PALAIS–SMALE CONDITION FOR CHIRAL FIELDS

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Abstract. The well known condition of compactness entered by R. Palais and S. Smale — condition (C) — can be proved traditionally in rare cases, especially if it is considered the problem about critical points for functional $f(u)$, $u \in E$ on the surface $\{u \in E : F(u) = 0\}$ with essentially nonlinear infinite dimensional $F : E \to E_1$. However it is possible to obtain the proof by consideration of special compactifications for bounded sets from E, and subsequent testing that the limit points of any pseudocritical sequence lie not in remainder above E, but in most E. Main application is a problem for spherical fields in the bounded domains.

0. Introduction. In some applications of Morse or Lusternik–Schnierelmann theories to the critical points for a functional $f(u)$, $u \in E$ (E is a Banach space) with constraint $\{F = 0\} \equiv \{u \in E : F(u) = 0, \quad F : E \to E_1\}$ one of the main technical difficulties is checkup for Palais–Smale compactness condition, named condition (C). Namely, whether will be some bounded sequence $\{u_i\}$ have a limit point, if values of the Euler–Lagrange operator tend to zero on $\{u_i\}$:

$$f'(u_i) + F'^{*}(u_i)\, r_i \to 0, \tag{1}$$

$r_i \in E_1^*$ are Lagrange multipliers. (It is a highly nontrivial problem — is Lagrange multiplier in the limiting equation at $f'(u)$ equal to zero, provided that regularity, i.e., $\mathrm{Im}F'(u) = E_1$, is absence. But its discussion leaves for a framework of the article.)

In case of finite dimensional constraints ($E_1 = \mathbb{R}^n$) one usually acting (after [PS], [Brow]) so: if $u_i, u_j \in \{u_i\}$ then

$$\begin{aligned}\langle f'(u_i) - f'(u_j), u_i - u_j\rangle + \langle r_i, F'(u_i)(u_i - u_j)\rangle \\ - \langle r_j, F'(u_j)(u_i - u_j)\rangle \longrightarrow 0.\end{aligned}$$

In the natural conditions of regularity and compactness F' it is possible to allocate subsequence on which $\langle r_i, F'(u_i)(u_i - u_j)\rangle \to 0$, then to make use, for example, of uniform monotonicity f'.

With some shifts this proof can be extended to case $\dim E_1 = \infty$, if F is the sum of linear surjection and compact nonlinear operator [Suv].

Partially supported by SFFI Ukraine, grant 1.4/127

However for essentially nonlinear F, without regularity, and for infinite dimensional E_1, such reasons are not proper. For their proof the compactness of F' and limitation of $\{r_i\}$ would be required. The choice of the space E_1 is largely arbitrary, and manipulating them, it is possible to achieve or first or second, but not first and second together: they are the inconsistent requirements in an infinite dimensional situation.

Nevertheless the proof can be received in the essentially other basis under the natural assumptions about (f, F, E) (statement in item 1). Namely, it is entered some extension $\overline{\gamma E}$ of the space E which is for bounded closed sets their compactification, narrower, than Stone–Čech's (see item 2); this compactification is constructing on some algebra $U(E) \subset C(E)$.

There is the limit point — point measure μ_0, limiting for $\{\mu_i\} = \{\gamma u_i\}$. Thus appears, that if μ_0 belongs to remainder $(\overline{\gamma E} \setminus \gamma E)$, then it does not lie in a centre of convolution algebra of measures (see item 3), a little that non-commutation μ_0 with some measure μ is displayed just on function f, i.e. $\langle [\mu_0, \mu], f \rangle_U \neq 0$.

One designation $\langle , \rangle$ is used for all pairs of the dual spaces with the space in the bottom index, when there is danger of mess.

For a strongly continuous F the movement in a direction of commutators does not change values of natural extinsion $\hat{F}$ of the operator F in U^*, i.e., does not remove for limits of constraint. Let us now have succeeded "to prolong" the smallness of values for a left-hand (1) uniformly to U^*, we have obtained there certain analogue of the higher smallness variation for functional near a critical point. Then the assumption for μ_0 belong to remainder which means linearity of variation along an appropriate commutator, results in the contradiction among powers of slowness.

So μ_0 is a measure, concentrated in usual point $u_0 \in E$. After that the fact u_0 is the limit point of a sequence $\{u_i\}$ — is trivial (see item 4).

In item 5 is shortly characterized as well as where properties of f, F, E are used. In item 6 is shown important and natural application: chiral fields in the bounded domains (similar problems see in [DNF],[Har],[Sch]). Functional

$$f(\mathbf{u}) = \int\limits_{\Omega} |\nabla \mathbf{u}|^2 \, dx$$

(possibly, more general), $F(\mathbf{u}) \equiv \sum u_i^2(x) - 1$, $\Omega \in \mathbb{R}^n$, $\mathbf{u} = \{u_1, \dots, u_m\}$; equation (1) turn into

$$\Delta u_i + (\sum_i |\nabla u_i|^2) \, u_i = 0; \tag{2}$$

boundary conditions is natural.

1. Statement of a problem. Let E and E_1 be the real uniformly convex Banach spaces. Functional f and operator $F : E \to E_1$ are Frechét boundedly uniformly differentiable, i.e., there are uniformly continuous derivatives and uniformly small Taylor remainders on bounded sets; continuous modules and upper bounds for remainders depend on the norm.

Functional f is even, convex and coercive; operator F is strongly continuous, i.e., F transforms every weak convergent sequence into strongly convergent.

The goal of this article is to obtain the following:

THEOREM. *Let $\{u_i\} \subset \{F = 0\}$ from (1) is the bounded sequence. Then it has a limit point.*

2. Expansion $\overline{\gamma E}$. We introduce countable-normed algebra of functions $U = U(E)$ consisting of unbounded (in general) functions, each of which is uniformly continuous on any bounded set: $|\varphi(u+h) - \varphi(u)| \leqslant \omega(\|h\|, \varphi, \|u\|)$ with usual properties for a module ω. The system of the seminorms are $p_n(u) = \sup\{|\varphi(u)|, \|u\| \leqslant n\}$, $n \in \mathbb{Z}$. Dual U^* can be interpreted as some space of measures on E (regular, finite, finitely additive, not necessarily positive [FT]) and $\gamma : E \to U^*$ is a calculation mapping: if $\mu_u = \gamma u, \quad u \in E$, then $\forall \varphi \in U \quad \langle \mu_u, \varphi \rangle = \varphi(u)$. The completion $\overline{\gamma E}$ is constructed by w^*-topology for the U^* (i.e., by weak dual).

PROPOSITION 1. *All measure supports are bounded.*

(It is like to appropriate Hewitt's result [FT].)

COROLLARY 1. *U^* is an algebra over convolution $\langle \mu_1 * \mu_2, \varphi \rangle = \langle \mu_1^y, \langle \mu_2^x, \varphi(x+y) \rangle \rangle$.*

This algebra is noncommutative - see item 3.

If the map $T : E \to E$ is a boundedly uniformly continuous then there is some linear operator $\mathcal{T} : U(E) \longrightarrow U(E)$ corresponding to T, namely, $(\mathcal{T}\varphi)(u) = \varphi(Tu)$. The last, in turn, have been corresponding to linear $\hat{T} = \mathcal{T}^* : U^*(E) \longrightarrow U^*(E)$. Similarly, with use $U(E)$, $U(E_1)$ we receive for $F : E \to E_1$ the linear operator $\hat{F} : U^*(E) \longrightarrow U^*(E_1)$. Truly linear T at such correspondence acquire the next additional property:

PROPOSITION 2. *If $T : E \to E$ is a continuous linear operator then $\hat{T}$ is a homomorphism, i.e., $\hat{T}(\mu_1 * \mu_2) = (\hat{T}\mu_1) * (\hat{T}\mu_2)$.*

We shall note one fact, describing essential difference of uniform compactification for bounded closed sets from their Stone–Čech compactification: it is possible to enter symmetric specifying the same topology in $\overline{\gamma E}$:

$$\rho(x, y) = \inf_{u_\alpha \to x,\, v_\alpha \to y} \overline{\lim_{\alpha}} \|u_\alpha - v_\alpha\|.$$

3. Main lemmas.

LEMMA 1. *If the point measure μ_1 is concentrated in a point of remainder then μ_1 does not belong to a centre of algebra U^*, i.e., $\exists (r \in U(E),\ \mu_2 \in U^*)$ so that $\langle [\mu_2, \mu_1], r \rangle \neq 0$.*

The following lemma formally exceed first, but first is used in the proof the second; μ_1 is the same measure.

LEMMA 2. *There is a family of measures $\mu_\varepsilon \in U^*$ such that for functional f (from item 1)*

(a) $\langle [\mu_1, \mu_\varepsilon], f \rangle = \alpha(\varepsilon) \neq 0$;
(b) μ_ε *are the w*-limits of convex combinations of point measures from compactifications for εneighborhoods of a zero in E (designation $\overline{\gamma S(\Theta, \varepsilon)}$);*
(c) $|\alpha(\varepsilon)| \geqslant c\,\varepsilon$, $c > 0$.

LEMMA 3. *For all $\mu \in U^*$ it is truly $\hat{F}[\mu_1, \mu] = 0$.*

With regard to the perturbations by commutators do not remove for limits of extended constraint in U^*, we have

COROLLARY 2. *If μ_1 is a limit point of set $\{\gamma u_i\}$, where $\{u_i\}$ are from the theorem, if ν_ε are point measures from $\overline{\gamma S(\Theta, \varepsilon)}$, then $|\langle [\mu_1, \nu_\varepsilon], f \rangle| = o(\varepsilon)$.*

4. About the theorem. If the limit point μ_1 lies in remainder, both powers of slowness for $\langle [\mu_1, \mu_\varepsilon], f \rangle$ from a lemma 2 and corollary 2 come to contradiction (a pass from point measures to their convex combinations and to the consequent limit is possible). Hence μ_1 is a measure, concentrated in some usual point $w \in E$.

As all linear functions and norm converge on appropriate subdirectedness $u_\alpha \to w$ in γE (however it is possible to manage by sequences, because of existence of symmetric) so w is the limit point for $\{u_i\}$ in the norm E.

5. About use of properties f, F, E. The completeness of E exploits in a lemma 1: if the point measure from remainder belongs to the centre then one succeeds in finding in $\overline{\gamma E}$ the set passing through it and $\gamma\Theta$, on which it is possible to introduce a linear structure $\mathbb{R}^1$ compatible with a linear structure of E. Thus E is embedded in some linear E_0 as proper dense subspace and with identical topology. It is impossible for complete E.

The uniform convexity of E exploits, in particular, in item 4.

Evenness and coerciveness for f are used in the proof of the fact: noncommutativity realization just on f (lemma 2.a), and the convexity is required in the proof of estimates (lemma 2.b,c): the standard inequalities connected with convexity, in correspondence $u+v \rightleftarrows \mu_u * \mu_v$ give set enough of inequalities to prove it in a cone of positive measures.

The strong continuity for F in a lemma 3 permits to approximate (in any bounded set) an arbitrary function of a kind $h(F(u))$, where $h \in U(E_1)$, by elements of algebra $WC(E)$ generated by unit and linear functionals: type

$$a_0 + a_1 \langle l_1, u \rangle + a_2 \langle l_2, u \rangle \langle l_3, u \rangle + \dots \ .$$

After it the proposition 2 is used.

Uniformity of f', F', uniformity of $U(E)$ are required in corollary 2 in order to "prolong" (1) into remainder.

6. Example. Let: $E = W^1_{2,m}(\Omega)$ be the space of vectors $\mathbf{u} = (u_1, \dots, u_m)$; the domain $\Omega \subset \mathbb{R}^n$ be bounded with regular $\partial\Omega$; the space E_1, for example, be $L_{\frac{n-1}{n-2},1}$. Then the embedding $E \hookrightarrow L_{\frac{2(n-1)}{n-2},m}$ is compact. If $F \equiv \sum u_i^2(x) - 1$ then $F : L_{\frac{2(n-1)}{n-2},m} \longrightarrow L_{\frac{n-1}{n-2},1}$ is boundedly uniformly continuous and has required properties for derivative, and at the expense of compactness of a specified embedding $F : E \to E_1$ is strongly continuous.

The typical functional (Euclidian action) is

$$f(\mathbf{u}) = \int_\Omega |\nabla \mathbf{u}|^2 \, dx,$$

and we shall here add it by term

$$f(\mathbf{u}) = \int_\Omega \left(|\nabla \mathbf{u}|^2 + \sum u_i^2 \right) dx,$$

that does not change a constrained $\{F = 0\}$ variational problem. The required properties f are trivial.

It is also clear, that it is possible to take more general

$$f(\mathbf{u}) = \int_\Omega \Phi(x, \mathbf{u}, \nabla \mathbf{u}) \, dx$$

with $E = W^1_{p,m}$, $p > 2$, smooth Φ and appropriate its behavior with respect to $\mathbf{u}, \nabla \mathbf{u}$.

The conclusion of a form for multiplier r from (1), reducing limiting (1) to (2) is simple (for smooth solutions and a priori nonzero Lagrange multiplier at $f'(u)$).

The boundary conditions is natural (Neumann).

A boundedness for $\{u_i\}$ probably is essential to not simple connected constraints in $\mathbb{R}^m$: the counterexample from [Har], though it does not concern to the Neumann problem, but gives enough basis for such assumption.

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Donetsk, Ukraine

Institute for Applied Mathematics
and Mechanics, UNAS,
R. Luxemburg Street 74,
340114 Donetsk, Ukraine.
Ph 0038 0622 510139,
Fax 0038 0622 552265
E–mail: suvorov@iamm.ac.donetsk.ua