-1}(\eta)}$$

and after the replacement of variables $\eta = \log \tau$, $\tau = e^{\eta}$, we come to (8.4). $\square$

Theorem 8.1. *Let $Q : \mathbb{H}(\varepsilon) \to [0,\infty]$, $\varepsilon \in (0,1)$, be a measurable function such that*

$$\int\limits_{\mathbb{H}(\varepsilon)} \Phi(Q(z))\ dh(z) \ < \ \infty \tag{8.15}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function with

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \ = \ \infty \tag{8.16}$$

for some $\delta_0 \ > \ \tau_0 \ := \ \Phi(0)$. Then

$$\int\limits_{0}^{\varepsilon} \frac{d\rho}{\rho q(\rho)} \ = \ \infty\,, \tag{8.17}$$

where $q(\rho)$ is the average of Q on the hyperbolic circle $h(z,0) = \rho$.

Proof. If $\Phi(0) \neq 0$, then Theorem 8.1 directly follows from Lemma 8.1 because Φ^{-1} is strictly increasing on the interval (τ_0,∞) and $\Phi^{-1}(\delta_0) > 0$. In the case $\Phi(0) = 0$, let us fix a number $\delta \in (0,\delta_0)$ and set $\Phi_*(t) = \Phi(t)$, if $\Phi(t) > \delta$, and $\Phi_*(t) = \delta$, if $\Phi(t) \leq \delta$. Then by (8.15) we have that $\int\limits_{\mathbb{H}(\varepsilon)} \Phi_*(Q(z))\ dh(z) < \infty$ because $|\Phi_*(t) - \Phi(t)| \leq \delta$ and the measure of $\mathbb{H}(\varepsilon)$ is finite. Moreover, $\Phi_*^{-1}(\tau) = \Phi^{-1}(\tau)$ for $\tau \geq \delta$ and then by (8.16) $\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi_*^{-1}(\tau)} = \infty$. Thus, (8.17) holds again by Lemma 8.1. $\square$

Remark 8.2. Note that condition (8.16) implies that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \ = \ \infty \qquad \forall\ \delta \ \in \ [0,\infty)\,. \tag{8.18}$$

but relation (8.18) for some $\delta \in [0,\infty)$, generally speaking, does not imply (8.16). Indeed, (8.16) evidently implies (8.18) for $\delta \in [0,\delta_0)$, and, for $\delta \in (\delta_0,\infty)$, we have that

$$0 \leq \int\limits_{\delta_0}^{\delta} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \leq \frac{1}{\Phi^{-1}(\delta_0)} \log \frac{\delta}{\delta_0} < \infty \tag{8.19}$$

because the function Φ^{-1} is non-decreasing and $\Phi^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi^{-1}(\tau) \equiv 0$ for all $\tau \in [0,\tau_0]$, $\tau_0 = \Phi(0)$, and hence (8.18) for $\delta \in [0,\tau_0)$, generally speaking, does not imply (8.16). If $\tau_0 > 0$, then

$$\int\limits_{\delta}^{\tau_0} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \qquad \forall\ \delta \in [0,\tau_0) \tag{8.20}$$

However, relation (8.20) gives no information on the function Q itself and, consequently, (8.18) for $\delta < \Phi(0)$ cannot imply (8.17) at all.

9. Other criteria for homeomorphic extension in prime ends

Theorem 7.1 has a magnitude of other consequences thanking to Theorem 8.1.

Theorem 9.1. *Under the hypothesis of Theorem 5.1, let*

$$\int\limits_{D(p_0,\varepsilon_0)} \Phi_{p_0}\left(K_f(p)\right)\ dh(p) < \infty \qquad \forall\ p_0 \in \partial D \tag{9.1}$$

for $\varepsilon_0 = \varepsilon(p_0)$ and a nondecreasing convex function $\Phi_{p_0} : [0,\infty) \to [0,\infty)$ with

$$\int\limits_{\delta(p_0)}^{\infty} \frac{d\tau}{\tau\Phi_{p_0}^{-1}(\tau)} = \infty \tag{9.2}$$

for $\delta(p_0) > \Phi_{p_0}(0)$. Then f is extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Proof. Indeed, in the case of the hyperbolic Riemann surfaces, (9.1) and (9.2) imply (7.1) by Theorem 8.1 and, after this, Theorem 9.1 becomes a direct consequence of Theorem 7.1. In the more simple case of the elliptic and parabolic Riemann surfaces, we similarly can apply Theorem 3.1 in [28] for the Euclidean plane instead of Theorem 8.1. $\square$

Corollary 9.1. *In particular, the conclusion of Theorem 9.1 holds if*

$$\int\limits_{D(p_0,\varepsilon_0)} e^{\alpha_0 K_f(p)}\, dh(p) \;<\; \infty \qquad \forall\, p_0 \in \partial D \tag{9.3}$$

for some $\varepsilon_0 = \varepsilon(p_0) > 0$ *and* $\alpha_0 = \alpha(p_0) > 0$.

Remark 9.1. Note that by Theorem 5.1 and Remark 5.1 in [16] condition (9.2) is not only sufficient but also necessary for a continuous extendibility to the boundary of all mappings f with the integral restriction (9.1).

Note also that by Theorem 2.1 in [28], see also Proposition 2.3 in [27], (9.2) is equivalent to every of the conditions from the following series:

$$\int\limits_{\delta(p_0)}^{\infty} H'_{p_0}(t)\ \frac{dt}{t} = \infty\ , \qquad \delta(p_0) > 0\ , \tag{9.4}$$

$$\int\limits_{\delta(p_0)}^{\infty} \frac{dH_{p_0}(t)}{t} = \infty\ , \qquad \delta(p_0) > 0\ , \tag{9.5}$$

$$\int\limits_{\delta(p_0)}^{\infty} H_{p_0}(t)\ \frac{dt}{t^2} = \infty\ , \qquad \delta(p_0) > 0\ , \tag{9.6}$$

$$\int\limits_{0}^{\Delta(p_0)} H_{p_0}\left(\frac{1}{t}\right)\ dt = \infty\ , \qquad \Delta(p_0) > 0\ , \tag{9.7}$$

$$\int\limits_{\delta_*(p_0)}^{\infty} \frac{d\eta}{H_{p_0}^{-1}(\eta)} = \infty\ , \qquad \delta_*(p_0) > H_{p_0}(0)\ , \tag{9.8}$$

where

$$H_{p_0}(t) = \log \Phi_{p_0}(t)\ . \tag{9.9}$$

Here the integral in (9.5) is understood as the Lebesgue–Stieltjes integral and the integrals in (9.4) and (9.6)–(9.8) as the ordinary Lebesgue integrals.

It is necessary to give one more explanation. From the right hand sides in the conditions (9.4)–(9.8) we have in mind $+\infty$. If $\Phi_{p_0}(t) = 0$ for $t \in [0, t_*(p_0)]$, then $H_{p_0}(t) = -\infty$ for $t \in [0, t_*(p_0)]$ and we complete the definition $H'_{p_0}(t) = 0$ for $t \in [0, t_*(p_0)]$. Note, the conditions (9.5) and (9.6) exclude that $t_*(p_0)$ belongs to the interval of integrability because in the contrary case the left hand sides in (9.5) and (9.6) are either equal to $-\infty$ or indeterminate. Hence we may assume in (9.4)–(9.7) that $\delta(p_0) > t_0$, correspondingly, $\Delta(p_0) < 1/t(p_0)$ where $t(p_0) := \sup\limits_{\Phi_{p_0}(t)=0} t$, set $t(p_0) = 0$ if $\Phi_{p_0}(0) > 0$.

The most interesting among the above conditions is (9.6), i.e. the condition:

$$\int\limits_{\delta(p_0)}^{\infty} \log \Phi_{p_0}(t)\ \frac{dt}{t^2} = +\infty \qquad \text{for some} \quad \delta(p_0) > 0 \,. \tag{9.10}$$

Finally, it is necessary to note the restriction on nondegeneracy of boundary components of domains in Theorem 5.1 as well as in all other theorems is not essential because this simplest case is included in our previous papers [30, 31].

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