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An operator approach to indefinite Stieltjes moment problem

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(*Presented by M. M. Malamud*)

Abstract. A function f meromorphic on $\mathbb{C}\backslash\mathbb{R}$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{Z}_+$), if f is symmetric with respect to $\mathbb{R}$ and the kernel $\mathsf{N}_\omega(z) := \frac{f(z)-\overline{f(\omega)}}{z-\overline{\omega}}$ has κ negative squares on $\mathbb{C}_+$. The generalized Stieltjes class $\mathbf{N}_\kappa^k$ ($\kappa, k \in \mathbb{Z}_+$) is defined as the set of functions $f \in \mathbf{N}_\kappa$, such that $zf \in \mathbf{N}_k$. The full indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ consists in the following: Given $\kappa, k \in \mathbb{Z}_+$, and a sequence $\mathbf{s} = \{s_i\}_{i=0}^\infty$ of real numbers, describe the set of functions $f \in \mathbf{N}_\kappa^k$, which satisfy the asymptotic expansion

$$f(z) = -\frac{s_0}{z} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right) \quad (z = -y \in \mathbb{R}_-,\ y\uparrow\infty)$$

for all n big enough. We associate to this expansion a special continued fraction, so-called generalized Stieltjes fraction, a three-term difference equations, generalized Stieltjes polynomials and a generalized Jacobi matrix $\mathfrak{J}_{[0,N]}$.

In the present paper we solve the indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ within the M.G. Krein theory of u-resolvent matrices applied to a Pontryagin space symmetric operator $A_{[0,N]}$ generated by $\mathfrak{J}_{[0,N]}$. The u-resolvent matrices of the operator $A_{[0,N]}$ are calculated in terms of generalized Stieltjes polynomials using the boundary triple's technique. Criterions for the problem $MP_\kappa^k(\mathbf{s})$ to be solvable and indeterminate are found. Explicit formulae for Pade approximants for generalized Stieltjes fraction in terms of generalized Stieltjes polynomials are also presented.

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for all n big enough. We associate to this expansion a special continued fraction, so-called generalized Stieltjes fraction, a three-term difference equations, generalized Stieltjes polynomials and a generalized Jacobi matrix $\mathfrak{J}_{[0,N]}$.

In the present paper we solve the indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ within the M.G. Krein theory of u-resolvent matrices applied to a Pontryagin space symmetric operator $A_{[0,N]}$ generated by $\mathfrak{J}_{[0,N]}$. The u-resolvent matrices of the operator $A_{[0,N]}$ are calculated in terms of generalized Stieltjes polynomials using the boundary triple's technique. Criterions for the problem $MP_\kappa^k(\mathbf{s})$ to be solvable and indeterminate are found. Explicit formulae for Pade approximants for generalized Stieltjes fraction in terms of generalized Stieltjes polynomials are also presented.

1. Introduction

The classical Stieltjes moment problem solved in [48] consists in the following: given a sequence of real numbers s_i ($i \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) find a positive measure σ with a support on $\mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+} t^i d\sigma(t) = s_i \qquad (i \in \mathbb{Z}_+). \tag{1.1}$$

It follows easily from (1.1) that the inequalities

$$S_n := (s_{i+j})_{i,j=0}^{n-1} \geq 0, \quad S_n^+ := (s_{i+j+1})_{i,j=0}^{n-1} \geq 0 \quad (n \in \mathbb{Z}_+) \tag{1.2}$$

are necessary for solvability of the moment problem (1.1). If the matrices S_n are nondegenerate for all $n \in \mathbb{Z}_+$, then the inequalities (1.2) are also sufficient for solvability of the moment problem (1.4), see [1, Appendix]. Let

$$D_n := \det\, S_n, \quad D_n^+ := \det\, S_n^+ \quad (n \in \mathbb{Z}_+).$$

In the pioneering paper [48] by T. Stieltjes a continued fraction

$$\cfrac{1}{-zm_1 + \cfrac{1}{l_1 + \dots \cfrac{1}{-zm_n + \cfrac{1}{l_n + \dots}}}} \tag{1.3}$$

$$m_n := \frac{D_n^2}{D_n^+ D_{n-1}^+}, \quad l_n := \frac{(D_{n-1}^+)^2}{D_n D_n^+} \quad (n \in \mathbb{Z}_+).$$

was associated with the sequence of moments $\{s_i\}_{i=0}^{\infty}$. The moment problem (1.1) is called determinate (indeterminate), if it has a unique (infinitely many) solutions. As was shown in [48] the moment problem (1.1) is indeterminate, if and only if

$$M := \sum_{i=1}^{\infty} m_i < \infty, \quad \text{and} \quad L := \sum_{i=1}^{\infty} l_i < \infty.$$

Although in [48] no mechanical interpretation for the fraction (1.3) was given, it was shown in [28] that solutions of the problem (1.1) can be interpreted as spectral functions of the so-called "Stieltjes strings", i.e. massless threads with countable sets of point masses. The truncated Stieltjes moment problem, i.e. the problem (1.1) with a finite set of data $\{s_i\}_{i=0}^{2n}$ was studied in [35, 41]. Matrix version of the Stieltjes moment problem was studied in [24].

For every measure $d\sigma$ on $\mathbb{R}_+$ the associated function

$$f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t-z} \qquad z \in \mathbb{C}\backslash\mathbb{R}_+$$

belongs to the class $\mathbf{N}$ of functions holomorphic on $\mathbb{C}\backslash\mathbb{R}$ with nonnegative imaginary part in $\mathbb{C}_+ := \{z : \operatorname{Im} z > 0\}$ and such that $f(\overline{z}) = \overline{f(z)}$ for $z \in \mathbb{C}_+$. Moreover, f belongs to the Stieltjes class $\mathbf{S}$ of functions $f \in \mathbf{N}$, which admit holomorphic and nonnegative continuation to $\mathbb{R}_-$. By M.G. Krein's criterion [33]

$$f \in \mathbf{S} \Longleftrightarrow f \in \mathbf{N} \quad \text{and} \quad zf \in \mathbf{N}.$$

Notice, that by the Hamburger–Nevanlinna theorem [1] a measure σ is a solution of the problem (1.1) if and only if the associated function f

satisfies the condition

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right) \quad \text{as } z\widehat{\rightarrow}\infty \tag{1.4}$$

for every $n \in \mathbb{N}$. The notation $z\widehat{\rightarrow}\infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

Indefinite version of the class $\mathbf{N}$ was introduced in [36].

Definition 1.1. [36]*A function f meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if the kernel $\mathsf{N}_\omega(z) := \frac{f(z)-\overline{f(\omega)}}{z-\overline{\omega}}$ has κ negative squares on $\mathbb{C}_+$, i.e. if for every set of $z_j \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($z_i \neq \overline{z}_j$, $i,j = 1,\ldots,n$) the form*

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j}\xi_i\overline{\xi}_j, \quad \xi_j \in \mathbb{C}$$

has at most κ and for some choice of z_j ($j = 1,\ldots,n$) exactly κ negative squares and

$$f(\overline{z}) = \overline{f(z)} \quad \text{for all } z \in \mathbb{C}_+ \cap \mathfrak{h}_f. \tag{1.5}$$

The generalized Stieltjes class $\mathbf{N}_\kappa^+$ was defined in [37] as the class of functions $f \in \mathbf{N}_\kappa$, such that $zf \in \mathbf{N}$. Similarly, in [11, 12] the class $\mathbf{N}_\kappa^k$ ($\kappa, k \in \mathbb{N}$) was introduced as the set of functions $f \in \mathbf{N}_\kappa$, such that $zf \in \mathbf{N}_k$, see also [22], where the class $\mathbf{N}_0^k$ was studied. Clearly, $\mathbf{N}_0^0 = \mathbf{S}$ and $\mathbf{N}_\kappa^0 = \mathbf{N}_\kappa^+$. The classes $\mathbf{S}^k := \mathbf{N}_0^k$ were introduced in [19, 22].

In the present paper we consider the following problems.

Truncated indefinite moment problem $MP_\kappa(\mathbf{s},\ell)$. Given $\ell,\kappa \in \mathbb{Z}_+$, and a finite sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ of real numbers, describe the set $\mathcal{M}_\kappa(\mathbf{s},\ell)$ of functions $f \in \mathbf{N}_\kappa$, which satisfy the asymptotic expansion

$$f(z) = -\frac{s_0}{z} - \cdots - \frac{s_\ell}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right) \quad \text{as } z\widehat{\rightarrow}\infty. \tag{1.6}$$

Truncated indefinite moment problem $MP_\kappa^k(\mathbf{s},\ell)$. Given $\ell,\kappa,k \in \mathbb{Z}_+$, and a sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ of real numbers, describe the set $\mathcal{M}_\kappa^k(\mathbf{s},\ell)$ of functions $f \in \mathbf{N}_\kappa^k$, which satisfy (1.6). A truncated moment problem is called *even* or *odd* regarding to the oddness of the number $\ell + 1$ of given moments.

Full indefinite moment problem $MP_\kappa(\mathbf{s})$. Given $\kappa \in \mathbb{Z}_+$, and an infinite sequence $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$, describe the set $\mathcal{M}_\kappa(\mathbf{s})$ of functions $f \in \mathbf{N}_\kappa$, which satisfy (1.6) for all $\ell \in \mathbb{N}$.

Full indefinite moment problem $MP_\kappa^k(\mathbf{s})$. Given $\kappa, k \in \mathbb{Z}_+$, and an infinite sequence $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$, describe the set $\mathcal{M}_\kappa^k(\mathbf{s}) := \mathcal{M}_\kappa(\mathbf{s}) \cap \mathbf{N}_\kappa^k$.

Indefinite moment problems $MP_\kappa(\mathbf{s})$ and $MP_\kappa^0(\mathbf{s})$ in the classes $\mathbf{N}_\kappa$ and $\mathbf{N}_\kappa^+ := \mathbf{N}_\kappa^0$, respectively, were studied in [39, 40] by the methods of extension theory of Pontryagin space symmetric operators developed in [37, 38]. In particular, it was shown in [39] that the moment problem $MP_\kappa(\mathbf{s})$ is solvable if the number $\nu_-(S_n)$ of negative eigenvalues of S_n does not exceed κ and $S_n^+ > 0$ for all $n \in \mathbb{N}$. Further applications of the operator approach to the moment problem $MP_\kappa^k(\mathbf{s})$ were given in [13]. A reproducing kernel approach to the moment problems $MP_\kappa(\mathbf{s})$ was presented in [23]. A step-by-step algorithm of solving the moment problems $MP_\kappa(\mathbf{s})$ was elaborated in [8, 9] and [2]. Applications of the Schur algorithm to degenerate moment problem in the class $\mathbf{N}_\kappa$ were given in [16].

Denote by $\nu_-(S)$ ($\nu_+(S)$) the number of negative (positive, resp.) eigenvalues of the matrix S. Let $\mathcal{H}$ be the set of finite or infinite real sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ and let $\mathcal{H}_{\kappa,\ell}$ be the set of sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell} \in \mathcal{H}$, such that

$$\nu_-(S_n) = \kappa \quad (n = [\ell/2] + 1). \tag{1.7}$$

let $\mathcal{H}_{\kappa,\ell}^k$ be the set of $\mathbf{s} = \{s_i\}_{i=0}^{\ell} \in \mathcal{H}_{\kappa,\ell}$, such that $\{s_{i+1}\}_{i=0}^{\ell-1} \in \mathcal{H}_{k,\ell-1}$, i.e.

$$\nu_-(S_n^+) = k \quad (n = [(\ell+1)/2]). \tag{1.8}$$

For an infinite sequence $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ one says $\mathbf{s} \in \mathcal{H}_\kappa^k$ (or $\mathbf{s} \in \mathcal{H}_\kappa^k$) if (1.7) (or (1.7) and (1.8)) is fulfilled for all $n \in \mathbb{N}$.

A number $n_j \in \mathbb{N}$ is called a *normal index* of the sequence $\mathbf{s}$, if $\det S_{n_j} \neq 0$. The ordered set of normal indices

$$n_1 < n_2 < \cdots < n_N$$

of $\mathbf{s}$ is denoted by $\mathcal{N}(\mathbf{s})$. A sequence $\mathbf{s}$ is called *regular* (see [17]), if

$$D_{n_j}^+ = \det S_{n_j}^+ \neq 0 \qquad \text{for} \quad (1 \le j \le N).$$

As was shown in [9] there exists a sequence of monic polynomials

$$a_i(z) = z^{\ell_i} + a_{\ell_i-1}^{(i)} z^{\ell_i-1} + \ldots + a_1^{(i)} z + a_0^{(i)}$$

of degree $\ell_i = n_{i+1} - n_i$ and real numbers $b_i \in \mathbb{R} \setminus \{0\}$, $i \in \mathbb{N}$, such that the convergents of the continued fraction

$$\cfrac{-b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \cdots - \cfrac{b_n}{a_n(z) - \ldots}}} \tag{1.9}$$

for sufficiently large n have the asymptotic expansion (2.1) for every $\ell \in \mathbb{N}$. This fact was known to L. Kronecker [42] and then it was reinvented in [8]. The pairs (a_i, b_i) are called atoms, see [26] and the continued fraction (1.9) is called the P–fraction, [45].

Consider the three-term recurrence relation (see [25])

$$b_i y_{i-1}(z) - a_i(z) y_i(z) + y_{i+1}(z) = 0, \tag{1.10}$$

associated with the sequence of atoms (a_i, b_i), $i \in \mathbb{N}$, and define polynomials $P_i(z)$ and $Q_i(z)$ of the first and second kind of the system (1.10) subject to the initial conditions

$$P_{-1}(z) \equiv 0,\ P_0(z) \equiv 1,\ Q_{-1}(z) \equiv -1,\ Q_0(z) \equiv 0. \tag{1.11}$$

Polynomials $P_i(z)$ and $Q_i(z)$ are called Lanzcos polynomials of the first and second kind. Moreover, the j-th convergent of the continued fraction (1.9) takes the form (see [26, Section 8.3.7])

$$f^{[j]}(z) = -\frac{Q_j(z)}{P_j(z)} \quad (1 \le j \le N).$$

As was shown in [9] the set $\mathcal{M}_\kappa(\mathbf{s}, 2n_N - 2)$ can be described in terms of the Lanzcos polynomials of the first and second kind.

Odd indefinite Stieltjes moment problems $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ for regular sequences $\mathbf{s}$ were studied in [18]. For this problem one step of the Schur algorithm is splited into two intermediate steps and this leads to the expansion of $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n_N - 2)$ into a generalized Stieltjes continued fraction

$$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1 + \cdots + \cfrac{1}{-zm_N(z) + \cfrac{1}{l_N + \tau(z)}}}}, \tag{1.12}$$

where m_j are real polynomials, $l_j \in \mathbb{R}\backslash\{0\}$ and τ is a parameter function from some generalized Stieltjes class $\mathbf{N}_{\kappa-\kappa_N}^{k-k_N}$, such that $\tau(z) = o(1)$ az $z\widehat{\rightarrow}\infty$. Such continued fractions were studied in [17]. Associated to the continued fraction (1.12) there is a system of difference equations (see [50, Section 1])

$$\begin{cases} y_{2j} - y_{2j-2} = l_j y_{2j-1}, \\ y_{2j+1} - y_{2j-1} = -zm_{j+1}(z) y_{2j} \end{cases} \tag{1.13}$$

Define the generalized Stieltjes polynomials $P_j^+(z)$ and $Q_j^+(z)$ of the first and second kind as solutions of the system (1.13) subject to the initial conditions

$$P_{-1}^+(z) \equiv 0,\ P_0^+(z) \equiv 1,\ Q_{-1}^+(z) \equiv 1,\ Q_0^+(z) \equiv 0.$$

The main result of [18] is the following

Theorem 1.2. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2n_N-2} \in \mathcal{H}_{\kappa,2n_N-2}^{k,reg}$, $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=1}^N$ *be the set of nomal indices,* $\kappa, k \in \mathbb{Z}_+, N \in \mathbb{N}$. *Then:*

(1) A nondegenerate odd moment problem $MP_\kappa^k(\boldsymbol{s}, 2n_N - 2)$ *is solvable, iff*

$$\kappa_N := \nu_-(S_{n_N}) \le \kappa \quad \text{and} \quad k_N := \nu_-(S_{n_N-1}^+) \le k.$$

(2) $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, 2n_N - 2)$ *iff* f *admits the representation*

$$f(z) = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N-2}^+(z)}{P_{2N-1}^+(z)\tau(z) + P_{2N-2}^+(z)}, \tag{1.14}$$

where $\tau(z) \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N}$ *and* $\frac{1}{\tau(z)} = o(z)$ *as* $z \widehat{\to} \infty$.

In what follows for every 2×2 matrix $\mathcal{W} = (w_{ij})_{i,j=1}^2$ we associate the linear-fractional transformation (LFT)

$$T_{\mathcal{W}}[\tau] := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}.$$

Denote by $\mathcal{W}_{[0,N-1]}^+(z)$ the coefficient matrix of the LFT (1.14)

$$\mathcal{W}_{[0,N-1]}^+(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N-2}^+(z) \\ P_{2N-1}^+(z) & P_{2N-2}^+(z) \end{pmatrix}.$$

Then the equality (1.14) can be rewritten in the form

$$f(z) = T_{\mathcal{W}_{[0,N-1]}^+(z)}[\tau(z)]. \tag{1.15}$$

The mvf $\mathcal{W}_{[0,N-1]}^+(z)$ admits the factorization

$$\mathcal{W}_{[0,N-1]}^+(z) = M_1(z)L_1 \ldots L_{N-1}M_N(z),$$

where the matrices $M_j(z)$ and L_j are defined by

$$M_j(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} \quad \text{and} \quad L_j = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix} \quad j = \overline{1, N}. \tag{1.16}$$

Similarly, the set of solutions of the moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$ can be described via the LFT (1.15) with the the coefficient matrix

$$\mathcal{W}_{[0,N-1]}^{++}(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N}^+(z) \\ P_{2N-1}^+(z) & P_{2N}^+(z) \end{pmatrix}$$

In the present paper we apply an operator approach to truncated and full indefinite moment problems. For this purpose we associate with the system of atoms $\{a_i, b_i\}$, $i \in \mathbb{N}$, the so-called monic generalized Jacobi matrix (GJM), see [9, 10]. This GJM $\mathfrak{J}_{[0,N-1]}$ generates a symmetric operator $A_{[0,N-1]}$ with deficiency indices $(1,1)$ acting in an indefinite inner product space $\mathfrak{H}_{[0,N-1]}$. Then we invoke to the theory of boundary triples developed in [20, 30, 32] and to the M.G. Krein theory of resolvent matrices elaborated in [21, 34] and [15], see Sections 2.2 and 4.1 in the present paper. We show that the matrices $\mathcal{W}_{[0,N-1]}^{+}(z)$ and $\mathcal{W}_{[0,N-1]}^{++}(z)$ are u-resolvent matrices of the operator $A_{[0,N-1]}$ corresponding to some boundary triples which are found explicitly in Section 4.2.

In the case of an infinite sequence $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$, it is shown that the coefficient matrix $\mathcal{W}_{[0,j]}^{+}(z)$ admits the factorization

$$\mathcal{W}_{[0,j]}^{+}(z) = \mathcal{W}_{[0,N-1]}^{++}(z)\mathcal{W}_{[N,j]}^{+}(z),$$

where $\mathcal{W}_{[N,j]}^{+}(z)$ is the resolvent matrix of some classical Stieltjes moment problem. This fact allows to derive some facts for indefinite moment problem from the classical ones. In particular, it is shown that an indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ is solvable if and only if

$$\nu_-(S_{n_j}) \le \kappa \quad \text{and} \quad \nu_-(S_{n_j}^+) \le k \quad \text{for all} \quad j \in \mathbb{N};$$

and $MP_\kappa^k(\mathbf{s})$ is indeterminate if and only if

$$M := \sum_{j=1}^{\infty} m_j(0) < \infty \quad \text{and} \quad L := \sum_{j=1}^{\infty} l_j < \infty. \tag{1.17}$$

If (1.17) is in force, then the mvf's $\mathcal{W}_{[0,j]}^{+}(z)$ are proved to converge to an entire mvf $\mathcal{W}_{[0,\infty]}^{+}(z)$ of order 1/2, which turns out to be an u-resolvent matrix of minimal operator $A_{\min}$ generated by the GJM $\mathfrak{J}_{[0,\infty)}$ in a Pontryagin space $\mathfrak{H}_{[0,\infty)}$. The LFT (1.15) generated by the mvf $\mathcal{W}_{[0,\infty)}^{+}(z)$ provides a description of the set $\mathcal{M}_\kappa^k(\mathbf{s})$ in Theorem 5.2.

In Section 6 it is shown that the convergents of the continued fraction (1.12) coincides with the diagonal and sub-diagonal Pade approximants of the corresponding formal power series.

2. Preliminaries

2.1. Generalized Nevanlinna functions

Recall that a function f meromorphic on $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ is said to be from the class $\mathbf{N}_\kappa$ $(\kappa \in \mathbb{Z}_+)$, if the kernel $\mathsf{N}_\omega(z)$ has κ negative squares on $\mathbb{C}_+ \cap \mathfrak{h}_f$ and (1.5) holds. In particular, the class $\mathbf{N} := \mathbf{N}_0$ consists of functions f holomorphic on $\mathbb{C}_+$, which map $\mathbb{C}_+$ into itself.

Notice that every real polynomial $m(z) = m_\nu z^\nu + \ldots + m_1 z + m_0$ of degree ν belongs to a class $\mathbf{N}_\kappa$, where the index $\kappa = \kappa_-(m)$ can be calculated by (see [37, Lemma 3.5])

$$\kappa_-(m) = \begin{cases} \left[\frac{\nu+1}{2}\right], & \text{if } m_\nu < 0 \text{ and } \nu \text{ is odd;} \\ \left[\frac{\nu}{2}\right], & \text{otherwise.} \end{cases}$$

Let $\mathbf{s} = \{s_i\}_{i=0}^{2n}$ be a sequence of real numbers and let $S_n := (s_{i+j})_{i,j=0}^{n-1}$ be a Hankel matrix of order n and denote $D_n := \det S_n$ $(n \in \mathbb{Z}_+)$.

A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_{\kappa,-\ell}$ if f admits the following asymptotic expansion at ∞

$$f(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} \cdots - \frac{s_\ell}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right) \quad \text{as} \quad z\widehat{\to}\infty. \tag{2.1}$$

The notation $z\widehat{\to}\infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$. Let us also set

$$\mathbf{N}_{\kappa,-\infty} := \bigcap_{\ell \geq 0} \mathbf{N}_{\kappa,-\ell}.$$

Denote by $\nu_-(S)$ $(\nu_+(S))$ the number of negative (positive, resp.) eigenvalues of the matrix S. Let $\mathcal{H}$ be the set of finite real sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ and let $\mathcal{H}_{\kappa,\ell}$ be the set of sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell} \in \mathcal{H}$, such that

$$\nu_-(S_n) = \kappa \quad (n = [\ell/2] + 1).$$

The index $\nu_-(S_n)$ for a Hankel matrix S_n can be calculated by the Frobenius rule (see [27, Theorem X.24]). In particular, if all the determinants $D_n := \det S_n$ $(n \in \mathbb{Z}_+)$ do not vanish, then $\nu_-(S_n)$ coincides with the number of sign alternations in the sequence

$$D_0 := 1, \quad D_1, \quad D_2, \ldots, \quad D_n.$$

Proposition 2.1. [37] *Let $f \in \mathbf{N}_\kappa$, $\kappa \in \mathbb{N}$. Then:*

(1) $f \in \mathbf{N}_\kappa \Longleftrightarrow -\frac{1}{f} \in \mathbf{N}_\kappa$.

(2) *If* $f \in \mathbf{N}_{\kappa,-\ell}$, *then there exists* $\kappa' \le \kappa$, *such that* $\{s_i\}_{i=0}^{\ell} \in \mathcal{H}_{\kappa',\ell}$.

Denote by $\mathcal{H}_{\kappa,\ell}^{k}$ the class of real sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell} \in \mathcal{H}_{\kappa,\ell}$, such that $\{s_{i+1}\}_{i=0}^{\ell-1} \in \mathcal{H}_{k,\ell-1}$, i.e. (1.8) holds. The following proposition is an easy corollary of Proposition 2.1, see also [11].

Proposition 2.2. *The following equivalences hold:*

(1) $f \in \mathbf{N}_{\kappa}^{k} \iff -\frac{1}{f} \in \mathbf{N}_{\kappa}^{-k}$.

(2) $f \in \mathbf{N}_{\kappa}^{k} \iff zf(z) \in \mathbf{N}_{k}^{-\kappa}$, *in particular,* $f \in \mathbf{N}_{\kappa}^{+} \iff zf(z) \in \mathbf{S}^{-\kappa}$.

If, in addition, $f \in \mathbf{N}_{\kappa}^{k}$ *has an asymptotic expansion* (1.6) *then* $\{s_i\}_{i=0}^{\ell} \in \mathcal{H}_{\kappa',\ell}^{k'}$ *with* $\kappa' \le \kappa$, $k' \le k$.

2.2. Pontryagin spaces, symmetric operators, boundary triples

Let $\mathfrak{H}$ be a Hilbert space and let G be a selfadjoint operator in $\mathfrak{H}$ such that $0 \in \rho(G)$ and the total multiplicity of negative eigenvalues of G is equal κ. The space $\mathfrak{H}$ with the indefinite inner product

$$[f,g] := (Gf,g) \quad f,g \in \mathfrak{H}$$

is called the Pontryagin space with negative index κ and is denoted by $(\mathfrak{H}, [\cdot,\cdot])$. A closed linear operator A in $(\mathfrak{H}, [\cdot,\cdot])$ is called *symmetric* in $(\mathfrak{H}, [\cdot,\cdot])$, if

$$[Af,g] = [f,Ag] \quad \text{for all} \quad f,g \in \operatorname{dom}(A).$$

A linear subspace $T \subset \mathfrak{H}^2$ is called a *linear relation* T in $\mathfrak{H}$, see [4]. In particular, the graph of the operator A in $(\mathfrak{H}, [\cdot,\cdot])$ is a linear relation in $\mathfrak{H}$. Identifying the operator A with its graph we will consider the set of linear operators as a subset of the set of linear relations in $\mathfrak{H}$. If the operator A is non-densely defined in $\mathfrak{H}$, then its adjoint $A^{[*]}$ can be defined as a linear relation in $\mathfrak{H}$ by the equality

$$A^{[*]} = \{\{g,g'\} \in \mathfrak{H}^2 : [Af,g] = [f,g'] \text{ for all } f \in \operatorname{dom} A\}.$$

An approach to extension theory of symmetric operators based on the notion of “abstract boundary conditions”, was proposed by Calkin [7], and later on it was developed independently in [30,32]. Recall the definition of the boundary triple from [32] (see also [20,21,46] for the present notations).

Definition 2.3. *A collection* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *consisting of a Hilbert space* $\mathcal{H}$ *and two linear mappings* Γ_0 *and* Γ_1 *from* $A^{[*]}$ *to* $\mathcal{H}$*, is said to be a boundary triple for* $A^{[*]}$ *if:*

(i) the abstract Green's identity

$$[f', g] - [f, g'] = \Gamma_1 \widehat{f}\,\overline{\Gamma_0 \widehat{g}} - \Gamma_0 \widehat{f}\,\overline{\Gamma_1 \widehat{g}}$$

holds for all $\widehat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}, \widehat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in A^{[*]}$*;*

(ii) the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : A^{[*]} \to \mathbb{C}^2$ *is surjective.*

Associated with a boundary triple Π there are two self-adjoint extensions of the operator A given by

$$A_0 = \ker \Gamma_0 \quad \text{and} \quad A_1 = \ker \Gamma_1.$$

Let $\mathfrak{N}_z := \ker(A^{[*]} - zI)$ and let us set

$$\widehat{\mathfrak{N}}_z := \left\{ \left(\begin{array}{c} f_z \\ z f_z \end{array} \right), \quad f_z \in \mathfrak{N}_z \right\} \subset A^{[*]}. \tag{2.2}$$

A symmetric operator A in $(\mathfrak{H}, [\cdot, \cdot])$ is called *simple*, if

$$\operatorname{span} \{\mathfrak{N}_z : z \neq \bar{z}\} = \mathfrak{H}. \tag{2.3}$$

Definition 2.4. *The abstract Weyl function of* A*, corresponding to the boundary triple* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is defined by*

$$M(z)\Gamma_0 \widehat{f}_z = \Gamma_1 \widehat{f}_z, \quad \widehat{f}_z \in \widehat{\mathfrak{N}}_z, \quad z \in \rho(A_0),$$

where $\widehat{\mathfrak{N}}_z$ *is defined by* (2.2)*.*

The notion of the Weyl function for a Hilbert space symmetric operator was introduced in [19–21,46] both for densely and nondensely defined operators. The definition of the Weyl function for a nondensely defined Pontryagin space symmetric operator was given in [15]. As was shown in [15] the Weyl function $M(z)$ of a symmetric operator A acting in a Pontryagin space $\mathfrak{H}$ with negative index κ, is well defined and belongs to the class $\mathbf{N}_{\kappa'}$ with $\kappa' \leq \kappa$.

A boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ allows to give a description of all self-adjoint extensions of A, which are disjoint with A_0 in the form

$$A_b = \ker(\Gamma_1 - b\Gamma_0), \quad b \in \mathbb{R}.$$

The resolvent set $\rho(A_b)$ of the linear relation A_b is defined as the set of points $z \in \mathbb{C}$, such that

$$\operatorname{ran}(A_b - zI) = \mathfrak{H} \quad \text{and} \quad \ker(A_b - zI) = \{0\}.$$

For a simple symmetric operator A the resolvent set of its extension A_b is characterized by the following statement

Proposition 2.5. [15]*Let A be a simple symmetric operator in $(\mathfrak{H}, [\cdot,\cdot])$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^{[*]}$ and $z \in \rho(A_0)$, $b \in \mathbb{R}$. Then*

$$z \in \rho(A_b) \Longleftrightarrow M(z) - b \neq 0.$$

2.3. Monic generalized Jacobi matrix

Let $a(z) = z^\ell + a_{l-1}z^{\ell-1} + \cdots + a_0$ be a monic real polynomial of degree ℓ, and let E_a and C_a be $\ell \times \ell$ matrices

$$E_a = \begin{pmatrix} a_1 & \cdots & a_{\ell-1} & 1 \\ \vdots & \iddots & \iddots & \\ a_{\ell-1} & \iddots & & \\ 1 & & & 0 \end{pmatrix}, C_a = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{\ell-1} \end{pmatrix}. \tag{2.4}$$

It follows from the equalities

$$E_aC_a = C_a^*E_a \tag{2.5}$$

(see [29, Chapter 12]) that the matrix E_aC_a is symmetric in the standard scalar product in $\mathbb{C}^\ell$.

Let us associate with the system of atoms $\{a_i, b_i\}$, $i \in \mathbb{N}$, the so-called monic generalized Jacobi matrix (GJM) (see [9, 10])

$$\mathfrak{J} = \begin{pmatrix} C_{a_0} & D_0 & & \\ B_1 & C_{a_1} & D_1 & \\ & B_2 & C_{a_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{2.6}$$

where the diagonal entries are companion matrices associated with the polynomials $a_i(z)$ (see [43]) and D_i and B_{i+1} are $\ell_i \times \ell_{i+1}$ and $\ell_{i+1} \times \ell_i$ matrices, respectively, determined by

$$D_i = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad B_{i+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_{i+1} & 0 & \cdots & 0 \end{pmatrix}, \quad i \in \mathbb{Z}_+. \tag{2.7}$$

The matrix $\mathfrak{J}$ defined by (2.6)– (2.7) is called a GJM associated with the system of atoms $\{a_i, b_i\}$, $i \in \mathbb{N}$.The shortened GJM $\mathfrak{J}_{[i,j]}$ is defined by

$$\mathfrak{J}_{[i,j]} = \begin{pmatrix} C_{a_i} & D_i & & \\ B_{i+1} & C_{a_{i+1}} & \ddots & \\ & \ddots & \ddots & D_{j-1} \\ & & B_j & C_{a_j} \end{pmatrix}, \quad i \leq j \text{ and } i, j \in \mathbb{Z}_+. \tag{2.8}$$

Let $P_i(z)$ and $Q_i(z)$ be the Lanzcos polynomials of the first and second kind determined by (1.10) and (1.11). The Lanzcos polynomials satisfy the following generalized Liouville–Ostrogradskii formula

$$Q_{i+1}(z)P_i(z) - Q_i(z)P_{i+1}(z) = \widetilde{b}_i. \tag{2.9}$$

Let $\mathfrak{H}_{[0,N]}$ be the indefinite inner product space of sequences from $\mathbb{C}^{n_{N+1}}$ endowed with the indefinite inner product

$$[x, y]_{[0,N]} = (G_{[0,N]}x, y), \;\; G_{[0,N]} = \operatorname{diag}(\widetilde{b}_0 E_0^{-1}, \widetilde{b}_1 E_1^{-1}, \dots, \widetilde{b}_N E_N^{-1}).$$

where

$$\widetilde{b}_0 = b_0 \quad \text{and} \quad \widetilde{b}_i = b_0 b_1 \dots b_i, \quad i \in \mathbb{N}.$$

It follows from the equalities (2.5) that the matrix $G_{[0,N]}\mathfrak{J}^T_{[0,N]}$ is self-adjoint in the standard scalar product in $\mathbb{C}^{n_{N+1}}$,

$$G_{[0,N]}\mathfrak{J}^T_{[0,N]} = \mathfrak{J}_{[0,N]}G_{[0,N]} \tag{2.10}$$

and hence the matrix $\mathfrak{J}^T_{[0,N]}$ generates a self-adjoint operator in $\mathfrak{H}_{[0,N]}$.

Let us set

$$\pi_{[0,N]}(z) = G^{-1}_{[0,N]} \begin{bmatrix} \pi_0(z) \\ \vdots \\ \pi_N(z) \end{bmatrix} \text{ and } \pi_i(z) = \begin{bmatrix} P_i(z) \\ zP_i(z) \\ \vdots \\ z^{l_i - 1}P_i(z) \end{bmatrix}, \; (i = \overline{0, N}). \tag{2.11}$$

Alongside with $\pi_{[0,N]}(z)$ let us define the vector-function

$$\xi_{[0,N]}(z) = G^{-1}_{[0,N]} \begin{bmatrix} \xi_0(z) \\ \vdots \\ \xi_N(z) \end{bmatrix} \text{ and } \xi_i(z) = \begin{bmatrix} Q_i(z) \\ zQ_i(z) \\ \vdots \\ z^{l_i - 1}Q_i(z) \end{bmatrix}, \; (i = \overline{0, N}).$$

Lemma 2.6. *For every $N \in \mathbb{N}$ the following equalities hold*

$$(\mathfrak{J}^\top_{[0,N]} - zI_{n_{N+1}})\pi_{[0,N]}(z) = -(\widetilde{b}_N)^{-1}P_{N+1}(z)e_{n_N}, \tag{2.12}$$

$$(\mathfrak{J}^\top_{[0,N]} - zI_{n_{N+1}})\xi_{[0,N]}(z) = e_0 - (\widetilde{b}_N)^{-1}Q_{N+1}(z)e_{n_N}. \tag{2.13}$$

2.4. A system of difference equations and generalized Stieltjes polynomials

In the present paper we will consider so-called regular sequences $\mathbf{s}$ from $\mathcal{H}_\kappa^k$ introduced in [17].

Definition 2.7. *The sequence $\mathbf{s} \in \mathcal{H}_\kappa^k$ is said to belong to the class $\mathcal{H}_\kappa^{k,reg}$ if one of the following equivalent conditions holds*

(1) $P_i(0) \neq 0$ *for every* $i \leq N$*;*

(2) $D^+_{n_i-1} \neq 0$ *for every* $i \leq N$*;*

(3) $D^+_{n_i} \neq 0$ *for every* $i \leq N$.

For a sequence $\mathbf{s}$ from $\mathcal{H}_\kappa^{k,reg}$ the following theorem holds, see [17].

Theorem 2.8. [17] *Let* $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$. *Then there exists a sequence of polynomials* $m_j(z)$ *and numbers* l_j *such that the* $2j-$*th convergent* $\frac{u_{2j}}{v_{2j}}$ *of the continued fraction*

$$\cfrac{1}{-zm_1(z)+\cfrac{1}{l_1+\dots\cfrac{1}{-zm_j(z)+\cfrac{1}{l_j+\dots}}}}. \tag{2.14}$$

coincides with the $j-$*th convergent of the* $P-$*fraction* (1.9) *corresponding to the sequence* $\mathbf{s}$. *The parameters* l_j *and* $m_j(z)$ *of the generalized* $S-$*fraction* (2.14) *are connected with the parameters* b_j *and* $a_j(z)$ *of the* $P-$*fraction* (1.9) *by the equalities*

$$b_0 = \frac{1}{d_1}, \quad a_0(z) = \frac{1}{d_1}\left(zm_1(z) - \frac{1}{l_1}\right), \tag{2.15}$$

$$b_j = \frac{1}{l_j^2 d_j d_{j+1}}, \quad a_j(z) = \frac{1}{d_{j+1}}\left(zm_{j+1}(z) - \left(\frac{1}{l_j} + \frac{1}{l_{j+1}}\right)\right), \tag{2.16}$$

where d_j *is the leading coefficient of* $m_j(z)$ $(j = 1, \dots, N-1)$.

The continued fraction (2.14) will be called a generalized $S-$fraction. In the case when $m_j(z) \equiv m_j$ are constant numbers it reduces to the classical S-fraction (1.3) and the formulas (2.15), (2.16) are well known, [48]. The parameters l_j and $m_j(z)$ in (2.14) can be calculated recursively by (2.15) and (2.16). Alternatively, m_j and l_j can be represented in terms of the sequence $\mathbf{s}$ (see [17]):

$$m_j(z)=\frac{(-1)^{\nu+1}}{D_\nu^{(j-1)}}\begin{vmatrix}0 & \dots & 0 & s_{\nu-1}^{(j-1)} & s_\nu^{(j-1)}\\ \vdots & & \dots & \dots & \vdots\\ s_{\nu-1}^{(j-1)} & \dots & \dots & \dots & s_{2\nu-2}^{(j-1)}\\ 1 & z & \dots & z^{\nu-2} & z^{\nu-1}\end{vmatrix}, \tag{2.17}$$

where $D_\nu^{(j)}:=\det S_\nu^{(j)}$, $\nu=n_j-n_{j-1}$ and

$$l_j=(-1)^{\nu+1}\frac{D_\nu^{(j-1)}}{\left(D_\nu^{(j-1)}\right)^+}\quad (j=1,\dots,N-1).$$

Let us consider a system of difference equations associated with the continued fraction (2.14)

$$\begin{cases} y_{2j}-y_{2j-2} &= l_j y_{2j-1},\\ y_{2j+1}-y_{2j-1} &= -zm_{j+1}(z)y_{2j}\end{cases}. \tag{2.18}$$

If the j–th convergent of this continued fraction is denoted by $\frac{u_j}{v_j}$, then u_j, v_j can be found as solutions of the system (2.18) (see [50, Section 1]) subject to the following initial conditions

$$u_{-1}\equiv 1,\quad u_0\equiv 0;\qquad v_{-1}\equiv 0,\quad v_0\equiv 1. \tag{2.19}$$

The first two convergents of the continued fraction (2.14) take the form

$$\frac{u_1}{v_1}=\frac{1}{-zm_1(z)}=T_{M_1}[\infty],\quad \frac{u_2}{v_2}=\frac{l_1}{-zl_1m_1(z)+1}=T_{M_1L_1}[0].$$

Similarly, the $(2j-1)$-th and $(2j)$-th convergents are given by

$$\frac{u_{2j-1}}{v_{2j-1}}=T_{\mathcal{W}_{2j-1}}[\infty],\quad \frac{u_{2j}}{v_{2j}}=T_{\mathcal{W}_{2j}}[0].$$

Definition 2.9. [18] *Let $\mathbf{s}\in\mathcal{H}_{\kappa,\ell}^{k,reg}$. Define polynomials $P_i^+(z)$, $Q_i^+(z)$ by*

$$\begin{aligned} &P_{-1}^+(z)\equiv 0,\quad P_0^+(z)\equiv 1,\qquad Q_{-1}^+(z)\equiv 1,\quad Q_0^+(z)\equiv 0,\\ &P_{2i-1}^+(z)=-\tfrac{1}{\widetilde{b}_{i-1}}\begin{vmatrix}P_i(z) & P_{i-1}(z)\\ P_i(0) & P_{i-1}(0)\end{vmatrix}\quad \text{and}\quad P_{2i}^+(z)=\tfrac{P_i(z)}{P_i(0)},\\ &Q_{2i-1}^+(z)=\tfrac{1}{\widetilde{b}_{i-1}}\begin{vmatrix}Q_i(z) & Q_{i-1}(z)\\ P_i(0) & P_{i-1}(0)\end{vmatrix}\quad \text{and}\quad Q_{2i}^+(z)=-\tfrac{Q_i(z)}{P_i(0)}.\end{aligned} \tag{2.20}$$

The polynomials $P_i^+(z)$, $Q_i^+(z)$ are called Stieltjes polynomials.

As was noticed in [18] Stieltjes polynomials are solutions of the system (2.18).

Proposition 2.10. *Let $\boldsymbol{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$ and let $P_i^+(z)$ and $Q_i^+(z)$ be the generalized Stieltjes polynomials defined by* (2.20). *Then solutions $\{u_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ of the system* (2.18), (2.19) *take the form*

$$u_i = Q_i^+(z), \quad v_i = P_i^+(z) \quad (i = -1, 0, \dots, N).$$

Remark 2.11. The Stieltjes polynomials satisfy the following properties

$$P_{2i-1}^+(0) = 0, \quad P_{2i-2}^+(0) = 1 \quad \text{and} \quad Q_{2i-1}^+(0) = 1.$$

By Definition 2.9 and (2.9)

$$P_{2i-1}^+(0) = -\frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} P_i(0) & P_{i-1}(0) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} = 0 \text{ and } P_{2i-2}^+(0) = \frac{P_i(0)}{P_i(0)} = 1.$$

$$Q_{2i-1}^+(0) = \frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} Q_i(0) & Q_{i-1}(0) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} = \frac{Q_i(0)P_{i-1}(0) - Q_{i-1}(0)P_i(0)}{\widetilde{b}_{i-1}} = 1.$$

Lemma 2.12. *Let $\boldsymbol{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$ and let $P_i(z)$ and $Q_i(z)$ be polynomials of the first and second kind associated with the monic GJM $\mathfrak{J}$. Then:*

(1) The distance l_i can be calculated by

$$l_i = -\frac{Q_i(0)}{P_i(0)} + \frac{Q_{i-1}(0)}{P_{i-1}(0)}. \tag{2.21}$$

(2) For every $N \in \mathbb{N}$ the following formula holds

$$-\frac{Q_N(0)}{P_N(0)} = \sum_{i=1}^{N} l_i. \tag{2.22}$$

Proof. (1) Considering (2.18) at $z = 0$, we obtain $Q_{2i}^+(0) = l_i Q_{2i-1}^+(0) + Q_{2i-2}^+(0)$ and hence

$$l_i Q_{2i-1}^+(0) = Q_{2i}^+(0) - Q_{2i-2}^+(0).$$

By Definition 2.9 and by the generalized Liouville–Ostrogradskii formula (2.9)

$$Q_{2i-1}^+(0) = \frac{1}{\widetilde{b}_{i-1}} (Q_i(0)P_{i-1}(0) - Q_{i-1}(0)P_i(0) = 1.$$

This implies (2.21).

(2) Summing the equalities (2.21) for $i = 1, \dots, N$ one obtains (2.22). $\square$

Lemma 2.13. *Let* $s \in \mathcal{H}_{\kappa,\ell}^{k,reg}$, *let* $P_i(z)$ *be the polynomials of the first kind associated with the monic GJM, let* $P_i^+(z)$ *and* $Q_i^+(z)$ *be Stieltjes polynomials of the first and second kind defined by* (2.20), $x_i = l_1 + \cdots + l_i$, $i \in \mathbb{N}$. *Then*

$$\sum_{i=0}^{\infty} d_{i+1} = \sum_{i=0}^{\infty} |P_i^2(0)| \widetilde{b}_i^{-1} \quad \text{and} \quad \sum_{i=2}^{\infty} d_{i+1} x_i^2 = \sum_{i=2}^{\infty} |Q_i^2(0)| \widetilde{b}_i^{-1}.$$

2.5. The class $\mathcal{U}_\kappa(J)$ and linear fractional transformations

Let $\kappa \in \mathbb{N}$ and let J be the 2×2 signature matrix $J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Definition 2.14. *A* 2×2 *mvf* $\mathcal{W}(z) = (w_{i,j}(z))_{i,j=1}^2$ *that is meromorphic in* $\mathbb{C}_+$ *is said to be in the class* $\mathcal{U}_\kappa(J)$ *of generalized* J*-inner mvf's if:*

(i) the kernel

$$\mathsf{K}_\omega^{\mathcal{W}}(z) = \frac{J - \mathcal{W}(z) J \mathcal{W}(\omega)^*}{-i(z - \bar{\omega})}$$

has κ *negative squares in* $\mathfrak{H}_{\mathcal{W}}^+ \times \mathfrak{H}_{\mathcal{W}}^+$ *and*

(ii) $J - \mathcal{W}(\mu) J \mathcal{W}(\mu)^* = 0$ *for a.e.* $\mu \in \mathbb{R}$,

where $\mathfrak{H}_{\mathcal{W}}^+$ *denotes the domain of holomorphy of* $\mathcal{W}$ *in* $\mathbb{C}_+$.

The set of meromorphic mvf's which satisfy only the first assumption (i) is denoted by $\mathcal{P}_\kappa(J)$. The class $\mathcal{P}(J) := \mathcal{P}_0(J)$ was introduced and studied by M.S. Livsič [44] in connection with the theory of characteristic functions of quasi-hermitian operators, see also [49], in the case of unbounded operators. A complete factorization theory for mvf's from the class $\mathcal{P}(J)$ was developed by V.P. Potapov [47]. The subclass of J-inner mvf's $\mathcal{U}(J)$ plays an improtant role in this theory. Notice that monodromy matrices of canonical systems and resolvent matrices of many interpolation problems belong to the class $\mathcal{U}(J)$, [5]. The definition and some properties of the class $\mathcal{U}_\kappa(J)$ are contained in [3].

Consider the linear fractional transformation (LFT)

$$T_{\mathcal{W}}[\tau] = (w_{11}\tau(z) + w_{12})(w_{21}\tau(z) + w_{22})^{-1}$$

associated with the mvf $\mathcal{W}(z)$. The LFT associated with the product $\mathcal{W}_1\mathcal{W}_2$ of two mvf's $\mathcal{W}_1(z)$ and $\mathcal{W}_2(z)$ coincides with the composition $T_{\mathcal{W}_1} \circ T_{\mathcal{W}_2}$.

As is known, if $\mathcal{W}_1 \in \mathcal{U}_{\kappa_1}(J)$ and $\mathcal{W}_2 \in \mathcal{U}_{\kappa_2}(J)$ then $\mathcal{W}_1\mathcal{W}_2 \in \mathcal{U}_{\kappa'}(J)$, where $\kappa' \le \kappa_1 + \kappa_2$. In the following statement a partial case, when the preceding inequality becomes equality, is considered, see [18].

Lemma 2.15. *Let $m_i(z)$ be real polynomials and $l_i \in \mathbb{R} \setminus \{0\}$ $(i = 1, \dots, j)$, let the mvf's $M_i(z)$, L_i, $\mathcal{W}_{2j-1}(z)$ and $\mathcal{W}_{2j}(z)$ be defined by* (1.16) *and*

$$\mathcal{W}_{2j-1}(z) = M_1(z)L_1 \dots L_{j-1}M_j(z), \mathcal{W}_{2j}(z) = M_1(z)L_1 \dots M_j(z)L_j.$$

Then:

(1) $M_i \in \mathcal{U}_{\kappa_-(m_i)}(J)$ and for every meromorphic function τ, such that $\tau(z)^{-1} = o(z)$ as $z\widehat{\to}\infty$ the following equivalence holds

$$\tau \in \mathbf{N}_{\kappa'}^{k'} \Longleftrightarrow T_{M_i}[\tau] \in \mathbf{N}_{\kappa_-(zm_i)+\kappa'}^{\kappa_-(m_i)+k'}, \quad \kappa', k' \in \mathbb{N}.$$

(2) $L_i \in \mathcal{U}(J)$ and for every meromorphic function τ, such that $\tau(z) = o(1)$ as $z\widehat{\to}\infty$ the following equivalence holds

$$\tau \in \mathbf{N}_{\kappa'}^{k'} \Longleftrightarrow T_{L_i}[\tau] \in \mathbf{N}_{\kappa'}^{\kappa_-(zl_i)+k'}.$$

(3) $\mathcal{W}_{2j-1} \in \mathcal{U}_{\kappa_j}(J)$, where

$$\kappa_j = \sum_{i=1}^{j} \kappa_-(zm_i(z)), \quad k_j = \sum_{i=1}^{j} \kappa_-(zm_i) + \sum_{i=1}^{j-1} \kappa_-(zl_i) \tag{2.23}$$

and for every meromorphic function τ, such that $\tau(z)^{-1} = o(z)$ as $z\widehat{\to}\infty$ the following equivalence holds

$$\tau \in \mathbf{N}_{\kappa'}^{k'} \Longleftrightarrow T_{\mathcal{W}_{2j-1}}[\tau] \in \mathbf{N}_{\kappa_j+\kappa'}^{k_j+k'}.$$

(4) $\mathcal{W}_{2j} \in \mathcal{U}_{\kappa_j}(J)$, where

$$\kappa_j = \sum_{i=1}^{j} \kappa_-(m_i), \quad k_j^+ = \sum_{i=1}^{j} \kappa_-(zm_i) + \sum_{i=1}^{j} \kappa_-(zl_i). \tag{2.24}$$

and for every meromorphic function τ, such that $\tau(z) = o(1)$ as $z\widehat{\to}\infty$ the following equivalence holds

$$\tau \in \mathbf{N}_{\kappa'}^{k'} \Longleftrightarrow T_{\mathcal{W}_{2j}}[\tau] \in \mathbf{N}_{\kappa_j+\kappa'}^{k_j^+ + k'}.$$

3. Boundary triples for the operator $A_{[0,j]}$

3.1. General case

Let us fix $j \in \mathbb{N}$ and define the operator $A_{[0,j]}$ in the Pontryagin space $\mathfrak{H}_{[0,j]}$ as the restriction of the operator $\mathfrak{J}^T_{[0,j]}$ to the domain

$$\mathrm{dom} A_{[0,j]} = \{f \in \mathfrak{H}_{[0,j]} : [f, e_{n_j}] = 0\} \tag{3.1}$$

As was shown in [10] the adjoint linear relation $A^{[*]}_{[0,j]}$ of $A_{[0,j]}$ has the following representation

$$A^{[*]}_{[0,j]} = \left\{ \widehat{f} = \begin{bmatrix} f \\ \mathfrak{J}^T_{[0,j]} f + c_f e_{n_j} \end{bmatrix} : \begin{array}{c} f \in \mathfrak{H}_{[0,j]} \\ c_f \in \mathbb{C} \end{array} \right\}. \tag{3.2}$$

Mention some properties of the operator $A_{[0,j]}$.

Proposition 3.1. *Let the operator $A_{[0,j]}$ be defined by* (3.1). *Then:*

1. *e_0 is a generating vector for the operator $A_{[0,j]}$;*
2. *$\sigma_p(A_{[0,j]}) = \emptyset$;*
3. *$H = \mathrm{ran}(A_{[0,j]} - z) \dotplus \mathrm{span}\{e_0\}$ for all $z \in \mathbb{C}$;*
4. *the operator $A_{[0,j]}$ is simple, see* (2.3).

Proof. (1) It follows from (2.8), (2.4) and (3.1) that $e_0 \in \mathrm{dom}(A^i_{[0,j]})$ for all $i = \overline{0, n_{j+1} - 1}$ and

$$\mathfrak{H}_{[0,j]} = \mathrm{span}\,\{\widetilde{e}_i : 0 \le i \le n_{j+1} - 1\}, \quad \widetilde{e}_i = A^i_{[0,j]} e_0. \tag{3.3}$$

(2) To prove the second statement let us assume that $z \in \sigma_p(A_{[0,j]})$ and $A_{[0,j]} f = zf$. Decomposing the vector f by vectors e_{n_k} from (3.3) one obtains

$$f = \sum_{i=0}^{n_{j+1}-1} \xi_k \widetilde{e}_i.$$

If m is the largest k for which $\xi_k \neq 0$ then the equality $A_{[0,j]} f = zf$ implies

$$\sum_{i=0}^{m} \xi_i \widetilde{e}_{i+1} = \sum_{i=0}^{m} z \xi_i \widetilde{e}_i$$

and hence one obtains $\xi_m = 0$. Therefore, $\xi_k = 0$ for all $i \le n_{j+1} - 1$.

(3) is implied by (3.3) and (2.6), since

$$(A_{[0,j]} - z)\widetilde{e}_i = \widetilde{e}_{i+1} - z\widetilde{e}_i, \quad i = 0, \dots, n_{j+1} - 2.$$

$\square$

For vectors

$$\widehat{f}=\begin{bmatrix} f \\ \mathfrak{J}_{[0,j]}^T f + c_f e_{n_j}\end{bmatrix} \text{ and } \widehat{g}=\begin{bmatrix} g \\ \mathfrak{J}_{[0,j]}^T g + c_g e_{n_j}\end{bmatrix} \in A_{[0,j]}^{[*]}.$$

define Wronskian $W_j[\widehat{f},\widehat{g}]$ by

$$W_j[\widehat{f},\widehat{g}] := \begin{vmatrix} c_f & c_g \\ f_j & g_j \end{vmatrix}, \quad f_j := [f, e_{n_j}], \quad g_j := [g, e_{n_j}]. \tag{3.4}$$

Proposition 3.2. *Vectors*

$$\widehat{\pi}_{[0,j]}(z) := \begin{bmatrix} \pi_{[0,j]}(z) \\ z\pi_{[0,j]}(z)\end{bmatrix}, \quad \widehat{\xi}_{[0,j]}(z) := \begin{bmatrix} \xi_{[0,j]}(z) \\ z\xi_{[0,j]}(z) + e_0\end{bmatrix} \quad (z \in \mathbb{C})$$

belong to $A_{[0,j]}^{[*]}$ *and admit the representations*

$$\widehat{\pi}_{[0,j]}(z) = \begin{bmatrix} \pi_{[0,j]}(z) \\ \mathfrak{J}_{[0,j]}^T \pi_{[0,j]}(z) + \widetilde{b}_j^{-1} P_{j+1}(z) e_{n_j}\end{bmatrix} \tag{3.5}$$

$$\widehat{\xi}_{[0,j]}(z) = \begin{bmatrix} \xi_{[0,j]}(z) \\ \mathfrak{J}_{[0,j]}^T \xi_{[0,j]}(z) + \widetilde{b}_j^{-1} Q_{j+1}(z) e_{n_j}\end{bmatrix}. \tag{3.6}$$

Wronskians $W_j[\widehat{f},\widehat{\pi}_{[0,j]}(z)]$ *and* $W_j[\widehat{f},\widehat{\xi}_{[0,j]}(z)]$ *can be found by*

$$W_j[\widehat{f},\widehat{\pi}_{[0,j]}(z)] := \begin{vmatrix} c_f & \widetilde{b}_j^{-1} P_{j+1}(z) \\ f_j & P_j(z) \end{vmatrix}, \, W_j[\widehat{f},\widehat{\xi}_{[0,j]}(z)] := \begin{vmatrix} c_f & \widetilde{b}_j^{-1} Q_{j+1}(z) \\ f_j & Q_j(z) \end{vmatrix}, \tag{3.7}$$

and the generalized Liouville–Ostrogradskii formula (2.9) *takes the form*

$$W_j[\widehat{\xi}_{[0,j]}(z), \widehat{\pi}_{[0,j]}(z)] = 1, \quad z \in \mathbb{C}. \tag{3.8}$$

Proof. The formula (3.5) is implied by (2.8), (2.10) and the equalities

$$\begin{aligned} \mathfrak{J}_{[0,j]}^T \pi_{[0,j]}(z) + \widetilde{b}_j^{-1} P_{j+1}(z) e_{n_j} &= G_{[0,j]}^{-1} \left(\mathfrak{J}_{[0,j]} \begin{bmatrix} \pi_0(z) \\ \vdots \\ \pi_j(z) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ P_{j+1}(z) \end{bmatrix} \right) \\ &= z\pi_{[0,j]}(z). \end{aligned}$$

It follows from (3.5) that $c_g = \widetilde{b}_j^{-1} P_{j+1}(z)$ for $\widehat{g} = \widehat{\pi}_{[0,j]}(z)$. Inserting this into (3.4) yields the first formula in (3.7). The proof of (3.6) and the seecond formula in (3.7) is similar.

The formula (3.8) is implied by (2.9), (3.7), (3.5) and (3.6). $\square$

Proposition 3.3. *Let* $\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}$, $\widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} \in A_{[0,j]}^{[*]}$. *Then*

$$[f', g] - [f, g'] = W_j[\widehat{f}, \overline{\widehat{g}}]. \tag{3.9}$$

Proof. Let vectors $\widehat{f}, \widehat{g} \in A_{[0,j]}^{[*]}$ be of the form

$$\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix} = \begin{bmatrix} f \\ \mathfrak{J}_{[0,j]}^T f + c_f e_{n_j} \end{bmatrix} \text{ and } \widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} = \begin{bmatrix} g \\ \mathfrak{J}_{[0,j]}^T g + c_g e_{n_j} \end{bmatrix}.$$

Since the matrix $\mathfrak{J}_{[0,j]}^T$ generates a self-adjoint operator in $\mathfrak{H}_{[0,j]}$, see (2.10), one gets from (3.4)

$$\begin{aligned} [f', g] - [f.g'] &= \left[\mathfrak{J}_{[0,j]}^T f + c_f e_{n_j}, g\right] - \left[f, \mathfrak{J}_{[0,j]}^T g + c_g e_{n_j}\right] \\ &= c_f \overline{g}_j - f_j \overline{c_g} = W_j[\widehat{f}, \overline{\widehat{g}}]. \end{aligned} \tag{3.10}$$

This completes the proof. □

The following Christoffel–Darboux formulas are implied by (3.9).

Corollary 3.4. *For all* $j \in \mathbb{Z}_+$, $z \in \mathbb{C}$ *and* $z_0 \in \mathbb{R}$ *the following formulas hold*

$$W_j[\widehat{\xi}_{[0,j]}(z), \widehat{\xi}_{[0,j]}(z_0)] = (z - z_0)[\xi_{[0,j]}(z), \xi_{[0,j]}(z_0)]_{\mathfrak{H}_{[0,j]}}, \tag{3.11}$$

$$W_j[\widehat{\xi}_{[0,j]}(z), \widehat{\pi}_{[0,j]}(z_0)] = 1 + (z - z_0)[\xi_{[0,j]}(z), \pi_{[0,j]}(z_0)]_{\mathfrak{H}_{[0,j]}},$$

$$W_j[\widehat{\pi}_{[0,j]}(z), \widehat{\xi}_{[0,j]}(z_0)] = -1 + (z - z_0)[\pi_{[0,j]}(z), \xi_{[0,j]}(z_0)]_{\mathfrak{H}_{[0,j]}},$$

$$W_j[\widehat{\pi}_{[0,j]}(z), \widehat{\pi}_{[0,j]}(z_0)] = (z - z_0)[\pi_{[0,j]}(z), \pi_{[0,j]}(z_0)]_{\mathfrak{H}_{[0,j]}}. \tag{3.12}$$

Theorem 3.5. *Let* $P_i(z)$ *be the polynomials of the first kind associated with the monic generalized Jacobi matrix* $\mathfrak{J}$. *Then:*

(1) the boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ *of the linear relation* $A_{[0,j]}^{[*]}$ *can be found by*

$$\Gamma_0 \widehat{f} = f_j = \left[f, e_{n_j}\right] \quad \textit{and} \quad \Gamma_1 \widehat{f} = c_f; \tag{3.13}$$

(2) the defect subspace of the operator $A_{[0,j]}$ *is given by*

$$\mathfrak{N}_z(A_{[0,j]}) = \operatorname{span} \pi_{[0,j]}(z),$$

where $\pi_{[0,N]}(z)$ *is given by* (2.11);

(3) the Weyl function and the γ-field of the operator $A_{[0,j]}$ corresponding to the boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ are given by

$$M(z) = \frac{P_{j+1}(z)}{\widetilde{b}_j P_j(z)}, \quad \gamma(z) = \frac{\pi_{[0,j]}(z)}{P_j(z)}. \tag{3.14}$$

Proof. (1) The Green's formula for the triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is derived from the equality (3.10):

$$[f', g] - [f, g'] = c_f \overline{[g, e_{n_j}]} - \overline{c_g}[f, e_{n_j}] = \Gamma_1 \widehat{f} \overline{\Gamma_0 \widehat{g}} - \Gamma_0 \widehat{f} \overline{\Gamma_1 \widehat{g}}. \tag{3.15}$$

(2) It follows from (3.2) and (2.11) that $\widehat{\pi}_{[0,j]}(z) \in A_{[0,j]}^{[*]}$ and hence the inclusion $\pi_{[0,j]}(z) \in \mathfrak{N}_z(A_{[0,j]})$ holds.

(3) Applying Γ_0 and Γ_1 to the defect vector $\widehat{f}_z := \widehat{\pi}_{[0,j]}(z)$, one obtains

$$\Gamma_0 \widehat{f}_z = [\pi_{[0,j]}(z), e_{n_j}] = P_j(z) \quad \text{and} \quad \Gamma_1 \widehat{f}_z = \widetilde{b}_j^{-1} P_{j+1}(z). \tag{3.16}$$

This proves the formulas (3.14). □

Remark 3.6. A similar construction of the boundary triple for "symmetric" GJM's was presented in [10]. Relations between the corresponding objects for monic and "symmetric" GJM's will be given in Section 5.2.

Theorem 3.7. *Let the operator $S_{[0,j]}$ in the space $\mathfrak{H}_{[0,j]}$ be defined as the restriction of the operator $\mathfrak{J}_{[0,j]}^T$ to the domain*

$$\mathrm{dom} S_{[0,j]} = \left\{ f \in \mathfrak{H}_{[0,j]} : [f, e_0] = 0 \right\}. \tag{3.17}$$

Then:

(1) the adjoint linear relation $S_{[0,j]}^{[*]}$ of $S_{[0,j]}$ has the following representation

$$S_{[0,j]}^{[*]} = \left\{ \widehat{f} = \begin{bmatrix} f \\ \mathfrak{J}_{[0,j]}^T f + d_f e_0 \end{bmatrix} : \begin{array}{c} f \in \mathfrak{H}_{[0,j]}, \\ d_f \in \mathbb{C} \end{array} \right\}; \tag{3.18}$$

(2) the boundary triple $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the linear relation $S_{[0,j]}^{[*]}$ can be found by

$$\Gamma_1 \widehat{f} = [f, e_0] \quad \text{and} \quad \Gamma_0 \widehat{f} = -d_f;$$

(3) the defect subspace $\mathfrak{N}_\lambda(S_{[0,j]})$ of the operator $S_{[0,j]}$ is spanned by

$$(\mathfrak{J}_{[0,j]}^T - z)^{-1} e_0 = \left(\xi_{[0,j]} - \frac{Q_{j+1}(z)}{P_{j+1}(z)} \pi_{[0,j]} \right);$$

(4) the Weyl function $m_{[0,j]}(z)$ of the operator $S_{[0,j]}$ corresponding to the boundary triple $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is given by

$$m_{[0,j]}(z) = [(\mathfrak{J}^T_{[0,j]} - z)^{-1}e_0, e_0] = -\frac{Q_{j+1}(z)}{P_{j+1}(z)}. \tag{3.19}$$

Moreover

$$m_{[0,j]}(z) = -\frac{s_0}{z} - \cdots - \frac{s_{2n_{j+1}-2}}{z^{2n_{j+1}-1}} + o\left(\frac{1}{z^{2n_{j+1}-1}}\right) \text{ as } z\widehat{\rightarrow}\infty,$$
$$s_i = \left[\left(\mathfrak{J}^T_{[0,j]}\right)^i e_0, e_0\right]. \tag{3.20}$$

Proof. (1) Assume that the linear operator $S_{[0,j]}$ in the space $\mathfrak{H}_{[0,j]}$ is defined as the restriction of the operator $\mathfrak{J}^T_{[0,j]}$ to the domain (3.17). Consequently, the adjoint linear relation $S^{[*]}_{[0,j]}$ is given by (3.18).

(2) Suppose, $\widehat{f} = \{f, \mathfrak{J}^T_{[0,j]}f + d_f e_0\}$ and $\widehat{g} = \{f, \mathfrak{J}^T_{[0,j]}g + d_g e_0\}$, then

$$\begin{aligned}[][f', g] - [f, g'] &= [\mathfrak{J}^T_{[0,j]}f + d_f e_0, g] - [f, \mathfrak{J}^T_{[0,j]}g + d_g e_0] = \\ &= d_f\overline{[g, e_0]} - \overline{d_g}[f, e_0] = \Gamma_1\widehat{f}\,\overline{\Gamma_0\widehat{g}} - \Gamma_0\widehat{f}\,\overline{\Gamma_1\widehat{g}}.\end{aligned}$$

(3) Let us set

$$f_z := \xi_{[0,j]}(z) - \frac{Q_{j+1}(z)}{P_{j+1}(z)}\pi_{[0,j]}(z).$$

Then it follows from (2.12) and (2.13) that $(\mathfrak{J}^T_{[0,j]} - z)f_z = e_0$. Hence

$$f_z = (\mathfrak{J}^T_{[0,j]} - z)^{-1}e_0 \tag{3.21}$$

and

$$\widehat{f_z} = \begin{bmatrix} f_z \\ zf_z \end{bmatrix} = \begin{bmatrix} f_z \\ \mathfrak{J}^T_{[0,j]}f_z - e_0 \end{bmatrix}. \tag{3.22}$$

Therefore, $f_z \in \mathfrak{N}_z(S_{[0,j]})$.

(4) It follows from (3.13) and (3.22)

$$\Gamma_0\widehat{f_z} = 1 \quad \text{and} \quad \Gamma_1\widehat{f_z} = -\frac{Q_{j+1}(z)}{P_{j+1}(z)}. \tag{3.23}$$

This proves the second formula in (3.19). The first formula is implied by (3.21) and (3.23). Due to (3.19), we obtain

$$m_{[0,j]}(z) = -\frac{[e_0, e_0]}{z} - \cdots - \frac{\left[\left(\mathfrak{J}^T_{[0,j]}\right)^{2n_{j+1}-2} e_0, e_0\right]}{z^{2n_{j+1}-1}} + o\left(\frac{1}{z^{2n_{j+1}-1}}\right)$$

as $z\widehat{\rightarrow}\infty$. Denote $s_i = \left[\left(\mathfrak{J}^T_{[0,j]}\right)^i e_0, e_0\right]$, we get (3.20). □

3.2. The case $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$

Definition 3.8. *A symmetric operator A in a Pontryagin space $(\mathfrak{H}, [\cdot,\cdot])$ is said to have k negative squares, if for every choice of $f_j \in \operatorname{dom} A$ the form*

$$\sum_{i,j=0}^{n} [Af_i, f_j]_{\mathfrak{H}} \xi_i \bar{\xi}_j$$

has at most k, and for some choice of $f_j \in \operatorname{dom} A$ exactly k negative squares.

Let a symmetric operator A in a Pontryagin space $(\mathfrak{H}, [\cdot,\cdot])$ have k negative squares. Recall [14], that a boundary triple $\Pi = \{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ for the linear relation $A^{[*]}$ is said to be *basic*, if the Weyl function $M(z)$ of A corresponding to the boundary triple Π satisfies the conditions

$$\lim_{iy\to 0} M(iy) = \infty, \qquad \lim_{iy\to\infty} M^+(iy) = 0. \tag{3.24}$$

Proposition 3.9. *Let $\boldsymbol{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$ and let $\mathcal{N}(\boldsymbol{s}) = \{n_i\}_{i=1}^{N+1}$. Then:*

1. *the operator $A_{[0,N]}$ has k negative squares;*
2. *a boundary triple $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ for the linear relation $A_{[0,N]}^{[*]}$ can be chosen as follows*

$$\Gamma_1^+ \widehat{f} = \frac{1}{P_N(0)} [f, e_{n_N}], \quad \Gamma_0^+ \widehat{f} = -P_N(0) c_f + \widetilde{b}_N^{-1} P_{N+1}(0)[f, e_{n_N}], \tag{3.25}$$

   where c_f is defined by the decomposition (3.2)*;*
3. *the corresponding Weyl function and the γ-field are given by*

$$M_{[0,N]}^+(z) = \frac{P_{2N}^+(z)}{P_{2N+1}^+(z)}, \quad \gamma^+(z) = \frac{\pi_{[0,N]}(z)}{P_{2N+1}^+(z)}; \tag{3.26}$$

4. *the boundary triple $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ for the linear relation $A_{[0,N]}^{[*]}$ is basic.*

Proof. (1) Arbitrary vector $f \in \operatorname{dom}(A_{[0,N]})$ can be expanded by vectors e_i $(0 \le i \le n_{N+1} - 2)$ from (3.3) as follows

$$f = \sum_{i=0}^{n_{N+1}-2} \xi_i e_i, \quad \xi_i \in \mathbb{C}.$$

By (3.20)

$$[A_{[0,N]}e_i, e_j]_{\mathfrak{H}_{[0,k]}} = s_{i+j+1}, \quad i,j = \overline{0,\dots,n_{N+1}-2}. \tag{3.27}$$

Since $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$ one obtains from (3.27)

$$\sum_{i,j=0}^{n_{N+1}-2} [A_{[0,N]}e_i, e_j]_{\mathfrak{H}_{[0,N]}}\xi_i\bar{\xi}_j = \sum_{i,j=0}^{n_{N+1}-2} s_{i+j+1}\xi_i\bar{\xi}_j$$

and hence the operator $A_{[0,N]}$ has k negative squares.

(2) Obviously the operators Γ_0^+, Γ_1^+ are connected with Γ_0, Γ_1 by

$$\Gamma_1^+\widehat{f} = \frac{1}{\beta}\Gamma_0 \quad \text{and} \quad \Gamma_0^+\widehat{f} = \beta\left(\alpha\Gamma_0 - \Gamma_1\right),$$

where $\beta = P_N(0)$ and $\alpha = \frac{\widetilde{b}_N^{-1}P_{N+1}(0)}{P_N(0)}$. Then, by [21, Proposition 1.7] $\{\mathbb{C},\Gamma_0^+,\Gamma_1^+\}$ is a boundary triple of the linear relation $A_{[0,N]}^{[*]}$. Hence, the corresponding Weyl function can be calculated by

$$\begin{aligned} M^+(z) &= \tfrac{1}{\beta^2(\alpha - M(z))} = \frac{1}{P_N^2(0)\left(\dfrac{P_{N+1}(0)}{\widetilde{b}_N P_N(0)} - \dfrac{P_{N+1}(z)}{\widetilde{b}_N P_N(z)}\right)} \\ &= -\frac{\dfrac{P_N(z)}{P_N(0)}}{\widetilde{b}_N^{-1}\left(P_{N+1}(z)P_N(0) - P_N(z)P_{N+1}(0)\right)} = \frac{P_{2N}^+(z)}{P_{2N+1}^+(z)}. \end{aligned}$$

(3) It follows from (3.16) that for $\widehat{f}_z = \widehat{\pi}_{[0,N]}(z)$

$$\begin{aligned} \Gamma_0^+\widehat{f}_z &= \frac{1}{\widetilde{b}_N}(P_{N+1}(0)P_N(z) - P_N(0)P_{N+1}(z)) = P_{2N+1}^+(z), \\ \Gamma_1^+\widehat{f}_z &= \frac{1}{P_N(0)}[\pi_{[0,N]}(z), e_{n_N}] = \frac{P_N(z)}{P_N(0)} = P_{2N}^+(z). \end{aligned} \tag{3.28}$$

This proves the formulas (3.26).

(4) It follows from (3.26) that the Weyl function $M^+(z)$ satisfies the conditions

$$\lim_{iy\to 0} M^+(iy) = \infty, \quad \lim_{iy\to\infty} M^+(iy) = 0.$$

Therefore, by the above definition (3.24) the boundary triple $\{\mathbb{C},\Gamma_0^+,\Gamma_1^+\}$ for the linear relation $A_{[0,N]}^{[*]}$ is basic. □

Remark 3.10. Recall that in [14] the boundary triple $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ for the linear relation $A^{[*]}$ was called basic, if the extensions $A_0 = A_K$ and $A_1 = A_F$ were the Krein and the Friedrichs extensions of $A_{[0,N]}$, respectively. We use here an equivalent definition to prevent introducing of extra notations A_F and A_K (see [14, Proposition 3.1]).

4. Resolvent matrix

4.1. Review of the M.G. Krein theory of resolvent matrix

Let A be a symmetric operator in a Pontryagin space $(\mathfrak{H}, [\cdot,\cdot])$ of negative index κ_0, let the defect indices of A be $(1,1)$, let a scale vector $u \in \mathfrak{H}$ be given and let $\widetilde{A}$ be a selfadjoint extension of A acting in a Pontryagin space $\widetilde{\mathfrak{H}}(\supset \mathfrak{H})$ of negative index κ satisfying the minimality condition

$$\widetilde{\mathfrak{H}} = \overline{\operatorname{span}}\{u, (\widetilde{A}-z)^{-1}u : z \in \rho(\widetilde{A})\}.$$

The function $[(\widetilde{A}-z)^{-1}u, u]_{\mathfrak{H}}$ is called the u-resolvent of the operator A of index κ.

A description of u-resolutions of a symmetric operator was given in [34, 38] in the framework of the theory of the u-resolvent matrix, which will be briefly presented below. A point $z \in \mathbb{C}$ is called (see [34, 38]) a u-regular point of the operator A if $\operatorname{ran}(A-z)$ is closed and

$$H = \operatorname{ran}(A-z) \dotplus \operatorname{span} u.$$

Denote by $\rho(A, u)$ the set of u-regular points of the operator A. Define two functionals $\mathcal{P}(z)$, $\mathcal{Q}(z) : H \to \mathbb{C}$ holomorphic in $\rho(A, u)$ by the formulas

$$f - (\mathcal{P}(z)f)u \in \operatorname{ran}(A-z), \quad \mathcal{Q}(z)f = [(A-z)^{-1}(f - (\mathcal{P}(z)f)u), u].$$

Define also two vector-valued functions $\mathcal{P}(z)^{[*]}$, $\mathcal{Q}(z)^{[*]}$ with values in $\mathfrak{H}$ by

$$[\mathcal{P}(z)^{[*]}, f] := \overline{\mathcal{P}(z)f}, \quad [\mathcal{Q}(z)^{[*]}, f] := \overline{\mathcal{Q}(z)f}.$$

Direct verification shows that for all $z \in \rho(A, u)$ we have

$$\begin{aligned} \widehat{\mathcal{P}}(z)^{[*]} &:= \{\mathcal{P}(z)^{[*]}, \bar{z}\mathcal{P}(z)^{[*]}\} \in A^{[*]}, \\ \widehat{\mathcal{Q}}(z)^{[*]} &:= \{\mathcal{Q}(z)^{[*]}, \bar{z}\mathcal{Q}(z)^{[*]} + u\} \in A^{[*]}. \end{aligned} \tag{4.1}$$

A description of u-resolvents of a Hilbert space symmetric operator with equal finite defect indices was obtained by M.G. Krein in [34], and for densely defined operator in a Pontryagin space, by M.G. Krein and H. Langer in [38]. An explicit formula for the resolvent matrix of a Hilbert space symmetric operator in boundary triple's notations was given in [20, 21]. For a nondensely defined symmetric operator in a Pontryagin space with the defect indices $(1,1)$ such a formula and the description of u-resolvents take the following form (see [15, Theorem 5.2]).

Theorem 4.1. *Let A be a symmetric operator with defect indices* $(1,1)$ *in a Pontryagin space $\mathfrak{H}$ of negative index κ_0, let $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^{[*]}$, let $u \in \mathfrak{H}$, let $\rho(A,u) \neq \emptyset$ and let*

$$\mathcal{W}(z) = \begin{pmatrix} -\Gamma_0 \widehat{\mathcal{Q}}(z^{[*]} & \Gamma_0 \widehat{\mathcal{P}}(z)^{[*]} \\ -\Gamma_1 \widehat{\mathcal{Q}}(z)^{[*]} & \Gamma_1 \widehat{\mathcal{P}}(z)^{[*]} \end{pmatrix}^*, \quad z \in \rho(A,u). \tag{4.2}$$

Then:

(1) $\mathcal{W}(z) = (w_{i,j}(z))_{i,j=1}^2$ *is holomorphic on $\rho(A,u)$, and the formula*

$$[(\widetilde{A} - z)^{-1} u, u] = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \quad z \in \rho(A,u) \cap \rho(\widetilde{A})$$

establishes a one-to-one correspondence between the set of all u-resolvents of A of index κ and the set of all $\tau \in \mathbf{N}_{\kappa-\kappa_0}$, such that

$$w_{21}(z)\tau(z) + w_{22}(z) \not\equiv 0. \tag{4.3}$$

(2) If, in addition, $\mathfrak{H}_0 := \mathfrak{H}[-]\operatorname{dom} A$ is not trivial and $A_0 = A \dotplus \{0\} \times \mathfrak{H}_0$ then $\widetilde{A}$ is an operator if and only if τ satisfies the Nevanlinna condition

$$\tau(iy) = o(y) \quad \text{as } y \to \infty. \tag{4.4}$$

(3) If, in addition, the operator A has k_0 negative squares and Π is a basic boundary triple, then the formula (4.2) *establishes a one-to-one correspondence between the set of all u-resolutions of the operator A of index κ, such that the extension $\widetilde{A}$ has k negative squares and the set of $\tau \in \mathbf{N}_{\kappa-\kappa_0}^{k-k_0}$, such that* (4.3) *holds.*

4.2. Resolvent matrix of the operator $A_{[0,j]}$

Theorem 4.2. [10, Theorem 3.14] *Let a boundary triple $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the operator $A_{[0,j]}^{[*]}$ be defined by* (3.13) *and let $u = e_0$. Then the corresponding $u-$resolvent matrix of the operator $A_{[0,j]}$ takes the following representation*

$$\mathcal{W}_{[0,j]}(z) = \begin{pmatrix} -Q_j(z) & -\widetilde{b}_j^{-1} Q_{j+1}(z) \\ P_j(z) & \widetilde{b}_j^{-1} P_{j+1}(z) \end{pmatrix}. \tag{4.5}$$

Proof. It follows from (3.5) and (3.6) that $\widehat{\mathcal{P}}_{[0,j]}(z)^{[*]} = \widehat{\pi}_{[0,j]}(\overline{z})$ and $\widehat{\mathcal{Q}}_{[0,j]}(z)^{[*]} = \widehat{\xi}_{[0,j]}(\overline{z})$. By (3.13) we obtain

$$\begin{aligned} \Gamma_1 \widehat{\mathcal{Q}}_{[0,j]}(z)^{[*]} = \widetilde{b}_j^{-1} Q_{j+1}(\overline{z}) \quad &\text{and} \quad \Gamma_0 \widehat{\mathcal{Q}}_{[0,j]}(z)^{[*]} = Q_j(\overline{z}), \\ \Gamma_1 \widehat{\mathcal{P}}_{[0,j]}(z)^{[*]} = \widetilde{b}_j^{-1} P_{j+1}(\overline{z}) \quad &\text{and} \quad \Gamma_0 \widehat{\mathcal{P}}_{[0,j]}(z)^{[*]} = P_j(\overline{z}). \end{aligned} \tag{4.6}$$

Substituting (4.6) in (4.2), we get (4.5). This completes the proof. □

Proposition 4.3. *Let a boundary triple* $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ *for the operator* $A_{[0,j]}^{[*]}$ *be defined by* (3.13) *and let* $u=e_0$*. Then the u-resolvent matrix* $\mathcal{W}_{[0,j]}(z)$ *admits the following factorization*

$$\mathcal{W}_{[0,j]}(z)=\mathcal{W}_0(z)\mathcal{W}_1(z)\dots\mathcal{W}_j(z), \tag{4.7}$$

where the elementary matrices are defined by

$$\mathcal{W}_0(z)=\begin{pmatrix}0 & -1\\ 1 & \dfrac{a_0(z)}{b_0}\end{pmatrix}\quad \text{and}\quad \mathcal{W}_i(z)=\begin{pmatrix}0 & -\widetilde{b}_{i-1}^{-1}\\ \widetilde{b}_{i-1} & \dfrac{a_i(z)}{b_i}\end{pmatrix},\ i=\overline{1,j}.$$

Proof. Prove (4.7) by induction

(1) $\mathcal{W}_{[0,0]}(z)=\mathcal{W}_0(z)$, i.e. (4.7) holds.

(2) Inductive step. Let (4.7) hold for some $i-1$, i.e.

$$\mathcal{W}_{[0,i-1]}(z)=\mathcal{W}_0(z)\mathcal{W}_1(z)\dots\mathcal{W}_{i-1}(z)=\begin{pmatrix}-Q_{i-1}(z) & -\widetilde{b}_{i-1}^{-1}Q_i(z)\\ P_{i-1}(z) & \widetilde{b}_{i-1}^{-1}P_i(z)\end{pmatrix}.$$

Then

$$\mathcal{W}_{[0,i-1]}(z)\mathcal{W}_i(z)=\begin{pmatrix}-Q_{i-1}(z) & -\widetilde{b}_{i-1}^{-1}Q_i(z)\\ P_{i-1}(z) & \widetilde{b}_{i-1}^{-1}P_i(z)\end{pmatrix}\begin{pmatrix}0 & -\widetilde{b}_{i-1}^{-1}\\ \widetilde{b}_{i-1} & \dfrac{a_i(z)}{b_i}\end{pmatrix}$$
$$=\begin{pmatrix}-Q_i(z) & -\widetilde{b}_i^{-1}(-b_iQ_{i-1}(z)+a_i(z)Q_i(z))\\ P_i(z) & \widetilde{b}_i^{-1}(-b_iP_{i-1}(z)+a_i(z)P_i(z))\end{pmatrix}$$
$$=\{\text{by } (1.10)\}=\begin{pmatrix}-Q_i(z) & -\widetilde{b}_i^{-1}Q_{i+1}(z)\\ P_i(z) & \widetilde{b}_i^{-1}P_{i+1}(z)\end{pmatrix}=\mathcal{W}_{[0,i]}(z).$$

This proves (4.7). □

Theorem 4.4. [10, Theorem 3.14] *Let* $j\in\mathbb{N}$ *and let the Wronskians* $W_j[\widehat{f},\widehat{\pi}_{[0,j]}(0)]$ *and* $W_j[\widehat{f},\widehat{\xi}_{[0,j]}(0)]$ *be defined by* (3.7)*. Then:*

(1) The formulas

$$\widetilde{\Gamma}_0\widehat{f}=W_j[\widehat{f},\widehat{\pi}_{[0,j]}(0)],\quad \widetilde{\Gamma}_1\widehat{f}=W_j[\widehat{f},\widehat{\xi}_{[0,j]}(0)],$$

define a boundary triple $\widetilde{\Pi}=\{\mathbb{C},\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$ *for the operator* $A_{[0,j]}^{[*]}$.

(2) The u-resolvent matrix of $A_{[0,j]}$ *corresponding to the boundary triple* $\widetilde{\Pi}$ *and the scale* $u=e_0$ *has the form*

$$\widetilde{W}_{[0,j]}(z)=\begin{bmatrix}-W_j[\widehat{\xi}_{[0,j]}(z),\widehat{\pi}_{[0,j]}(0)] & -W_j[\widehat{\xi}_{[0,j]}(z),\widehat{\xi}_{[0,j]}(0)]\\ W_j[\widehat{\pi}_{[0,j]}(z),\widehat{\pi}_{[0,j]}(0)] & W_j[\widehat{\pi}_{[0,j]}(z),\widehat{\xi}_{[0,j]}(0)]\end{bmatrix}. \tag{4.8}$$

(3) The Weyl function corresponding to the boundary triple $\widetilde{\Pi}$ has the form

$$\widetilde{M}_{[0,j]}(z) = \frac{W_j[\widehat{\pi}_{[0,j]}(z), \widehat{\xi}_{[0,j]}(0)]}{W_j[\widehat{\pi}_{[0,j]}(z), \widehat{\pi}_{[0,j]}(0)]}.$$

Proof. The proof of (1) is based on the identities (3.15) and

$$W_j[\widehat{f}, \widehat{\xi}_{[0,j]}(0)]W_j[\widehat{g}, \widehat{\pi}_{[0,j]}(0)]^* - W_j[\widehat{f}, \widehat{\pi}_{[0,j]}(0)]W_j[\widehat{g}, \widehat{\xi}_{[0,j]}(0)]^* = W_j[\widehat{f}, \overline{\widehat{g}}].$$

Applying the operators $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ to

$$\widehat{f} = \widehat{\mathcal{Q}}_{[0,j]}(z)^{[*]} = \widehat{\xi}_{[0,j]}(\overline{z}) \quad \text{and} \quad \widehat{g} = \widehat{\mathcal{P}}_{[0,j]}(z)^{[*]} = \widehat{\pi}_{[0,j]}(\overline{z})$$

one obtains the equalities

$$\widetilde{\Gamma}_0\widehat{f} = W_j[\widehat{\xi}_{[0,j]}(\overline{z}), \widehat{\pi}_{[0,j]}(0)], \quad \widetilde{\Gamma}_1\widehat{f} = W_j[\widehat{\xi}_{[0,j]}(\overline{z}), \widehat{\xi}_{[0,j]}(0)].$$

$$\widetilde{\Gamma}_0\widehat{g} = W_j[\widehat{\pi}_{[0,j]}(\overline{z}), \widehat{\pi}_{[0,j]}(0)], \quad \widetilde{\Gamma}_1\widehat{g} = W_j[\widehat{\pi}_{[0,j]}(\overline{z}), \widehat{\xi}_{[0,j]}(0)], \tag{4.9}$$

In view of (4.2) this yields (4.8). The second part of the formula (4.8) is implied by (3.11)–(3.12).

The statement (3) follows from (4.9). □

Corollary 4.5. *The formula* (4.8) *for the u-resolvent matrix of $A_{[0,j]}$ corresponding to the boundary triple $\widetilde{\Pi}$ can be rewritten as*

$$\widetilde{W}_{[0,j]}(z) = -I + z\begin{bmatrix} -[\xi_{[0,j]}(z), \pi_{[0,j]}(0)]_{[0,j]} & -[\xi_{[0,j]}(z), \xi_{[0,j]}(0)]_{[0,j]} \\ [\pi_{[0,j]}(z), \pi_{[0,j]}(0)]_{[0,j]} & [\pi_{[0,j]}(z), \xi_{[0,j]}(0)]_{[0,j]} \end{bmatrix}. \tag{4.10}$$

The Weyl function $\widetilde{M}_{[0,j]}(z)$ corresponding to the boundary triple $\widetilde{\Pi}$ is equal to

$$\widetilde{M}_{[0,j]}(z) = \frac{P_{j+1}(z)Q_j(0) - P_j(z)Q_{j+1}(0)}{P_{j+1}(z)P_j(0) - P_j(z)P_{j+1}(0)}$$

and has the properties

$$\lim_{x\uparrow 0} \widetilde{M}_{[0,j]}(x) = \infty, \quad \lim_{x\downarrow -\infty} \widetilde{M}_{[0,j]}(x) = \frac{Q_j(0)}{P_j(0)}.$$

4.3. The case $\mathbf{s} \in \mathcal{H}^{k,reg}_{\kappa,2n_j-2}$

Theorem 4.6. [10, Theorem 3.14] *Let $\mathbf{s} = \{s_i\}_{i=0}^{2n_j-2} \in \mathcal{H}^{k,reg}_{\kappa,2n_j-2}$, let $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ be a boundary triple of the linear relation $A_{[0,j]}^{[*]}$. Then:*

(1) The corresponding u-resolvent matrix of the operator $A_{[0,j]}$ can be found by

$$\mathcal{W}^{+}_{[0,j]}(z)=\begin{pmatrix} Q^{+}_{2j+1}(z) & Q^{+}_{2j}(z)\\ P^{+}_{2j+1}(z) & P^{+}_{2j}(z)\end{pmatrix}. \tag{4.11}$$

(2) $\mathcal{W}^{+}_{[0,j]}\in\mathcal{U}_{\kappa_j}(J)$, where κ_j are calculated by (2.23).

Proof. It follows from (4.1) that $\widehat{\mathcal{P}}_{[0,j]}(z)^{[*]}=\widehat{\pi}_{[0,j]}(\overline{z})$ and $\widehat{\mathcal{Q}}_{[0,j]}(z)^{[*]}=\widehat{\xi}_{[0,j]}(\overline{z})$. Calculating entries of $\mathcal{W}^{+}_{[0,j]}(z)$, we obtain

$$\begin{gathered}\Gamma_0^{+}\widehat{Q}_{[0,j]}(z)^{[*]}=\tfrac{1}{\widetilde{b}_j}(P_{j+1}(0)Q_j(\overline{z})-P_j(0)Q_{j+1}(\overline{z}))=-Q^{+}_{2j+1}(\overline{z}),\\ \Gamma_1^{+}\widehat{Q}_{[0,j]}(z)^{[*]}=\tfrac{1}{P_j(0)}[\xi_{[0,j]}(\overline{z}),e_{n_j}]=\tfrac{Q_j(\overline{z})}{P_j(0)}=-Q^{+}_{2j}(\overline{z}).\end{gathered}\tag{4.12}$$

Inserting (4.12) and (3.28) in (4.2), one obtains (4.11). □

In the following statement relations between the resolvent matrices $\widetilde{\mathcal{W}}_{[0,j]}(z)$ and $\mathcal{W}^{+}_{[0,j]}(z)$ is established.

Proposition 4.7. *Let $\mathbf{s}=\{s_i\}_{i=0}^{2n_j-2}\in\mathcal{H}^{k,reg}_{\kappa,2n_j-2}$. The resolvent matrices $\widetilde{\mathcal{W}}_{[0,j]}(z)$ and $\mathcal{W}^{+}_{[0,j]}(z)$ of the operator $A_{[0,j]}$ are related by*

$$\mathcal{W}^{+}_{[0,j]}(z)=\widetilde{\mathcal{W}}_{[0,j]}(z)V_{[0,j]},\quad \text{where}\quad V_{[0,j]}=\begin{pmatrix}-1 & \frac{Q_j(0)}{P_j(0)}\\ 0 & -1\end{pmatrix} \tag{4.13}$$

Proof. It follows from (4.8) and (3.7) that

$$\widetilde{\mathcal{W}}_{[0,j]}(z)=\begin{pmatrix}-\frac{Q_{j+1}(z)P_j(0)-Q_j(z)P_{j+1}(0)}{\widetilde{b}_j} & -\frac{Q_{j+1}(z)Q_j(0)-Q_j(z)Q_{j+1}(0)}{\widetilde{b}_j}\\ \frac{P_{j+1}(z)P_j(0)-P_j(z)P_{j+1}(0)}{\widetilde{b}_j} & \frac{P_{j+1}(z)Q_j(0)-P_j(z)Q_{j+1}(0)}{\widetilde{b}_j}\end{pmatrix}$$

and hence

$$\widetilde{\mathcal{W}}_{[0,j]}(z)V_{[0,j]}=\begin{pmatrix}\frac{Q_{j+1}(z)P_j(0)-Q_j(z)P_{j+1}(0)}{\widetilde{b}_j} & -\frac{Q_j(z)(Q_{j+1}(0)P_j(0)-P_{j+1}(0)Q_j(0))}{\widetilde{b}_jP_j(0)}\\ -\frac{P_{j+1}(z)P_j(0)-P_j(z)P_{j+1}(0)}{\widetilde{b}_j} & \frac{P_j(z)(Q_{j+1}(0)P_j(0)-P_{j+1}(0)Q_j(0))}{\widetilde{b}_jP_j(0)}\end{pmatrix}$$

Now (4.13) follows from the Liouville–Ostrogradskii formula (2.9). □

Notice that the mvf $\mathcal{W}^{+}_{[0,j]}(z)$ coincides with the mvf $\mathcal{W}^{+}_{2j+1}(z)$ introduced in [18]. Recall that $\mathcal{W}^{+}_{2j+1}(z)$ admits the following factorization

$$W^{+}_{2j-1}(z)=M_1(z)L_1\dots L_{j-1}M_j(z) \tag{4.14}$$

where the matrices $M_i(z)$, L_i are defined by

$$L_i = \begin{pmatrix} 1 & l_i \\ 0 & 1 \end{pmatrix} \quad i = \overline{1, j-1}, \quad M_i(z) = \begin{pmatrix} 1 & 0 \\ -zm_i(z) & 1 \end{pmatrix}, \quad i = \overline{1, j}. \tag{4.15}$$

and the polynomials m_i and the numbers l_i are defined by (2.17).

In the following proposition we introduce one more resolvent matrix which will be used as a frame for the description of solutions of even moment problem.

Proposition 4.8. *Let $\boldsymbol{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$, $\mathcal{N}(\boldsymbol{s}) = \{n_i\}_{i=1}^{j}$ and let a boundary triple $\Pi^{++} = \{\mathbb{C}, \Gamma_0^{++}, \Gamma_1^{++}\}$ for the linear relation $A_{[0,j]}^{[*]}$ be defined by*

$$\Gamma_1^{++}\widehat{f} = \Gamma_1^{+}\widehat{f} + l_j\Gamma_0^{+}\widehat{f}, \quad \Gamma_0^{++}\widehat{f} = \Gamma_0^{+}\widehat{f},$$

where the boundary triple $\Pi^{+} = \{\mathbb{C}, \Gamma_0^{+}, \Gamma_1^{+}\}$ is given by (3.2). Then

(1) the corresponding u-resolvent matrix $\mathcal{W}_{[0,j]}^{++}(z)$ is given by

$$\mathcal{W}_{[0,j]}^{++}(z) = \begin{pmatrix} Q_{2j-1}^{+}(z) & Q_{2j}^{+}(z) \\ P_{2j-1}^{+}(z) & P_{2j}^{+}(z) \end{pmatrix}. \tag{4.16}$$

(2) $\mathcal{W}_{[0,j]}^{++} \in \mathcal{U}_{\kappa_j}(J)$, where κ_j are calculated by (2.24).

The above two propositions allow to formulate the following factorization result for the resolvent matrix $\mathcal{W}_{[0,j]}^{+}(z)$.

Theorem 4.9. *Let $\mathbf{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_{\kappa}^{k,reg}$ and let $N \in \mathbb{N}$ be big enough so that the equalities*

$$\nu_-(S_{n_N}) = \nu_-(S_{n_j}) \quad \text{and} \quad \nu_-(S_{n_N}^{+}) = \nu_-(S_{n_j}^{+})$$

hold for all $j > N$, $j \in \mathbb{N}$. Let $\widetilde{\mathfrak{J}}_{[N,j]}$ be a Jacobi matrix

$$\widetilde{\mathfrak{J}}_{[N,j]} = \mathfrak{J}_{[N,j]} + \operatorname{diag}\left(\frac{1}{m_N l_{N-1}}, 0, \dots, 0\right).$$

Then the resolvent matrix $\mathcal{W}_{[0,j]}^{+}(z)$ of $A_{[0,j]}$ admits the factorization

$$\mathcal{W}_{[0,j]}^{+}(z) = \mathcal{W}_{[0,N-1]}^{++}(z)\mathcal{W}_{[N,j]}^{+}(z), \tag{4.17}$$

where $\mathcal{W}_{[0,N-1]}^{++}(z)$ is the resolvent matrix of the operator $A_{[0,N-1]}$ of the form (4.16) corresponding to the boundary triple $\Pi_{[0,N-1]}^{++}$, and $\mathcal{W}_{[N,j]}^{+}(z)$ is the resolvent matrix of the operator

$$\widetilde{A}_{[N,j]} = \widetilde{\mathfrak{J}}_{[N,j]}|_{\operatorname{dom}(\widetilde{A}_{[N,j]})}, \quad \operatorname{dom}(\widetilde{A}_{[N,j]}) = \{f \in \mathfrak{H}_{[N,j]} : [f, e_{n_j}] = 0\}$$

corresponding to the boundary triple $\Pi_{[N,j]}^{+}$ of the form (3.25).

Proof. By Proposition 4.8 the resolvent matrix of the operator $\widetilde{A}_{[N,j]}$ corresponding to the boundary triple $\Pi^+_{[N,j]}$ admits the factorization

$$\mathcal{W}^+_{[N,j]}(z) = \widetilde{M}_{N+1}\widetilde{L}_{N+1}\dots\widetilde{M}_{j+1},$$

where $\widetilde{M}_i$ and $\widetilde{L}_i$ are matrices generated by numbers $\widetilde{m}_i$, $\widetilde{l}_i$ defined by (1.16). By [17, Theorem 4.1]

$$\widetilde{m}_i = m_i,\ (i = N+1,\dots,j+1),\quad \widetilde{l}_i = l_i,\ (i = N+2,\dots,j),$$

and the equalities

$$\frac{1}{\widetilde{m}_{N+1}\widetilde{l}_{N+1}} = -\widetilde{a}_N(0) = -a_N(0) - \frac{1}{m_{N+1}l_N} = \frac{1}{m_{N+1}l_{N+1}}$$

imply that $\widetilde{l}_{N+1} = l_{N+1}$. Therefore,

$$\mathcal{W}^+_{[N,j]}(z) = M_{N+1}L_{N+1}\dots M_{j+1}. \tag{4.18}$$

It follows from (4.14) that $\mathcal{W}^+_{[0,j]}(z)$ admits the factorization (4.17), where

$$\mathcal{W}^{++}_{[0,j-1]}(z) = M_1L_1\dots M_jL_j,\quad \mathcal{W}^{++}_{[0,N-1]}(z) = M_1L_1\dots M_NL_N, \tag{4.19}$$

The statement is implied now by (4.18) and (4.19). □

4.4. Truncated indefinite moment problem

Consider the truncated indefinite moment problems $MP_\kappa(\mathbf{s},\ell)$ and $MP^k_\kappa(\mathbf{s},\ell)$. The moment problem $MP_\kappa(\mathbf{s},\ell)$ is called *even* or *odd* regarding to the oddness of the number $\ell+1$ of given moments. The odd moment problem $MP^k_\kappa(\mathbf{s},2n-2)$ is called nondegenerate, if $D_n = \det S_n \neq 0$. The moment problem $MP_\kappa(\mathbf{s},\ell)$ was studied in [13, 23, 39, 40]. Recall the following description of the set $\mathcal{M}_\kappa(\mathbf{s},2n_N-2)$ from [10, Proposition 3.31].

Proposition 4.10. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2n_N-2} \in \mathcal{H}_{\kappa,2n_N-2}$ *and let* $\mathcal{N}(\mathbf{s}) = \{n_i\}_{i=1}^N$ *be the set of normal indices of* $\mathbf{s}$ *and let* $\mathcal{W}_{[0,N-1]} = (w_{ij})_{i,j=1}^2$. *The problem* $MP_\kappa(\boldsymbol{s},2n_N-2)$ *is solvable if and only if* $\kappa_N := \nu_-(S_{n_N}) \le \kappa$ *and the formula*

$$f(z) = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \tag{4.20}$$

establishes a one-to-one correspondence between the class $\mathcal{M}_\kappa(\boldsymbol{s},2n_N-2)$ *and the set of functions* $\tau \in N_{\kappa-\kappa_N}$, *which satisfy the Nevanlinna condition* (4.4).

The operator approach to $MP_\kappa(\mathbf{s}, 2n_N - 2)$ in [39] is based on the formula

$$f(z) = [(\widetilde{A} - zI)^{-1}e_0, e_0]_{[0,N-1]} \tag{4.21}$$

which describes the set $\mathcal{M}_\kappa(\mathbf{s}, 2n_N - 2)$ when $\widetilde{A}$ ranges over the set of all single-valued self-adjoint extensions of $A_{[0,N-1]}$. In fact, in [39] a full moment problem is considered. The description (4.20) is based on this formula and on Theorem 4.1. Another proof of this formula was given in [8, 9] and [10] by applications of the Schur algorithm.

The problem $MP_\kappa^k(\mathbf{s}, 2n_N)$ was posed and solved by the methods of extension theory in [13, Theorem 4.14].

Theorem 4.11. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2n_N-2} \in \mathcal{H}_{\kappa,2n_N-2}^{k,reg}$, $n = n_N$ *and let the matrix* $S_n = (s_{i+j})_{i,j=0}^{n-1}$ *be nondegenerate and* $\kappa_N := \nu_-(S_n) \le \kappa$. *Then the problem* $MP_\kappa(\boldsymbol{s}, 2n - 2)$ *is solvable and the formula*

$$F(z) = [(\widetilde{A} - z)^{-1}u, u]_{[0,N-1]}$$

establishes a one-to-one correspondence between the set of all solutions of the truncated indefinite moment problem $MP_\kappa(\mathbf{s}, 2n_N - 2)$ *and the set of all* u*-resolvents of the symmetric operator* $A_{[0,N-1]}$ *of index* κ *generated by single-valued self-adjoint extensions* $\widetilde{A}$.

If, in addition, the matrix $S_{n_N-1}^+ = (s_{i+j+1})_{i,j=0}^{n_N-2}$ *is nondegenerate and* $k_N := \nu_-(S_{n_N-1}^+) \le k$, *then the problem* $MP_\kappa^k(\boldsymbol{s}, 2n_N - 2)$ *is solvable and the formula* (4.21) *establishes a one-to-one correspondence between the set of all solutions of the truncated indefinite moment problem* $MP_\kappa^k(\boldsymbol{s}, 2n_N - 2)$ *and the set of all* u*-resolvents of the symmetric operator* $A_{[0,N-1]}$ *of index* κ, *such that the extension* $\widetilde{A}$ *has* k *negative squares.*

Proof of Theorem 1.2. Since $\mathbf{s} = \{s_i\}_{i=0}^{2n_N-2} \in \mathcal{H}_{\kappa,2n_N-2}^{k,reg}$, the matrix $S_{n-1}^+ = (s_{i+j+1})_{i,j=0}^{n-2}$ is nondegenerate and $k_N := \nu_-(S_{n-1}^+) \le k$ by Theorem 4.11 the problem $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ is solvable and the formula (4.21) establishes a one-to-one correspondence between the set of all solutions of the truncated indefinite moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ and the set of all u-resolvents of the symmetric operator $A_{[0,N-1]}$ of index $\kappa - \kappa_N$, such that the extension $\widetilde{A}$ has k negative squares.

Consider now the boundary triple $\Pi_{[0,N-1]}^+ = \{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ for the operator $A_{[0,N-1]}$. By Proposition 3.9 the boundary triple $\Pi_{[0,N-1]}^+$ is basic. Therefore, by Theorem 4.11 the set of all u-resolvents in (4.21) is described by the formula

$$[(\widetilde{A} - z)^{-1}u, u]_{[0,N-1]} = T_{\mathcal{W}_{[0,N-1]}^+}[\tau(z)],$$

where $\tau \in \mathbf{N}^{k-k_N}_{\kappa-\kappa_N}$. Moreover, by Theorem 4.1 an u-resolvent $[(\widetilde{A}-z)^{-1}u,u]_{[0,N-1]}$ is generated by a single-valued self-adjoint extensions $\widetilde{A}$ if and only if $\tau(z)=o(1)$ as $z\widehat{\to}\infty$. This completes the proof of Theorem 1.2. □

In the following proposition we recall a description of solutions of the even indefinite Stieltjes moment problem presented in [18].

Theorem 4.12. *Let $\mathbf{s} \in \mathcal{H}^{k,reg}_{\kappa,2n_N-1}$, $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=1}^N$ and let the u-resolvent matrix $\mathcal{W}^{++}_{[0,N-1]}(z)$ be given by* (4.16). *Then:*

(1) A nondegenerate even moment problem $MP^k_\kappa(\mathbf{s},2n_N-1)$ is solvable, iff

$$\kappa_N := \nu_-(S_{n_N}) \le \kappa \quad \textit{and} \quad k_N := \nu_-(S^+_{n_N}) \le k.$$

In this case the set $\mathcal{M}^k_\kappa(\mathbf{s},2n_N-1)$ is parametrized by the formula

$$f(z) = \frac{Q^+_{2N-1}(z)\tau(z) + Q^+_{2N}(z)}{P^+_{2N-1}(z)\tau(z) + P^+_{2N}(z)},$$

where $\tau(z) \in \mathbf{N}^{k-k_N}_{\kappa-\kappa_N}$ *and* $\tau(z) = o(1)$ *as* $z\widehat{\to}\infty$.

(2) The matrix $\mathcal{W}^{++}_{[0,N-1]}(z)$ admits the following factorization

$$\mathcal{W}^{++}_{[0,N-1]}(z) = M_1(z)L_1 \dots M_{N-1}(z)L_{N-1},$$

where the matrices $M_i(z)$, L_i are defined by (4.15).

5. The full indefinite moment problem $MP^k_\kappa(\mathbf{s})$

5.1. Description of solutions

Let us associate with the infinite sequence $\mathbf{s} = \{s_i\}_{i=0}^\infty$ an indefinite inner product space $\mathfrak{H}_{[0,\infty)}$ of infinite sequences endowed with the inner product

$$[x,y]_{[0,\infty)} = (G_{[0,\infty)}x,y), \quad G_{[0,\infty)} = \operatorname{diag}(\widetilde{b}_0E_0^{-1}, \widetilde{b}_1E_1^{-1}, \dots).$$

If $\mathbf{s} \in \mathcal{H}_\kappa$, then the space $\mathfrak{H}_{[0,\infty)}$ is a Pontryagin space (see [6]), with

$$\max\{\operatorname{ind}_-\mathfrak{H}_{[0,\infty)}, \operatorname{ind}_+\mathfrak{H}_{[0,\infty)}\} = \kappa.$$

Associated to the matrix $\mathfrak{J}^T_{[0,\infty)}$ there is a *minimal operator* $A_{\min}$, defined as the closure of the operator $\mathfrak{J}^T_{[0,\infty)}$ restricted originally to the set of finite sequences. The operator $A_{\min}$ turns out to be symmetric in the space $\mathfrak{H}_{[0,\infty)}$.

Alongside with the minimal operator we consider also the *maximal operator* $A_{\max}$, defined as the restriction of $\mathfrak{J}_{[0,\infty)}^T$ to the domain

$$\mathrm{dom}(A_{\max}) := \{x \in \mathfrak{H}_{[0,\infty)} : \mathfrak{J}_{[0,\infty)}^T x \in \mathfrak{H}_{[0,\infty)}\}.$$

As is known [10,39] the operator $A_{\min}$ is self-adjoint in $\mathfrak{H}_{[0,\infty)}$, if and only if $A_{\min} = A_{\max}$. One can easily see that $A_{\max} = A_{\min}^{[*]}$. It follows from the equalities (2.12) that the defect subspace $\mathfrak{N}_z(A_{\min}) := \ker(A_{\max} - zI)$ is either a one dimensional space generated by $\pi(z)$ or is trivial. Combining this observation with Lemma 2.13 one obtains the following

Proposition 5.1. *Let $\boldsymbol{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_\kappa$. Then the following statements are equivalent:*

(1) The moment problem $MP_\kappa(\boldsymbol{s})$ is indeterminate.

(2) The deficiency indices of the operator $A_{\min}$ are equal to one.

(3) The "moment of inertia" of the string is finite

$$\sum_{j=1}^{\infty} x_j^2 m_j(0) < \infty, \quad \textit{where} \quad x_j = l_1 + \cdots + l_j.$$

The results of [39] on indefinite moment problem can be reformulated in notations connected with a monic GJM J as follows.

Theorem 5.2. *Let $\nu_-(S_n) = \kappa_1 \le \kappa$ for all $n \in \mathbb{N}$, $n \ge n_N$, and let the moment problem $MP_\kappa(\boldsymbol{s})$ be indeterminate. Then:*

(1) For all $f \in \mathfrak{H}_{[0,\infty)}$ there exist finite limits

$$W_\infty[f, \pi(\lambda_0)] = \lim_{j\to\infty} W_j[f, \pi(\lambda_0)],$$
$$W_\infty[f, \xi(\lambda_0)] = \lim_{j\to\infty} W_j[f, \xi(\lambda_0)].$$

(2) A boundary triple for $A_{\max}$ can be defined by the equalities

$$\widetilde{\Gamma}_0 \widehat{f} = W_\infty[\widehat{f}, \widehat{\pi}(0)], \quad \widetilde{\Gamma}_1 \widehat{f} = W_\infty[\widehat{f}, \widehat{\xi}(0)]. \tag{5.1}$$

(3) The corresponding resolvent matrix takes the form

$$\widetilde{\mathcal{W}}_{[0,\infty)}(\lambda) = \begin{pmatrix} -W_\infty[\widehat{\xi}(\lambda), \widehat{\pi}(0)] & -W_\infty[\widehat{\xi}(\lambda), \widehat{\xi}(0)] \\ W_\infty[\widehat{\pi}(\lambda), \widehat{\pi}(0)] & W_\infty[\widehat{\pi}(\lambda), \widehat{\xi}(0)] \end{pmatrix}.$$

The mvf $\widetilde{\mathcal{W}}_{[0,\infty)}(z) = (\widetilde{w}_{ij}(z))_{i,j=1}^2$ is entire of minimal exponential type.

(4) The Weyl function of A, corresponding to the boundary triple (5.1), *takes the form*

$$\widetilde{M}_{[0,\infty]}(z) = -\frac{W_\infty[\widehat{\pi}(z), \widehat{\pi}(0)]}{W_\infty[\widehat{\pi}(z), \widehat{\xi}(0)]}.$$

(5) The formula

$$f(z) = \frac{\widetilde{w}_{11}(z)\tau(z) + \widetilde{w}_{12}(z)}{\widetilde{w}_{21}(z)\tau(z) + \widetilde{w}_{22}(z)}$$

establishes a one-to-one correspondence between the class $\mathcal{M}_\kappa(\boldsymbol{s})$ and the set of functions $\tau \in N_{\kappa-\kappa_1}$.

Assume now that $\mathbf{s} = \{s_j\}_{j=0}^\infty \in \mathcal{H}_{\kappa_1}^{k_1}$ and let N be big enough, so that

$$\nu_-(S_j) = \kappa_1 = \nu_-(S_{n_N}), \quad \nu_-(S_j^{(1)}) = k_1 = \nu_-(S_{n_N}^{(1)}) \quad \text{for all} \quad j \geq n_N.$$

Then in view of Theorem 4.9 the indefinite Stieltjes moment problem $\mathcal{M}_\kappa^k(\mathbf{s})$ can be reduced to a classical Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$, where the induced sequence $\mathbf{s}^{(N)}$ can be calculated recursively as in [18, Theorems 3.3, 3.5]. Alternatively, the sequence $\mathbf{s}^{(N)}$ can be found as the sequence of coefficients of the series expansion $-\sum\limits_{i=0}^{\infty} \frac{s_i^{(N)}}{z^{i+1}}$ corresponding to the continued fraction

$$\cfrac{1}{-zm_{N+1} + \cfrac{1}{l_{N+1} + \cfrac{1}{-zm_{N+2} + \cfrac{1}{\ddots}}}}.$$

Notice that the sequence $\mathbf{s}^{(N)}$ belongs to the class $\mathcal{H}_0^0$.

Theorem 5.3. *Let $\boldsymbol{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa_0}^{k_0}$ for some $\kappa_0, k_0 \in \mathbb{N}$. Then the moment problem $MP_\kappa^k(\boldsymbol{s})$ with $\kappa, k \in \mathbb{N}$ is solvable if and only if*

$$\kappa_0 \leq \kappa, \quad \textit{and} \quad k_0 \leq k;$$

and the moment problem $MP_\kappa^k(\boldsymbol{s})$ is indeterminate if and only if

$$\sum_{j=1}^\infty m_j(0) < \infty \quad \textit{and} \quad \sum_{j=1}^\infty l_j < \infty. \tag{5.2}$$

If (5.2) *holds then:*

(1) The sequence of resolvent matrices $\mathcal{W}^{+}_{[0,n]}(z)$ *converges to an entire mvf* $\mathcal{W}^{+}_{[0,\infty)}(z) = (w^{+}_{ij}(z))^{2}_{i,j=1}$ *of minimal exponential type.*

(2) The mvf $\mathcal{W}^{+}_{[0,\infty)}(z)$ *is the resolvent matrix of the operator* $A_{\max}$ *corresponding to the boundary triple*

$$\Gamma^{+}_{0}\widehat{f} = -W_{\infty}[\widehat{f},\widehat{\pi}(0)], \quad \Gamma^{+}_{1}\widehat{f} = -W_{\infty}[\widehat{f},\widehat{\xi}(0)] - LW_{\infty}[\widehat{f},\widehat{\pi}(0)]; \tag{5.3}$$

(3) The formula

$$f(z) = \frac{w^{+}_{11}(z)\tau(z) + w^{+}_{12}(z)}{w^{+}_{21}(z)\tau(z) + w^{+}_{22}(z)}$$

establishes a one-to-one correspondence between the class $\mathcal{M}^{k}_{\kappa}(\boldsymbol{s})$ *and the set of functions* $\tau \in N^{k-k_1}_{\kappa-\kappa_1}$.

Proof. **1.** *Verification of criterion* (5.2)*:* It follows from Proposition 4.7 that there is a one-to-one correspondence between solutions f of the problem $\mathcal{M}^{k_0}_{\kappa_0}(\mathbf{s})$ and solutions φ of the problem $\mathcal{M}^{0}_{0}(\mathbf{s}^{(N)})$ given by LFT

$$f(z) = T_{\mathcal{W}^{++}_{[0,N]}(z)}[\varphi(z)]. \tag{5.4}$$

Therefore, the indefinite moment problem $\mathcal{M}^{k_0}_{\kappa_0}(\mathbf{s})$ is indeterminate if and only if the classical Stieltjes problem $\mathcal{M}^{0}_{0}(\mathbf{s}^{(N)})$ is indeterminate. As is known, see [31, Appendix II.13] the problem $\mathcal{M}^{0}_{0}(\mathbf{s}^{(N)})$ is indeterminate if and only if

$$\sum_{j=N+1}^{\infty} m_j < \infty \quad \text{and} \quad \sum_{j=N+1}^{\infty} l_j < \infty,$$

i.e. if (5.2) holds. Notice that for the classical Stieltjes moment problem $\mathcal{M}^{0}_{0}(\mathbf{s}^{(N)})$ the corresponding masses $m^{(N)}_{j}$ and lengthes $l^{(N)}_{j}$ are constants, which coincide with m_{j+N} and l_{j+N}, respectively.

Now it remains to notice that the set of solutions f of the problem $\mathcal{M}^{k}_{\kappa}(\mathbf{s})$ and the set of solutions φ of the problem $\mathcal{M}^{k-k_0}_{\kappa-\kappa_0}(\mathbf{s}^{(N)})$ are also connected by the LFT (5.4) and since $\mathbf{s}^{(N)} \in \mathcal{H}^{0}_{0}$ the problem $\mathcal{M}^{k-k_0}_{\kappa-\kappa_0}(\mathbf{s}^{(N)})$ is indeterminate if and only if the problem $\mathcal{M}^{0}_{0}(\mathbf{s}^{(N)})$ is indeterminate (see [14]), which leads again to the condition (5.2).

2. *Verification of (1):* The convergence of the sequence of the resolvent matrices $\mathcal{W}^{+}_{[0,n]}(z)$ is implied by Theorem 5.2 and the formula (4.13)

$$\mathcal{W}^{+}_{[0,n]}(z) = \widetilde{\mathcal{W}}_{[0,n]}(z) \begin{pmatrix} -1 & \frac{Q_n(0)}{P_n(0)} \\ 0 & -1 \end{pmatrix},$$

where $\frac{Q_n(0)}{P_n(0)} = -L := -\sum\limits_{j=0}^{n} l_j$ in view of (2.22). In view of the limit mvf $\mathcal{W}^+_{[0,\infty]}(z)$ is connected with $\widetilde{\mathcal{W}}_{[0,\infty]}(z)$ by the equality

$$\mathcal{W}^+_{[0,\infty]}(z) = \widetilde{\mathcal{W}}_{[0,\infty]}(z) \begin{pmatrix} -1 & -L \\ 0 & -1 \end{pmatrix}, \tag{5.5}$$

3. *Verification of (2):* The formulas (5.3) for the triple $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ can be rewritten as

$$\Gamma_0^+ \widehat{f} = -\widetilde{\Gamma}_0 \widehat{f}, \quad \Gamma_1^+ \widehat{f} = -\widetilde{\Gamma}_1 \widehat{f} - L\widetilde{\Gamma}_0 \widehat{f}. \tag{5.6}$$

By Theorem 5.2 this implies that $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ is a boundary triple for $A_{\max}$. Moreover, it follows from (5.6) that the resolvent matrix of $A_{\min}$ corresponding to the boundary triple $\{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ is connected with the resolvent matrix $\widetilde{\mathcal{W}}_{[0,\infty]}(z)$ by the equality (5.5) and hence it coincides with $\mathcal{W}^+_{[0,\infty]}(z)$.

4. *Verification of (3):* The last statement is implied by the formula (5.4) and the description of solutions of the problem $\mathcal{M}^{k-k_0}_{\kappa-\kappa_0}(\mathbf{s}^{(N)})$ given in [14]. $\square$

5.2. Padé approximants

Definition 5.4. *The $[L/M]$ Pade approximant for a formal power series*

$$-\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}} \tag{5.7}$$

is a ratio $f^{[n/k]}(z) = \frac{A^{[n/k]}(1/z)}{B^{[n/k]}(1/z)}$ of polynomials $A^{[n/k]}$, $B^{[n/k]}$ of formal degree n, k, respectively, such that $B^{[n/k]}(0) \neq 0$ and

$$f^{[n/k]}(z) + \sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}} = O\left(\frac{1}{z^{n+k+1}}\right) \quad \text{as } z\widehat{\to}\infty.$$

Explicit formula for diagonal Pade approximants was found in [9]. Here we give another proof of this formula.

Proposition 5.5. *Let $\mathbf{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$. Then the $[n/n]$ Pade approximant for a formal power series (5.7) exists if $n \in \mathcal{N}(\mathbf{s})$ and*

$$f^{[n_j/n_j]}(z) = -\frac{Q_j(z)}{P_j(z)}, \quad j \in \mathbb{N}.$$

Proof. It follows from (2.20) and Proposition 4.8 that the function

$$-\frac{Q_j(z)}{P_j(z)} = \frac{Q^+_{2j}(z)}{P^+_{2j}(z)} = T_{\mathcal{W}^{++}_{[0,j]}(z)}[0] \in \mathcal{M}(\mathbf{s}, 2n_j - 1)$$

belongs to $\mathcal{M}(\mathbf{s}, 2n_j - 1)$. Therefore, the function $-\frac{Q_j(z)}{P_j(z)}$ has the asymptotic

$$-\frac{Q_j(z)}{P_j(z)} = -\frac{s_0}{z} - \cdots - \frac{s_{2n_j-1}}{z^{2n_j}} + O\left(\frac{1}{z^{2n_j+1}}\right) \quad \text{as } z\widehat{\rightarrow}\infty.$$

Next, $A(z) := z^{n_j} Q_j(\frac{1}{z})$, $B(z) := z^{n_j} P_j(\frac{1}{z})$ are polynomials of formal degree n_j and $B(0) = 1$. By Definition 5.4 the function $-\frac{Q_j(z)}{P_j(z)}$ is the $[n_j/n_j]$ Pade approximant for the formal power series (5.7). □

The following formula for sub-diagonal Pade approximants in terms of generalized Stieltjes polynomials can be proved similarly.

Proposition 5.6. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$. *Then the* $[n_j/n_j - 1]$ *Pade approximants for the formal power series* (5.7) *exists and has the form*

$$f^{[n_j/n_j-1]}(z) = \frac{Q^+_{2j-1}(z)}{P^+_{2j-1}(z)}, \quad j \in \mathbb{N}.$$

Appendix.
Relations between monic and symmetric GJM's

The exposition of all results in [10] is based on so-called symmetric GJM's. To make a connection between [10] and the present paper we present in this appendix some formulas which relate symmetric and monic GJM's and the corresponding polynomials of the first and second type.

Recall that the symmetric GJM H associated with the sequence of atoms (a_i, b_i) $(i \in \mathbb{Z}_+)$ was defined in [10] by the formulas

$$H = \begin{pmatrix} C^T_{a_0} & \theta_1 \widehat{B}_1 & & \\ \widehat{B}_1 & C^T_{a_1} & \theta_2 \widehat{B}_2 & \\ & \widehat{B}_2 & C^T_{a_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \text{ where } \widehat{B}_i = \begin{pmatrix} 0 & 0 & \cdots & \sqrt{|b_i|} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{A.1}$$

$\theta_i = \mathrm{sign}(b_i)$ and the entries C_{a_i} are defined by (2.4), $i \in \mathbb{N}$. The shortened symmetric GJM $H_{[0,j]}$ is defined by

$$H_{[0,j]} = \begin{pmatrix} C_{a_0}^T & \theta_1 \widehat{B}_1 & & \\ \widehat{B}_1 & C_{a_1}^T & \ddots & \\ & \ddots & \ddots & \theta_j \widehat{B}_j \\ & & \widehat{B}_j & C_{a_j}^T \end{pmatrix}.$$

Lemma A.1. *Let $\mathfrak{J}$ be a monic GJM, corresponding to a sequence of atoms (a_i, b_i), $i \in \mathbb{Z}_+$, and let the matrix Ψ be defined by*

$$\Psi = \mathrm{diag}(I_{\ell_0}, \sqrt{|b_1|} I_{\ell_1}, \sqrt{|b_1 b_2|} I_{\ell_2}, \dots). \tag{A.2}$$

Then the monic and symmetric GJM's $\mathfrak{J}$ and H are connected by

$$H^T := \Psi^{-1} \mathfrak{J} \Psi.$$

Similarly, the matrices $J_{[0,j]}$ and $H_{[0,j]}$ are connected by the formula

$$\begin{gathered} H_{[0,j]}^T := \Psi_{[0,j]}^{-1} J_{[0,j]} \Psi_{[0,j]}, \\ \Psi_{[0,j]} = \mathrm{diag}(I_{\ell_0}, \sqrt{|b_1|} I_{\ell_1}, \dots, \sqrt{|b_1 \dots b_j|} I_{\ell_j}). \end{gathered}$$

Proof. The equality (A.1) is obtained from (2.6) and (A.2) by direct computations. □

Remark A.2. Let $\widehat{\mathfrak{H}}_{[0,j]}$ be the indefinite inner product space of sequences from $\mathbb{C}^{n_{j+1}}$ endowed with the indefinite inner product

$$[x, y]_{[0,j]} = (\widehat{G}_{[0,j]} x, y), \quad \widehat{G}_{[0,j]} = \mathrm{diag}(\theta_0 E_0, \theta_1 E_1, \dots, \theta_j E_j).$$

where

$$E_{a_i} = \begin{pmatrix} a_1^{(i)} & \cdots & a_{\ell_i - 1}^{(i)} & 1 \\ \vdots & \iddots & \iddots & \\ a_{\ell_i - 1}^{(i)} & \iddots & & \\ 1 & & & 0 \end{pmatrix} \quad i = \overline{0, j}.$$

It follows from the equalities (see [29, Chapter 12])

$$E_a C_a = C_a^* E_a$$

that the matrix $\widehat{G}_{[0,j]} H_{[0,j]}$ is self-adjoint in the standard scalar product in $\mathbb{C}^{n_{j+1}}$, and hence the matrix $H_{[0,j]}$ generates a self-adjoint operator in $\widehat{\mathfrak{H}}_{[0,j]}$.

The polynomials $\widehat{P}_j(z)$ and $\widehat{Q}_j(z)$ of the first and second kind of H were defined in [10] as solutions of the following recurrent relations

$$\theta_j\sqrt{|b_j|}y_{j-1}(z) - a_j(z)y_j(z) + \sqrt{|b_{j+1}|}y_{j+1}(z) = 0 \qquad (b_0 = \varepsilon_0) \quad \text{(A.3)}$$

subject to the initial conditions

$$\widehat{P}_{-1}(z) \equiv 0,\ \widehat{P}_0(z) \equiv 1,\ \widehat{Q}_{-1}(z) \equiv -1,\ \widehat{Q}_0(z) \equiv 0,$$

Lemma A.3. *Let J be the monic GJM and let $P_j(z)$ and $Q_j(z)$ be the Lanzcos polynomials of the first and second kind of J. Let H be the symmetric GJM. Then the polynomials of the first and second kind of H can be found by*

$$\begin{aligned} \widehat{P}_0(z) = P_0(z) \quad &\text{and} \quad \widehat{P}_j(z) = \tfrac{1}{\sqrt{|b_1 \ldots b_j|}} P_j(z), \\ \widehat{Q}_0(z) = Q_0(z) \quad &\text{and} \quad \widehat{Q}_j(z) = \tfrac{1}{\sqrt{|b_1 \ldots b_j|}} Q_j(z). \end{aligned} \qquad \text{(A.4)}$$

Proof. Substituting (A.4) into (A.3) one obtains in view of (1.10) for $j = 0$

$$-a_0(z)\widehat{P}_0(z) + \sqrt{|b_1|}\widehat{P}_1(z) = -a_0(z)P_0 + P_1(z) = 0,$$

and for arbitrary $j \in \mathbb{N}$ the left part of (A.3) takes the form

$$\frac{b_j P_{j-1} - a_j(z)P_j(z) + P_{j+1}(z)}{\sqrt{|b_1 \ldots b_j|}} = 0.$$

The equalities (A.4) for the polynomials $\widehat{Q}_j(z)$ are proved similarly. □

Remark A.4. The Liouville–Ostrogradskii formula (2.9) for polynomials $\widehat{P}_j$ and $\widehat{Q}_j$ takes the form

$$\theta_0 \ldots \theta_j \sqrt{|b_{j+1}|}\left(\widehat{Q}_{j+1}(z)\widehat{P}_j(z) - \widehat{Q}_j(z)\widehat{P}_{j+1}(z)\right) = 1.$$

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