THE CHARACTERISTIC SYSTEM FOR THE EULER - POISSON'S EQUATIONS

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Abstract

In this paper we investigate the nonlinear system naturally connected with the Euler - Poisson's equations. The solutions of this system may be used for description of the singular points to the Euler - Poisson's equations.

The properties of the solutions of the Euler - Poisson's equations ([1], [2]) depend on the singular points of these solutions. For example, the classical case of the S.Kovalevskaya ([3]) was found on the way of investigation of the single-valued solutions. On another hand it is proved ([4],[5]) that the branching of the solutions implies the absence of single-valued first integrals.

We hope that complete information about the singular points will permit to get new results in the solid body problems. In this paper we present in detail the first part of the method to reseach the singular points ([6],[7]). It is solving of the characteristic system naturally appearing from the Euler - Poisson's equations. We emphasize that the conditions of the Euler, Lagrange, Kovalevskaya and Grioli cases appear in this stage of the method and without the investigation of the differential equations.

Let's write the Euler - Poisson's equations in the following form:

$$\begin{cases} A\,\dot{p} = Ap \times p + \gamma \times r \\ \dot{\gamma} = \gamma \times p, \end{cases} \tag{1}$$

here $p = (p_1, p_2, p_3) \in \mathbf{C}^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbf{C}^3$, $Ap = (A_1p_1, A_2p_2, A_3p_3)$, $A_i > 0$, $r = (r_1, r_2, r_3) \in \mathbf{R}^3$.

We use the notation $z(t) = (p(t), \gamma(t))$ too.

Define $\mathbf{C}$-scalar production in $\mathbf{C}^3$: $\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$.

Notate $\|z(t)\| = \langle p, \bar{p}\rangle^{1/2} + \langle \gamma, \bar{\gamma}\rangle^{1/4}$.

We use the circle replacement of indices $\sigma = (1, 2, 3)$ for writing the sumes or products (for example, $\sum_\sigma A_1A_2 = A_1A_2 + A_2A_3 + A_3A_1$, $\prod_\sigma A_1 = A_1A_2A_3$),) and expressions which differ one from another only by the circle replacement of indices ($\dot{\gamma} = \gamma \times p$, can be writed $\dot{\gamma}_1 = p_3\gamma_2 - p_2\gamma_3, \sigma$).

Introduce the notations $B_{ij} = A_i - A_j$, $C_{ij} = 2A_i - A_j$, $D_{ij} = A_i + A_j$ too.

Let $t_* \in \mathbf{C}$ be a singular point of the solution $z(t)$ of the system (1) (i.e. t_* is a singular point of the coordinate functions of $z(t)$). Get rid of the branch in t_*, if any, by the representation $z(t) = \hat{z}\,(\ln(t - t_*))$, where $\hat{z}\,(\tau)$ is single-valued function when Re $\tau \to -\infty$.

The system (1) is transformed into:

$$\begin{cases} A\dot{\hat{p}} = e^{\tau}\ (A\hat{p}\times\ \hat{p}+\ \hat{\gamma}\times r) \\ \dot{\hat{\gamma}} = e^{\gamma}\ (\ \hat{\gamma}\times\ \hat{p}), \end{cases}$$

where the derivative is taken by τ.

In order to make the right part of the equation independent of τ we make replacement of variable again, setting $\tilde{p}\ (\tau) = e^{\tau}\ \hat{p}\ (\tau)$, $\tilde{\gamma}\ (\tau) = e^{2\tau}\ \hat{\tau}\ (\tau)$ and then we have:

$$\begin{cases} A\dot{\tilde{p}} = A\tilde{p}\times\ \tilde{p}+\ \tilde{\gamma}\times r + A\tilde{p} \\ \dot{\tilde{\gamma}} = \tilde{\gamma}\times\ \tilde{p} + 2\tilde{\gamma}\ . \end{cases} \tag{2}$$

In this case the dependence on between the solutions (1) and (2) is expressed by the corelations

$$p(t) = \frac{1}{t-t_*}\ \tilde{p}\ (\ln(t-t_*)), \quad \gamma(t) = \frac{1}{(t-t_*)^2}\ \tilde{\gamma}\ (\ln(t-t_*)) \tag{3}$$

Assertion 1 *The solution $p(t)$, $\gamma(t)$ of the system (1) does not have the singularity in the point t_*, if and only if the corresponding solution (2) by (3) have the asymptotic behaviour $\tilde{p}\ (\tau) \sim \tilde{p}_0\ e^{\tau}$, $\tilde{\gamma}\ (\tau) \sim \tilde{\gamma}_0\ e^{2\tau}$, when $Re\ \tau \to -\infty$.*

Proof. If $\left\|\tilde{z}\ (\tau)\right\|$ isn't separated from zero then we can neglect the quadratic part of (2) in the suitable moment, and the solution $\tilde{z}\ (\tau)$ turns out to be exponently decreasing: $\tilde{p}\sim\ p_0 e^{\tau}, \tilde{\gamma}\sim\ \gamma_0 e^{2\tau}$ if $Re\tau \to -\infty$. According to (3) we have $p \sim p_0, \gamma \sim \gamma_0, t \to t_*$, consequently $(p(t), \gamma(t))$ does not have the singularity in the point t_*.

If $\left\|\tilde{z}\ (\tau)\right\|$ is separated from zero when $Re\tau \to -\infty$ then according (3) $\|z(t)\| \to \infty, t \to t_*$ □

The solution $\tilde{z}\ (\tau)$ which does not have the asymtotic behaviour $(\tilde{p}_0\ e^{\tau}, \tilde{\gamma}_0\ e^{2\tau})$, $Re\ \tau \to -\infty$, are first, the constant solutions and, second, have trajectories entering singular points.

This fact is fundamental: we can completely investigate the singular points of the differential equation but at the same time we cannot say that all singular points of the solution of (1) can be obtained in such a way.

Definition 1 *We call the system (the solution of which are singular points of (2))*

$$\begin{cases} A\overset{\sim 0}{p}\times\ \overset{\sim 0}{p}+\ \overset{\sim 0}{\gamma}\times r + A\overset{\sim 0}{p} = 0 \\ \overset{\sim 0}{\gamma}\times\ \overset{\sim 0}{p} + 2\overset{\sim 0}{\gamma} = 0. \end{cases} \tag{4}$$

characteristic (for Euler - Poisson's equations).

Now for convenience we write (p, γ) instead of $(\overset{\sim 0}{p}, \overset{\sim 0}{\gamma})$ in this paragraph.

Assertion 2 *If all A_i are different, $r_1r_2r_3 \neq 0$, then characteristic system is equivalent to two systems*

$$\begin{cases} Ap \times p + Ap = 0 \\ \gamma = 0. \end{cases}, \tag{5}$$

$$\begin{cases} \langle Ap, Ap\rangle = -\langle Ap, r\rangle^2 \langle p, r\rangle^{-2} \\ \langle Ap, p\rangle = -2\langle Ap, r\rangle \langle p, r\rangle^{-1} \\ \langle p, p\rangle = -4 \\ \gamma = -(Ap \times p)\langle p, r\rangle^{-1}, p \times \gamma \neq -2\gamma. \end{cases} \tag{6}$$

Proof. Necessity. Let's obtain the following relations from (4):

$$0 = \langle \gamma \times p, \gamma\rangle + 2\langle \gamma, \gamma\rangle = 2\langle \gamma, \gamma\rangle,$$

$$0 = \langle \gamma \times p, p\rangle + 2\langle \gamma, p\rangle = 2\langle \gamma, p\rangle,$$

$$0 = \langle Ap \times p, \gamma\rangle + \langle \gamma \times r, \gamma\rangle + \langle Ap, \gamma\rangle = \langle p \times \gamma, Ap\rangle + \langle Ap, \gamma\rangle = 3\langle Ap, \gamma\rangle,$$

then we see that vectors Ap, p, γ are linearly dependent.

If p and Ap are proportional then two from three coordinates of p (or Ap) are equal to zero. In this case we should have $\gamma \times r + Ap = 0 \Rightarrow \langle Ap, r\rangle = 0$, but it is impossible because all coordinates of r are not equal to zero by condition. So, $\gamma = \nu_1 Ap + \nu_2 p$; multiply this equivalence by Ap and p.

$$\begin{cases} \nu_1\langle Ap, Ap\rangle + \nu_2\langle Ap, p\rangle = 0 \\ \nu_1\langle Ap, p\rangle + \nu_2\langle p, p\rangle = 0 \end{cases}$$

Now we have $\nu_1 = \nu_2 = 0$ and the system (5) is true; or we have $\langle Ap, Ap\rangle\langle p, p\rangle = \langle Ap, p\rangle^2$, where $\langle p, p\rangle = -4$, ($0, \pm\sqrt{-\langle p, p\rangle}$ is eigenvalue of the eigenvector γ of the linear transformation $\xi \to p \times \xi$)

On the one hand we have $\langle \gamma, p\rangle = \langle \gamma, Ap\rangle = 0$, therefore $\gamma = \nu Ap \times p$. On the other hand we have $p \times \gamma = 2\gamma$ and $\gamma = \nu_1 Ap + \nu_2 p$, thus, $\nu = -\frac{\nu_1}{2}$ and $Ap \times p = -2Ap - \frac{\nu_2}{\nu_1}p$ ($\nu_1 \neq 0$, because in other case $\gamma = 0$) .

Substitute $Ap \times p$ in (4) by this presentation expression and multiply by $\nu_1 r$. We obtain the relation

$$-\nu_1\langle Ap, r\rangle - 2\nu_2\langle p, r\rangle = 0.$$

Add to the obtained equation

$$\nu_1\langle Ap, p\rangle + \nu_2\langle p, p\rangle = 0,$$

then we get

$$-2\langle Ap, r\rangle = \langle p, r\rangle\langle Ap, p\rangle.$$

At last multiply the first equation of characteristic system by p having γ as linear combination of Ap p . It will give

$$\nu_1\langle p \times Ap, r\rangle + \langle Ap, p\rangle = 0$$

or by substitution $p \times Ap$ from (4) we have

$$\nu_1\langle Ap, r\rangle + \langle Ap, p\rangle = 0.$$

Now it is evident that $\nu_1 = 2\langle p, r\rangle^{-1}$, and $\gamma = -(Ap \times p)\langle p, r\rangle^{-1}$. Note that $\langle p, r\rangle \neq 0$ because in other case $\langle Ap, r\rangle = 0$, and then $r \times \gamma = 0$ and $\langle \gamma, \gamma\rangle \neq 0$ that is the contradiction to the condition.

Sufficiency. If p, γ is the solution of the system (5) then p, γ is the solution of (4) too.

Now let p, γ be the solution of the system (6). Then vector $\xi_0 = \langle p, p\rangle Ap - \langle Ap, p\rangle p$ is normal to p and Ap which are not proportional (because of $\gamma = -(Ap \times p)\langle p, r\rangle^{-1} \neq 0$). The vectors $\xi_0, p \times \xi_0$ are proportional to γ and, therefore, ξ_0, γ are eigenvectors of the linear transformation $\xi_0 \to p \times \xi_0$, namely, $p \times \xi_0 = 2\xi_0, p \times \gamma = 2\gamma$. Using the expression for ξ_0 in the last equation we have

$$Ap \times p = -2p - \frac{1}{2}\langle Ap, p\rangle p \;\Rightarrow$$

$$\langle Ap \times p + \gamma \times r + Ap, r\rangle = -\langle Ap, r\rangle - \frac{1}{2}\langle Ap, p\rangle \langle p, r\rangle = 0.$$

It is evident that $\langle Ap \times p + \gamma \times r + Ap, \gamma\rangle = 0$.

We want to prove that $\langle Ap \times p + \gamma \times r + Ap, p\rangle = 0$. This equality is equivalent to

$$\langle \gamma \times r, p\rangle + \langle Ap, p\rangle = 0,$$

$$2\langle \gamma, r\rangle + \langle Ap, p\rangle = 0,$$

but $\gamma = -(Ap \times p)\langle p, r\rangle^{-1}$, therefore,

$$-2\langle Ap \times p, r\rangle + \langle Ap, p\rangle \langle p, r\rangle = -2\langle Ap \times p, r\rangle - 2\langle Ap, r\rangle = 0,$$

So if the vectors r, γ, p are linear independent then the assertion is proved.

Let $\langle p \times \gamma, r\rangle = 2\langle \gamma, r\rangle = 0$; then this condition is equivalent to $\langle Ap \times p, r\rangle = 0$ or $\langle Ap, r\rangle = 0$.

In this case $\langle Ap, Ap\rangle = \langle Ap, p\rangle = 0$, consequently $Ap \times \gamma = 0 \Rightarrow Ap \times p = -2Ap$. The vector r equals to linear combination of p and Ap. Let $r = \mu_1 Ap + \mu_2 p$,
$\mu_2 \neq 0$. Then

$$Ap \times p + \gamma \times r + Ap = 0 \Leftarrow \gamma \times r = Ap \Leftarrow 2\langle p, r\rangle^{-1} Ap \times (\mu_1 Ap + \mu_2 p) = Ap \Leftarrow$$

$$\frac{2\mu_2 Ap \times p}{\mu_2 \langle p, p\rangle} = Ap$$

So the assertion is proved. □

Assertion 3 *The solution of the system (5) is as follows:*

) $p = 0$;

) (if all A_i are different)

$$p_1 = \sqrt{\frac{A_2 A_3}{B_{12} B_{31}}}, \sigma,$$

here if (p_1, p_2, p_3) is solution of (5) then other solutions are

$$(-p_1, -p_2, p_3), (-p_1, p_2, -p_3), (p_1, -p_2, -p_3).$$

Proof. The solution $z = 0$ is evident. The other solutions we find by transformation the system $\langle Ap, p\rangle = \langle Ap, Ap\rangle = 0$ to a non-homogeneous one. □

Assertion 4 *The solution of the system (6) (all A_i are different, $r_1 r_2 r_3 \neq 0$) may be found if the solution of the equation*

$$\sum_\sigma r_1(A_1-\alpha)\sqrt{(2A_2-\alpha)(2A_3-\alpha)B_{23}}=0 \tag{7}$$

or the equation of the 8th power

$$\sum_\sigma [r_1^4 B_{23}^2 (A_1-\alpha)^4 (2A_2-\alpha)^2 (2A_3-\alpha)^2 -$$

$$2r_2^2 r_3^2 B_{12} B_{31} (A_2-\alpha)^2 (A_3-\alpha)^2 (2A_1-\alpha) \prod_\sigma (2A_1-\alpha)] = 0. \tag{8}$$

is known.

PROOF. Let $\alpha = \langle Ap, r\rangle \langle p, r\rangle^{-1}$. Then the system (6) has the form

$$\begin{cases} \langle Ap, Ap\rangle = -\alpha^2 \\ \langle Ap, p\rangle = -2\alpha \\ \langle p, p\rangle = -4. \end{cases}$$

This system is linear as to p_i^2. Its solution is

$$p_1^2 = \frac{(2A_2-\alpha)(2A_3-\alpha)}{B_{12}B_{31}}, \sigma \tag{9}$$

Let's recall what is α. We obtain the equation

$$\sum_\sigma A_1 r_1 \sqrt{\frac{(2A_2-\alpha)(2A_3-\alpha)}{B_{12}B_{31}}} = \alpha \sum_\sigma r_1 \sqrt{\frac{(2A_2-\alpha)(2A_3-\alpha)}{B_{12}B_{31}}},$$

which equals to (7).

To receive the polynomial by α we use the identity

$$(\sum_\sigma a_1) \prod_\sigma (a_1 - a_2 - a_3) = \sum_\sigma (a_1^4 - 2a_2^2 a_3^2).$$

□

Now we propose another method for solving the characteristic system which gives a number of impotant relations.

Assertion 5 *The solutions z of the characteristic system satisfy the following relations*

$$\begin{cases} \mathcal{H} = \frac{1}{2}\langle Ap, p\rangle + \langle \gamma, r\rangle = 0 \\ \mathcal{M} = \langle Ap, \gamma\rangle = 0 \\ \mathcal{T} = \langle \gamma, \gamma\rangle = 0 \\ \mathcal{D} = \langle p, \gamma\rangle = 0 \\ \mathcal{E} = \sum_\sigma [r_1 B_{23} \gamma_2 \gamma_3 (C_{21} B_{31} \gamma_3^2 - C_{31} B_{12} \gamma_2^2)] = 0 \end{cases}$$

the last relation takes place if is not true

$$\mathcal{E}_0 = \langle A\gamma, \gamma\rangle = 0.$$

PROOF. The relations $\mathcal{T}=0, \mathcal{M}=0, \mathcal{D}=0$ were obtained in the proof of the assertion 2.

$$0=\langle Ap\times p+\gamma\times r+Ap,p\rangle=2\mathcal{H}.$$

Lets prove the last relation. Note that

$$p=\frac{2A\gamma\times\gamma}{\langle A\gamma,\gamma\rangle},\quad \text{if } \langle A\gamma,\gamma\rangle\neq 0. \tag{10}$$

Indeed, it is clear that $p=\lambda A\gamma\times\gamma$ because $A\gamma$ and γ are not proportional and $\langle p,\gamma\rangle=\langle p,A\gamma\rangle=0$. Moreover, $-4=\langle p,p\rangle=\lambda\langle p,A\gamma\times\gamma\rangle=-2\lambda\langle A\gamma,\gamma\rangle$.

Substitute p in (4)

$$\frac{4\gamma_1\prod_\sigma(B_{12}\gamma_3)}{\langle A\gamma,\gamma\rangle^2}+r_3\gamma_2-r_2\gamma_3+\frac{2A_1B_{23}\gamma_2\gamma_3}{\langle A\gamma,\gamma\rangle}=0,\sigma.$$

Then by multiplying these equations by r_1,σ and adding them we obtain the relation which is equivalent the relation $\mathcal{E}=0$.

$$4\prod_\sigma(B_{12}\gamma_3)\sum_\sigma(\gamma_1r_1)+2\sum_\sigma(A_1B_{23}r_1\gamma_2\gamma_3)\sum_\sigma(A_1\gamma_1^2)=0. \tag{11}$$

Then we obtain

$$\sum_\sigma[2B_{12}B_{23}B_{31}\gamma_2\gamma_3(-\gamma_2^2-\gamma_3^2)+A_1B_{23}\gamma_2\gamma_3(-B_{12}\gamma_2^2+B_{31}\gamma_3^2)]r_1=0$$

$$\sum_\sigma r_1B_{23}\gamma_2\gamma_3[(A_1B_{31}-2B_{12}B_{31})\gamma_3^2-(A_1B_{12}+2B_{12}B_{31})\gamma_2^2]=0$$

$$\sum_\sigma r_1B_{23}\gamma_2\gamma_3(C_{21}B_{31}\gamma_3^2-C_{31}B_{12}\gamma_2^2)=0.$$

□

So, we have the following method for solving the characteristic system:
at first to find the vector $\lambda\gamma,(\lambda\in\mathbf{C})$, as the solution of the system

$$\begin{cases} \sum_\sigma\gamma_1^2=0 \\ \sum_\sigma r_1B_{23}\gamma_2\gamma_3(C_{21}B_{31}\gamma_3^2-C_{31}B_{12}\gamma_2^2)=0, \end{cases} \tag{12}$$

then to find p,

$$p=\frac{2A\gamma\times\gamma}{\langle A\gamma,\gamma\rangle}$$

and, finally, to find γ , using any non-homogeneous relation of the characteristic system.

Remark 1 *By condition $r_3=0$ the characteristic system (4) has symmetry*

$$S_3:(p_1,p_2,p_3,\gamma_1,\gamma_2,\gamma_3)\longleftrightarrow(-p_1,-p_2,p_3,\gamma_1,\gamma_2,-\gamma_3). \tag{13}$$

Theorem 1 *The characteristic system (4) has following solutions*
0) $(p,\gamma)=0$;
1) by condition $\prod_\sigma B_{12}\neq 0$

$$\gamma_1=0,\ p_1=\sqrt{\frac{A_2A_3}{B_{12}B_{31}}},\sigma$$

here, if (p_1, p_2, p_3) *is the solution of (5) then the other solutions are*

$$(-p_1, -p_2, p_3), (-p_1, p_2, -p_3), (p_1, -p_2, -p_3),$$

moreover,

a) by condition $\prod_\sigma r_1 \neq 0$*, there are 8 solutions (taking into account the multiplisity of the roots) which can be obtained (see (6), (8), (9)) if we know the roots of the polinomial (8);*

b) by condition $r_3 = 0, r_1 r_2 \neq 0$*, there are 2 solutions lying on the axes of symmetry* S_3 *(see (13))*

$$p_1 = p_2 = \gamma_3 = 0,\ \ p_3 = \pm 2i,\ \ \gamma_1 = \frac{2A_3}{r_1 \pm ir_2},\ \ \gamma_2 = \pm\frac{2A_3 i}{r_1 \pm ir_2}; \tag{14}$$

b$_1$*) on condition* $r_1^2 B_{23} = r_2^2 B_{31}$ *(Grioli's case), there is a pair os the* S_3- *symmetric solutions*

$$p_1 = \frac{\mp 2A_2 i}{D_{12}B_{12}}\sqrt{A_1^2\frac{B_{23}}{B_{31}} + A_2^2},\ \ p_2 = \frac{\mp 2A_1 i}{D_{12}B_{12}}\sqrt{A_1^2 + A_2^2\frac{B_{31}}{B_{23}}},\ \ p_3 = \frac{2A_1A_2}{D_{12}\sqrt{B_{23}B_{31}}},$$

$$\gamma_1 = \frac{2A_1^2A_2}{D_{12}^2 r_1},\ \ \gamma_2 = \frac{2A_1A_2^2}{D_{12}^2 r_2},\ \ \gamma_3 = \frac{\pm 2A_1A_2 i}{D_{12}^2\ \ r_1}\sqrt{A_1 + A_2^2\frac{B_{31}}{B_{23}}};$$

b$_2$*) by condition* $r_1^2 B_{23} \neq r_2^2 B_{31}$ *there are 3 pairs* S_3 *– symmetric solutions which can be obtained as follows: we find the roots* $\frac{\gamma_1}{\gamma_2}$ *of the polinomial of the 3rd power*

$$r_2 B_{31}\frac{\gamma_1}{\gamma_2}\left(A_2 B_{31}\frac{\gamma_1^2}{\gamma_2^2} - C_{12}B_{23}\right) + r_1 B_{23}\left(C_{21}B_{31}\frac{\gamma_1^2}{\gamma_2^2} - A_1 B_{23}\right) = 0,$$

and then we use the relations

$$\frac{\gamma_2}{\gamma_3} = \sqrt{1 - \frac{\gamma_1^2}{\gamma_2^2}}, p = \frac{2A\gamma\times\gamma}{\langle A\gamma, \gamma\rangle}, \gamma = \frac{-Ap\times p}{\langle p, r\rangle};$$

c) by condition $r_2 = r_3 = 0, r_1 \neq 0$ *there are 4 solutions lying on the axises of* S_2, S_3 *– symmetries (see (13))*

$$p_1 = p_2 = \gamma_3 = 0,\ \ p_3 = \pm 2i,\ \ \gamma_1 = \frac{2A_3}{r_1},\ \ \gamma_2 = \pm\frac{2A_3 i}{r_1}, \tag{15}$$

$$p_1 = p_3 = \gamma_2 = 0,\ \ p_2 = \pm 2i,\ \ \gamma_1 = \frac{2A_2}{r_1},\ \ \gamma_3 = \mp\frac{2A_2 i}{r_1} \tag{16}$$

and the 4 S_2, S_3 *-symmetric solutions*

$$p_1 = \frac{\sqrt{C_{21}B_{31}}\sqrt{C_{31}B_{12}}}{B_{12}B_{31}},\ \ p_2 = \frac{\sqrt{A_1B_{23}}\sqrt{C_{31}B_{12}}}{B_{12}B_{23}},\ \ p_3 = \frac{\sqrt{C_{21}B_{31}}\sqrt{A_1B_{23}}}{B_{23}B_{31}},$$

$$\gamma_1 = \frac{A_1}{r_1},\ \ \gamma_2 = \frac{A_1\sqrt{C_{21}B_{31}}}{r_1\sqrt{A_1B_{23}}},\ \ \gamma_3 = \frac{A_1\sqrt{C_{31}B_{12}}}{r_1\sqrt{A_1B_{23}}},$$

here the signs $\sqrt{C_{21}B_{31}}, \sqrt{C_{31}B_{12}}, \sqrt{A_1B_{23}}$ *are taken freely but equally in all the formulas;*

d) by condition $r = 0$ *(Euler's case), other solutions are absent;*

2) by condition $A_1 = A_2, r_2 = 0,$

a) by condition $r_1 \neq 0$

$$p_1 = p_3 = \gamma_2 = 0, p_2 = \pm 2i, \gamma_1 = \frac{\pm 2A_1 i}{r_3 \pm ir_1}, \gamma_3 = \frac{\pm 2A_1}{r_3 \pm ir_1};$$

a_1*) by condition* $C_{31} \neq 0$

$$p_1 = \mp \frac{2A_3 r_3 i}{C_{31} r_1}, p_2 = \frac{2A_3 r_3}{C_{31} r_1}, p_3 = \pm 2i, \gamma_1 = \frac{2A_3}{r_1}, \gamma_2 = \pm \frac{2A_3 i}{r_1}, \gamma_3 = 0; \tag{17}$$

a_2*) by condition* $C_{31} = 0$

$a_{2.1}$*) by condition* $r_3 \neq 0$*, other solutions are absent;*

$a_{2.2}$*) by condition* $r_3 = 0$ *(Kovalevskaya's case), an one-parameter set of the points*

$$\gamma_1 = \frac{A_1}{r_1}, \gamma_2 = \pm \frac{A_1 i}{r_1}, \gamma_3 = 0, p_3 = \pm 2i, p_2 = \pm p_1 i; \tag{18}$$

b) by condition $r_1 = 0, r_3 \neq 0$ *(Lagrange's case), an one-parameter set of the points*

$$\gamma_3 = \frac{2A_1}{r_3}, p_3 = 0, p_2 = \frac{\gamma_1 r_3}{A_1}, p_1 = -\frac{\gamma_2 r_3}{A_1}, \gamma_1^2 + \gamma_2^2 = -4\frac{A_1^2}{r_3^2};$$

c) by condition $r = 0$ *(confluent Euler's case) any solutions are absent. (the above mentioned cases see in [1], [2]).*

PROOF. 1. Is respected to assertion 3.

1. The coefficient at α^8 in (8) equals

$$\sum_\sigma (r_1^4 B_{23}^2 - 2r_2^2 r_3^2 B_{12} B_{31}) = (\sum_\sigma r_1 \sqrt{B}_{23}) \prod_\sigma (r_1 \sqrt{B}_{23} - r_2 \sqrt{B}_{31} - r_3 \sqrt{B}_{12}).$$

This coefficient isn't equal to zero because all A_i are different and $r_1 r_2 r_3 \neq 0$. Hence there exist exactly 8 roots of the equation (8). Suppose that these roots are different. Every root α_0 corresponds to 4 pairs $(p, -p)$ of the representation (9). But only one pair satisfies the condition $\alpha_0 \langle p, r \rangle = \langle Ap, r \rangle$.

In fact, if $p_i = 0$ for some i , then by (9) $p_j = 0$ for some another j and then Ap, p are proportional. According to proof of assertion 2 it is impossible if $r_1 r_2 r_3 \neq 0$. So all p_i are not equal to zero.

The left part of (8) we can represent in the form (preliminary fixing the branches for the expressions $\sqrt{(2A_1 - \alpha)(2A_2 - \alpha) B_{12}, \sigma}$)

$$\mathcal{P}_8(\alpha) = (\sum_\sigma r_1 (A_1 - \alpha) \sqrt{(2A_2 - \alpha)(2A_3 - \alpha) B_{23}} \prod_\sigma [r_1 (A_1 - \alpha) \sqrt{(2A_2 - \alpha)(2A_3 - \alpha) B_{23}}$$

$$-r_2 (A_2 - \alpha) \sqrt{(2A_3 - \alpha)(2A_1 - \alpha) B_{31}} - r_3 (A_3 - \alpha) \sqrt{(2A_1 - \alpha)(2A_2 - \alpha) B_{12}}] \tag{19}$$

Since $p_i \neq 0$ then $\mathcal{P}_8(\alpha)'_{\alpha=\alpha_0} \neq 0$ if and only if only one factor of (19) equals to zero, when $\alpha = \alpha_0$. But we supposed that all roots of (8) are different, hence only one pair $(-p, p)$ satisfies the condition $\alpha_0 \langle p, r \rangle = \langle Ap, r \rangle$.

The condition $p \times \gamma = 2\gamma$ determines the only one possible p. So there exists one to one correspondence between roots of the polynomial $\mathcal{P}_8(\alpha)$ and roots of the characteristic systems.

Suppose now that the polynomial $\mathcal{P}_8(\alpha)$ has the multiple root. Remined the definition of the multiplicity of the root α_0: it is number of the roots α_0' of the polynomial $\mathcal{P}_8(\alpha)+\epsilon$ ($\epsilon \approx 0$) near the α_0. Thus it is sufficiently to prove the correspondence between roots of (6) and (8) in generel case but it is already done.

1.b. **Lemma**. Let $\prod_\sigma B_{12} \neq 0$. Then all solutions $(p, \gamma), \gamma \neq 0$ of the characteristic system have following form:

$$p = \frac{2A\gamma' \times \gamma'}{\langle A\gamma', \gamma' \rangle}, \gamma = \lambda\gamma',$$

where γ' – is a root of the system (12) such that $\langle A\gamma', \gamma' \rangle \neq 0$, λ – is some constant.

Proof of lemma. Let (p, γ) be a solution of the characteristic system. We want to prove that $\langle A\gamma, \gamma \rangle \neq 0$,

Indeed, if $\langle \gamma, \gamma \rangle = 0$, then $A\gamma \times \gamma \neq 0$. But in this case p is proportional to γ because $\langle \gamma, \gamma \rangle = \langle A\gamma, \gamma \rangle = 0 = \langle \gamma, p \rangle = \langle A\gamma, p \rangle$, consequently, $2\gamma = p \times \gamma = 0$. We have contradiction with the condition $\gamma \neq 0$, hence $\langle A\gamma, \gamma \rangle \neq 0$.

So, if (p, γ) is a root of the characteristic system, $\gamma \neq 0$ then (γ, p) satisfy (12) and (10) .

Let now that γ' be a solution of (12) and $\langle A\gamma', \gamma' \rangle \neq 0$. We take

$$p = \frac{2A\gamma' \times \gamma'}{\langle A\gamma', \gamma' \rangle}.$$

Second equation of the system (12) can be represented in the form

$$\langle Ap \times p, r \rangle + \langle Ap, r \rangle = 0$$

or

$$\langle Ap \times p + \gamma' \times r + Ap, r \rangle = 0.$$

We have too

$$\langle Ap \times p + \gamma' \times r + Ap, \gamma' \rangle = 0,$$

because $Ap \times p$ is propotional γ'. The vectors r, γ' are non-proportional therefor

$$Ap \times p + \gamma' \times r + Ap = \mu\gamma' \times r;$$

and taking $\lambda = 1 - \mu$, we get

$$Ap \times p + \lambda\gamma' \times r + Ap = 0.$$

Second equation of the characteristic system is satisfied by $(p, \lambda\gamma')$:

$$p \times \lambda\gamma' = \frac{2A\gamma' \times \gamma'}{\langle A\gamma', \gamma' \rangle} \times \lambda\gamma' =$$

$$\frac{\lambda}{\langle A\gamma', \gamma' \rangle}[\langle 2A\gamma', \gamma' \rangle \gamma' - 2 \langle \gamma', \gamma' \rangle A\gamma'] = 2\lambda\gamma'.$$

The lemma is proved.

So let $\prod_\sigma B_{12} \neq 0, r_1 r_2 \neq 0, r_3 = 0$. In this case (12) has the form (if we substitute $\gamma_3^2 = -\gamma_1^2 - \gamma_2^2$):

$$\begin{cases} r_1 B_{23}\gamma_2\gamma_3(C_{21}B_{31}\gamma_1^2 - A_1 B_{23}\gamma_2^2) - r_2 B_{31}\gamma_3\gamma_1(C_{12}B_{23}\gamma_2^2 - A_2 B_{31}\gamma_1^2) = 0 \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0. \end{cases} \quad (20)$$

We see the root $\gamma_3 = 0$, hence $\gamma = \lambda(1, \pm i, 0)$ and by (10) $p_1 = p_2 = 0, p_3 = \pm 2i$. Then we have $2A_3 = -\frac{1}{2}\langle Ap, p\rangle = \langle \gamma, r\rangle = \lambda(r_1 \pm ir_2)$, and

$$\gamma_1 = \frac{2A_3}{r_1 \pm ir_2}, \gamma_2 = \frac{\pm 2A_3 i}{r_1 \pm ir_2}, \gamma_3 = 0$$

Let now $\gamma_3 \neq 0$.

1b$_1$) $\langle A\gamma, \gamma\rangle = 0$ is realized for some root γ of system (12). Then $\langle \gamma, r\rangle = 0$ (see (11)) and $r_1^2 B_{23} = r_2^2 B_{31}$.

In this case $\gamma_1 r_1 + \gamma_2 r_2 = \gamma_1 r_2 B_{31} + \gamma_2 r_1 B_{23}$. must devide first equation of system (20). As a result we get

$$r_2 A_2 B_{31}^2 \gamma_1^3 + r_1 B_{23} B_{31} C_{21} \gamma_1^2 \gamma_2 - r_2 B_{23} B_{31} C_{12} \gamma_1 \gamma_2^2 - r_1 A_1 B_{23}^2 \gamma_2^3 =$$

$$(B_{31} r_2 \gamma_1 + B_{23} r_1 \gamma_2)^2 (A_2 r_2 B_{31} \gamma_1 - A_1 r_1 B_{23} \gamma_2) r_2^{-2} B_{31}^{-1}.$$

So, according the lemma, by condition $\langle A\gamma', \gamma'\rangle \neq 0$ the unique solution of (12) (within proportionality) is

$$\gamma' = (A_1 B_{23} r_1, A_2 B_{31} r_2, \pm i r_1 \sqrt{(A_1^2 B_{23} + A_2^2 B_{31}) B_{23}})\ ,$$

$$p_1 = \pm \frac{2B_{23} A_2 B_{31} r_1 r_2 i \sqrt{(A_1^2 B_{23} + A_2^2 B_{31}) B_{23}}}{B_{23} B_{31} (-A_1^2 B_{23} r_1^2 + A_2^2 B_{31} r_2^2)} = \mp \frac{2A_2 i}{(A_1 + A_2) B_{12}} \sqrt{A_1^2 \frac{B_{23}}{B_{31}} + A_2^2}\ ,$$

$$p_2 = \pm \frac{2B_{31} A_1 B_{23} r_1^2 i \sqrt{(A_1^2 B_{23} + A_2^2 B_{31}) B_{23}}}{B_{23} B_{31} (-A_1^2 B_{23} r_1^2 + A_2^2 B_{31} r_2^2)} = \mp \frac{2A_1 i}{(A_1 + A_2) B_{12}} \sqrt{A_1^2 + A_2^2 \frac{B_{31}}{B_{23}}}\ ,$$

$$p_3 = \frac{2A_1 A_2 B_{12} B_{23} B_{31} r_1 r_2}{B_{23} B_{31} (-A_1^2 B_{23} r_1^2 + A_2^2 B_{31} r_2^2)} = \frac{2A_1 A_2}{(A_1 + A_2)\sqrt{B_{23} B_{31}}}.$$

By assertion 2 the solution of characteristic system satisfies (5) or (6) if Ap and p are non-proportional. Hence $\langle \gamma, r\rangle = -\frac{1}{2}\langle Ap, p\rangle = \alpha$, where α is root of (7)

$$r_1(A_1 - \alpha)\sqrt{(2A_2 - \alpha)(2A_3 - \alpha) B_{23}} = -r_2 (A_2 - \alpha)\sqrt{(2A_3 - \alpha)(2A_1 - \alpha) B_{31}} \Leftrightarrow$$

$$\overset{\alpha \neq 2A_3}{\Longleftrightarrow} r_1^2 (A_1 - \alpha)^2 (2A_2 - \alpha) B_{23} = r_2^2 (A_2 - \alpha)^2 (2A_1 - \alpha) B_{31} \Leftrightarrow$$

$$\Leftrightarrow \alpha = \frac{2A_1 A_2}{A_1 + A_2}.$$

The root $\alpha = 2A_3$ corresponds to the solution (14), consequently $\alpha = 2A_1 A_2 (A_1 + A_2)^{-1}$ corresponds to the solution which we consider now. Lt $\gamma = \lambda \gamma'$.

$$2A_1 A_2 (A_1 + A_2)^{-1} = \lambda \langle \gamma', r\rangle = \lambda (A_1 B_{23} r_1^2 + A_2 B_{31} r_2^2) \Rightarrow \lambda = \frac{2A_1 A_2}{(A_1 + A_2)^2 r_1^2 B_{23}},$$

consequently,

$$\gamma_1 = \frac{2A_1^2 A_2}{(A_1 + A_2)^2 r_1}, \gamma_2 = \frac{2A_1 A_2^2}{(A_1 + A_2)^2 r_2}, \gamma_3 = \pm \frac{2A_1 A_2}{(A_1 + A_2)^2 r_1} \sqrt{A_1^2 + A_2^2 \frac{B_{31}}{B_{23}}}\ ,$$

1b$_2$. It follows from the lemma.

1c. Two solutions of (12) are evident:

$$\gamma_3 = 0 \Rightarrow \gamma = \lambda(1, \pm i, 0) \Rightarrow p_1 = p_2 = 0, p_3 = \pm 2i \Rightarrow \lambda r_1 = -\frac{1}{2}\langle Ap, p\rangle = 2A_3 \Rightarrow$$

$$\gamma_1 = \frac{2A_3}{r_1}, \gamma_2 = \pm\frac{2A_3 i}{r_1}, \gamma_3 = 0,$$

$$\gamma_2 = 0 \Rightarrow \gamma = \lambda(1, 0, \pm i) \Rightarrow p_1 = p_3 = 0, p_2 = \pm 2i \Rightarrow \lambda r_1 = -\frac{1}{2}\langle Ap, p\rangle = 2A_2 \Rightarrow$$

$$\gamma_1 = \frac{2A_2}{r_1}, \gamma_3 = \pm\frac{2A_2 i}{r_1}, \gamma_2 = 0,$$

Moreover, $C_{21}B_{31}\gamma_3^2 - C_{31}B_{12}\gamma_2^2 = 0$ (see (12)) $\Rightarrow$

$$\gamma = \lambda(\sqrt{A_1B_{23}}, \sqrt{C_{21}B_{31}}, \sqrt{C_{31}B_{12}})$$

; and we see that

$$\langle A\gamma, \gamma\rangle = \lambda^2(A_1^2B_{23} + A_2B_{23}C_{21} + A_3B_{12}C_{31}) = -2\lambda^2\prod_\sigma B_{12}.$$

$$p_1 = \frac{\sqrt{C_{21}B_{31}}\sqrt{C_{31}B_{12}}}{-B_{12}B_{31}}, p_2 = \frac{\sqrt{A_1B_{23}}\sqrt{C_{31}B_{12}}}{-B_{12}B_{23}}, p_3 = \frac{\sqrt{C_{21}B_{31}}\sqrt{A_1B_{23}}}{-B_{23}B_{31}}$$

From equation (7) we get $\langle\gamma, r\rangle = \alpha = A_1$ (roots $\alpha = 2A_2, \alpha = 2A_3$ are correspond to solutions (15), (16) of the characteristic system).

So, $\lambda\sqrt{A_1B_{23}}r_1 = A_1$, consequently,

$$\gamma_1 = \frac{A_1}{r_1}, \gamma_2 = \frac{A_1\sqrt{C_{21}B_{31}}}{r_1\sqrt{A_1B_{23}}}, \gamma_3 = \frac{A_1\sqrt{C_{31}B_{12}}}{r_1\sqrt{A_1B_{23}}}$$

1d. It follows from asseertion 3.

2. If $A_1 = A_2 = A_3$ then we can choose $r_1 = r_2 = 0$, and then we have confluent case of Euler($r_3 = 0$) or special case of Lagrange ($r_3 \neq 0$), (see lower).

So, $A_1 = A_2 \neq A_3, r_1 \neq 0, r_3 \neq 0$ and wee can choose $r_2 = 0$.

Since $\langle Ap, \gamma\rangle = \langle p, \gamma\rangle = 0, A_1 = A_2 \neq A_3$, then $p_3\gamma_3 = 0$.

Let at first $p_3 = 0, \gamma_3 \neq 0$, then $\langle A\gamma, \gamma\rangle \neq 0$ and from (12) we get $r_1B_{23}\gamma_2C_{21}B_{31} = 0 \Rightarrow \gamma_2 = 0$.

From the relation $p_1\gamma_3 - p_3\gamma_1 + 2\gamma_1 = 0$ we get $p_1 = 0$. Consequently,($\langle p, p\rangle = -4$)$p_2 = \pm 2i$.

From the characteristic system we get $\gamma_1/\gamma_3 = p_2/p2 = \pm i$.

The solutions $(p, \lambda\gamma'), \gamma' = (\pm i, 0, 1)$ which we have within λ satisfy the condition $\gamma \times p = 2\gamma$.

The condition $Ap\times p + \gamma\times r + Ap = 0 \Leftrightarrow \gamma\times r + Ap = 0$ is true if $\lambda = 2A_1(r_3 \pm ir_1)^{-1}$.

Now we see the case $\gamma_3 = 0 \Rightarrow \lambda(1, \pm i, 0)$. Since $p_3\gamma_2 - p_2\gamma_3 + 2\gamma_1 = 0$, then $p_3 = \pm 2i$; moreover from $p_2\gamma_1 - p_1\gamma_2 + 2\gamma_3 = 0$ we obtain $p_1/p_2 = \gamma_1/\gamma_2 = \mp i$.

The condition $p \times \gamma = 2\gamma$ is equivalent (in this case) to the condition $\gamma_1 = \lambda, \gamma_2 = \pm\lambda i, \gamma_3 = 0, p_1 = \mu, p_2 = \pm\mu i, p_3 = \pm 2i$, and the condition $Ap \times p + \gamma \times r + Ap = 0-$ is equivalent to

$$\begin{cases} -2\mu B_{23} \pm \lambda i r_3 + \mu A_1 = 0 \\ \pm 2\mu i B_{31} - \lambda r_3 \pm \mu A_2 i = 0 \\ \mp\lambda r_1 i \pm 2A_3 i = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \lambda = 2A_3/r_1, \mu C_{31} \pm \lambda i r_3 = 0$.

2a$_1$. We get the root (17).

2a$_{21}$. Because $C_{31} = 0$ the other solutions are absent;

2a$_{22}$. $C_{31} = 0, r_3 = 0$, hence μ is an arbitrary constant and we get (18).

2b. The characteristic system in the case $r_1 = r_2 = 0, r_3 \neq 0$ has the form

$$\begin{cases} B_{23}p_2p_3 + r_3\gamma_2 + A_1p_1 = 0 \\ B_{31}p_1p_3 - r_3\gamma_1 + A_2p_2 = 0 \\ A_3p_3 = 0 \\ \gamma \times p + 2\gamma = 0, \end{cases}$$

or equivalent

$$\begin{cases} p_1 = -A_1^{-1}\gamma_2 r_3, p_2 = -A_1^{-1}\gamma_1 r_3, p_3 = 0 \\ -A_1^{-1} r_3\gamma_1\gamma_3 = 2\gamma_1 = 0 \\ -A_1^{-1} r_3\gamma_2\gamma_3 = 2\gamma_2 = 0 \\ A_1^{-1} r_3\gamma_1^2 + A_1^{-1} r_3\gamma_2^2 + 2\gamma_3 = 0. \end{cases}$$

Then we obtain $\gamma_1 = \gamma_2 = 0 \Leftrightarrow (p, \gamma) = 0$, or

$$\begin{cases} \gamma_3 = 2A_1 r_3^{-1} \\ \gamma_1^2 + \gamma_2^2 = -4A_1^2 r_3^{-2} \end{cases}$$

2c. The characteristic system in the case $r = 0$ has the form

$$\begin{cases} B_{23}p_2p_3 + A_1p_1 = 0 \\ B_{31}p_1p_3 + A_2p_2 = 0 \\ B_{12}p_1p_2 + A_3p_3 = 0 \\ \gamma \times p + 2\gamma = 0. \end{cases}$$

Since $B_{12} = 0$, then $p_3 = 0$, and $p_1 = p_2 = 0, 2\gamma = -\gamma \times p = 0$.
The theorem is proved. □

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