ON TOPOLOGICAL DEGREE TO SOME CLASS OF MULTIVALUED MAPPINGS AND ITS APPLICATIONS

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In this work the elements of topological degree theory has been developed for the class of multivalued maps from the reflexive Banach space to its dual one. In particular, this class contains the maps which generated by the inclusions with partial derivatives, by variational inequalities etc. This research is based on the results of [1–3].

1. **The multivalued mappings.** Let X be a reflexive Banach space, X^* be its topological dual space, $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ be the duality pairing, $A : X \to 2^{X^*}$ be a multivalued mapping (2^{X^*} is the totality of all subset of the space X^*). We define $\mathrm{Dom}(A) = \{y \in X | A(y) \neq \varnothing\}$, and the map A is called the strong if $\mathrm{Dom}(A) = X$. We associated with A the lower and upper support functions $[A(y), \xi]_- = \inf\limits_{d\in A(y)} \langle d, \xi\rangle$ and $[A(y), \xi]_+ = \sup\limits_{d\in A(y)} \langle d, \xi\rangle$, where $y, \xi \in X$. If $y \notin \mathrm{Dom}(A)$ then $[A(y), \xi]_- = +\infty$, $[A(y), \xi]_+ = -\infty$ for each $\xi \in X$. Moreover, $\|A(y)\|_+ = \sup\limits_{d\in A(y)} \|d\|_{X^*}$, $\|A(y)\|_- = \inf\limits_{d\in A(y)} \|d\|_{X^*}$ and $\|\varnothing\|_- = \|\varnothing\|_+ = 0$.

Lemma 1.1. *Let $A, A_1, A_2 : X \to 2^{X^*}$. The following statements hold for each $y \in \mathrm{Dom}(A)$, $\xi_1, \xi_2 \in X$:*

$$[A(y), \xi_1 + \xi_2]_+ \geq [A(y), \xi_1]_+ + [A(y), \xi_2]_-;$$
$$[A(y), \xi_1 + \xi_2]_- \leq [A(y), \xi_1]_+ + [A(y), \xi_2]_-;$$
$$\|A_1(y) + A_2(y)\|_+ \geq \begin{cases} |\|A_1(y)\|_+ - \|A_2(y)\|_+| \\ |\|A_1(y)\|_+ - \|A_2(y)\|_-| \end{cases}.$$

Lemma 1.2. *For each $y \in \mathrm{Dom}(A)$ the following equalities hold:*

$$\|\overline{\mathrm{co}}A(y)\|_+ = \|A(y)\|_+, \qquad \|\overline{\mathrm{co}}A(y)\|_- = \|A(y)\|_-.$$

Definition 1.1. The mapping $A : G \subset X \to 2^{X^*}$ satisfies the condition $\alpha_0(G)$ if from $G \ni y_n \to y$ weakly on X and

$$\overline{\lim_{n\to\infty}} [A(y_n), y_n - y]_- \leq 0 \tag{1}$$

it follows that $y_n \to y$ strongly on X.

Let D be an arbitrary open bounded set from X, ∂D be its boundary, $\overline{D} = D \cup \partial D$.

Definition 1.2. The mapping $A : \overline{D} \to 2^{X^*}$ satisfies the condition $\alpha(G)$, where $\overline{D} \supset G$, if from $\mathrm{Dom}(A) \cap G \ni y_n \to y$ weakly on X and from (1) it follows that $y_n \to y$ strongly on X.

Remark 1.1. For single-valued mappings the conditions $\alpha_0(G)$ and $\alpha(G)$ had been introduced by I.V.Skrypnik [2]. Like to them condition $(S)_+$ had been introduced by F.Browder [4].

We say that the continuous function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ belong to the class Φ if $\tau^{-1}C(r_1, \tau r_2) = 0$ for each $r_1, r_2 > 0$ as $\tau \to +0$.

Definition 1.3. The mapping $A : \mathrm{Dom}(A) \subset X \to 2^{X^*}$ is called the operator of semibounded variation (s.b.v), if for any $R > 0$ and each $y_1, y_2 \in X$ such that $\|y_i\|_X \leq R$ $(i = 1, 2)$ the following inequality holds:

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_X),$$

where C belong to Φ, and $\|\cdot\|'_X$ is compact seminorm with respect to the norm $\|\cdot\|_X$.

Remark 1.2. Let $A = A_1 + A_2$, where $A_1 : X \to 2^{X^*}$ is a monotone map, $A_1 : Y \to Y^*$ is a locally Lipschitz operator, Y is a Banach space, moreover, the embedding $X \subset Y$ is compact. Then A is the map of semibounded variation.

Proposition 1.1. *Let the mapping $A_0 : G \subset X \to 2^{X^*}$ satisfies the condition $\alpha(G)$, $A_1 : X \to 2^{X^*}$ be a map of s.b.v. Then $A = A_0 + A_1$ satisfies the condition $\alpha(G)$ if* $\mathrm{Dom}(A_0) \cap \mathrm{Dom}(A_1) = \mathrm{Dom}(A) \neq \varnothing$.

Definition 1.4. The mapping $A : \mathrm{Dom}(A) \subset X \to 2^{X^*}$ is called $(G; F)$–pseudomonotone, where $F \subset X$, if from $\mathrm{Dom}(A) \cap G \ni y_n \to y \in \mathrm{Dom}(A)$ weakly in X and (1) it follows that

$$\varliminf_{n\to\infty} [A(y_n), y_n - \xi]_- \geq [A(y), y - \xi]_- \qquad \forall \xi \in F.$$

If $G = F = X$ the map A is called pseudomonotone.

Remark 1.3. The multivalued pseudomonotone mappings were studied also in [3,6].

Proposition 1.2. *Let the mapping A_0 satisfies the conditions from Proposition 1.1, $A_1 : X \to 2^{X^*}$ be a $(G; F)$–pseudomonotone mapping, moreover,* $\mathrm{Dom}(A_0) \cap \mathrm{Dom}(A_1) \neq \varnothing$. *Then $A = A_0 + A_1$ satisfies the condition $\alpha(G)$.*

Proposition 1.3. *Finite union of maps $\{A_i : G \subset X \to 2^{X^*}\}_{i=1}^n$ which satisfy the condition $\alpha(G)$, is closed w.r.t. the positive multiplication and w.r.t. the intersection. If, moreover,* $\bigcap_{i=1}^n \mathrm{Dom}(A_i) \neq \varnothing$ *then the map $\sum_{i=1}^n A_i$ satisfies the condition $\alpha(G)$.*

Definition 1.5. The mapping $A : X \to 2^{X^*}$ is called

a) demiclosed if graph A is closed in $X \times X^*$ with respect to the strong convergence on X and weak one on X^*;

b) radially upper semicontinuous at point $y_0 \in \mathrm{Dom}(A)$ if for any $\xi, h \in X$ the function $[0, 1] \ni t \mapsto [A(y_0 + th), \xi]_+$ is upper semicontinuous at point 0. The mapping A is radially upper semicontinuous (r.u.c.) if it is r.u.c. at each point of $\mathrm{Dom}(A)$.

Remark 1.4. R.u.c. multivalued mappings are the generalization of upper hemicontinuous mappings (see [5]). Moreover, each r.u.c. mapping of s.b.v. is upper hemicontinuous.

Proposition 1.4. *Let* $A : X \to 2^{X^*}$ *be a closed-convex-valued maps and one of following conditions holds:*

a) A is strong, r.u.c. map of s.b.v.;

b) A is pseudomonotone map and $\mathrm{Dom}(A)$ *is closed.*

Then A is demiclosed mapping.

We recall that the map $A : X \to 2^{X^*}$ is bounded if for each bounded $B \subset X$ there exists $k > 0$ such that $\|A(y)\|_+ \le k \ \forall y \in B$.

Definition 1.6. We call that the mapping $A : \overline{D} \subset X \to 2^{X^*}$ belong to class $\mathfrak{U}_0(D; G)$ (respectively, $\mathfrak{U}(D; G)$) if it is bounded, demiclosed and satisfies the condition $\alpha_0(G)$ (respectively, $\alpha(G)$). We will write $\mathfrak{U}_0(D)$ and $\mathfrak{U}(D)$ in place of $\mathfrak{U}_0(D; \overline{D})$ and $\mathfrak{U}(D; \overline{D})$ (i.e. as $G = \overline{D}$).

2. **The degree of the mapping** A**.** First we consider the case when X is a reflexive separable Banach space. Let $\mathrm{Cv}(X^*)$ be a totality of nonempty closed convex subset from X^*, D be an arbitrary bounded open subset in X with the boundary ∂D. We assume that

2a) $A : \overline{D} \subset X \to \mathrm{Cv}(X^*)$;

2b) $A \in \mathfrak{U}_0(D; \partial D)$;

2c) $A(y) \not\ni 0$ for each $y \in \partial D$.

Let $\{h_i\}_{i=1}^{\infty}$ be an arbitrary complete system of X, moreover, for any n the elements $h_1, \dots, h_n$ are linear independent. Let X_n be the span of $\{h_1, \dots, h_n\}$, $J_n : X_n \to X$ be the inclusion map, $J_n^* : X^* \to X_n^*$ be its dual. Under each n we define the finite-dimensional multivalued mapping A_n associated with A by formula

$$A_n(y) = \bigcup_{d \in A} \Big\{ \sum_{i=1}^{n} \langle d(y), h_i \rangle h_i \Big\} = J_n^* A(J_n y)$$

$$\forall y \in \overline{D_n} = D \cap X_n. \quad (2)$$

The symbol $d \in A$ denote that d is a selector of multivalued map A.

Theorem 2.1. *Let the multivalued mapping A satisfies conditions 2a)–2c). Then there exists N such that as* $n \ge N$ *the following statements hold:*

1) the inclusion $A_n(y) \ni 0$ *has not any solution on* ∂D_n*;*

2) the degree $\deg(A_n, \overline{D_n}, 0)$ *of the map* A_n *set* $\overline{D_n}$ *with respect to* $0 \in X_n$ *is defined and is not depended on* $n \ge N$.

The proof is based on following statements.

Lemma 2.1. *The mapping* $A_n : \overline{D_n} \to \mathrm{Cv}(X_n^*)$ *is upper continuous and compact.*

Let us consider auxiliary mapping $\tilde{A}_n(y) = A_{n-1}(y) + \langle \xi_n, y \rangle h_n$ where the element $\xi_n \in X^*$ such that $\langle \xi_n, h_i \rangle = 0$ for each $i < n$ and $\langle \xi_n, h_n \rangle = 1$.

Lemma 2.2. *There exists N such that for each* $n \ge N$

a) $\deg(A_{n-1}, \overline{D_{n-1}}, 0) = \deg(\tilde{A}_n, \overline{D_n}, 0)$*;*

b) $\deg(A_n, \overline{D_n}, 0) = \deg(\tilde{A}_n, \overline{D_n}, 0)$.

Remark 2.1. From Theorem 2.1. it follows that there exists the limit

$$\lim_{n \to \infty} \deg(A_n, \overline{D_n}, 0) = D(\{h_i\}).$$

Theorem 2.2. *Under conditions 2a)–2c) the limit $D(\{h_i\})$ is independent on the choice of $\{h_i\}$.*

From Theorems 2.1. and 2.2. it follows the naturalism of the following definition.

Definition 2.2. For the mapping $A : X \to 2^{X^*}$ which satisfies conditions 2a)–2c) the number

$$\operatorname{Deg}(A, \overline{D}, 0) = \lim_{n\to\infty} \deg(A_n, \overline{D_n}, 0)$$

is called its degree of the set $\overline{D}$ with respect to the point $0 \in X^*$, where A_n and D_n are defined by (2).

Above we assume that X is separable, let us show that this requirement is not necessary. Let now X be a reflexive Banach space, $F(X)$ be the totality of its finite-dimensional subspaces, $F \in F(X)$ and $h_1, \ldots, h_\nu$ be some basis in F. We define the finite-dimensional mapping

$$A_F(y) = \bigcup_{d\in A} \{\sum_{i=1}^{\nu} \langle d(y), h_i\rangle h_i\} \quad \forall y \in \overline{D_F} = D \cap F. \tag{3}$$

Theorem 2.3. *Let the mapping $A : \overline{D} \subset X \to 2^{X^*}$ satisfies conditions 2a),2c) and $A \in \mathfrak{U}(D; \partial D)$. Then there exists $F_0 \in F(X)$ such that for each $F \in F(X)$ $(F_0 \subset F)$ the following properties are:*

1) the inclusion $A_F(y) \ni 0$ has not any solution on ∂D_F;

2) $\deg(A_F, \overline{D_F}, 0) = \deg(A_{F_0}, \overline{D_{F_0}}, 0)$,

where deg *is the degree of the finite-dimensional map, A_F and D_F are defined by* (3).

Lemma 2.3. *There exists $F_0 \in F(X)$ such that for each $F \in F(X)$ $(F_0 \subset F)$ the set*

$$Z^F_{F_0} = \bigcup_{d\in A} Z^F_{F_0}(d) = \varnothing,$$

where $Z^F_{F_0}(d) = \big\{y \in \partial D_F | \langle d(y), \xi\rangle = 0 \, \forall \xi \in F_0\big\}$.

Definition 2.2. For the mapping $A : X \to 2^{X^*}$ which satisfies conditions of Theorem 2.3. the number

$$\operatorname{Deg}(A, \overline{D}, 0) = \deg(A_{F_0}, \overline{D_{F_0}}, 0)$$

is called its degree of the set $\overline{D}$ with respect to $0 \in X^*$, where A_{F_0} and D_{F_0} are defined by (3) and $F_0 \in F(X)$ is choiced by Theorem 2.3.

3. **The basis properties of degree.** In this section we are restricted to the case when X is separable.

Definition 3.1. The mapping $A : [0,1] \times (\overline{D} \subset X) \to 2^{X^*}$ satisfies the condition $\alpha_{0,t}(\partial D)$ if for arbitrary subsequences $\{y_n\} \subset \partial D$, $\{t_n\} \subset [0,1]$ from $y_n \to y_0$ weakly on X, $A(t_n, y_n) \ni d_n \to d_0$ weakly on X^* and

$$\overline{\lim_{n\to\infty}} [A(t_n, y_n), y_n - y_0]_- \le 0$$

it follows that $y_n \to y$ strongly on X.

Definition 3.2. The mapping $A_0, A_1 : \overline{D} \subset X \to 2^{X^*}$ of class $\mathfrak{U}(D;\partial D)$ which satisfy the condition 2c) is called gomotopic in $\overline{D}$ if there exists the bounded map $A : [0,1] \times \overline{D} \to 2^{X^*}$ which satisfy the following conditions:

1) $A(0,\cdot) = A_0$, $A(1,\cdot) = A_1$;

2) A satisfies the condition $\alpha_{0,t}(\partial D)$;

3) $A(t,y) \not\ni 0$ for each $t \in [0,1]$ and for each $y \in \partial D$;

4) A is demiclosed, i.e. if $t_n \to t_0$, $y_n \to y_0$ strongly in X and $A(t_n, y_n) \ni d_n \to d_0$ weakly on X^*, then $d_0 \in A(t_0, y_0)$.

Theorem 3.1. *Let A_0 and A_1 be multivalued mappings of the class $\mathfrak{U}(D;\partial D)$ which satisfy conditions 2a), 2c). If, in additional,A_0 and A_1 are gomotopic on $\overline{D}$, then* $\mathrm{Deg}(A_0, \overline{D}, 0) = \mathrm{Deg}(A_1, \overline{D}, 0)$.

Theorem 3.2. *Let $A : \overline{D} \subset X \to 2^{X^*}$ be the map of class $\mathfrak{U}_0(D)$, $A(y) \not\ni 0$ for each $y \in \overline{D}$ and 2a) is. Then* $\mathrm{Deg}(A, \overline{D}, 0) = 0$.

Corollary 3.1. Let $A : \overline{D} \subset X \to 2^{X^*}$ be the map of class $\mathfrak{U}_0(D)$ and it satisfies properties 2a), 2b). For the inclusion $A(y) \ni 0$ has at least one solution on D it is sufficiently that $\mathrm{Deg}(A, \overline{D}, 0) \neq 0$.

Theorem 3.3. *Let $A : \overline{D} \subset X \to \mathrm{Cv}(X^*)$ be the map of class $\mathfrak{U}_0(D;\partial D)$, $0 \in \overline{D} \setminus \partial D$ and*

$$[A(y), y]_- \geq 0 \qquad \forall y \in \partial D.$$

Then $\mathrm{Deg}(A, \overline{D}, 0) = 1$.

Theorem 3.4. *Let D be a symmetric bounded neighborhood of zero, $A : \overline{D} \subset X \to$ $\mathrm{Cv}(X^*)$ be the map of class $\mathfrak{U}(D;\partial D)$ and $0 \notin A(\partial D)$. In additional let*

$$A(y) \cap \lambda A(-y) \neq \varnothing \qquad \forall y \in \partial D \text{ and } \lambda \in [0,1].$$

Then $\mathrm{Deg}(A, \overline{D}, 0)$ *is odd number.*

Theorem 3.5. *Let D_1 and D_2 be a nonintersecting open subset on D, in additional,*

$$A(y) \not\ni 0 \qquad \forall y \in \overline{D} \setminus (D_1 \cup D_2),$$

where $A \in \mathfrak{U}_0(D;\partial D)$ and 2a) holds.

Then $\mathrm{Deg}(A, \overline{D}, 0) = \mathrm{Deg}(A, \overline{D_1}, 0) + \mathrm{Deg}(A, \overline{D_2}, 0)$.

Remark 3.1. The statements of this section allow the natural extension in the case of nonseparable spaces.

4. **The degree for pseudomonotone maps.** Let D be some bounded open subset on X, $A_0 : \overline{D} \subset X \to \mathrm{Cv}(X^*)$ be the map of class $\mathfrak{U}(D;\partial D)$. We assume that $A : \overline{D} \subset X \to \mathrm{Cv}(X^*)$ is $(\partial D; X)$–pseudomonotone, demiclosed, bounded mapping and $0 \notin \overline{A(\partial D)}$. In additional, there exists $\delta_0 > 0$ such that $\|A(y)\|_- \geq \delta_0$ for each $y \in \partial D$. We consider the map $\mathfrak{A}_\varepsilon = \varepsilon A_0 + A : \overline{D} \subset X \to \mathrm{Cv}(X^*)$. Let $M = \sup\limits_{y \in \overline{D}} \|A_0(y)\|_+$. Obviously, $M < \infty$ and since Proposition 1.2. for each $\varepsilon > 0$ $\mathfrak{A}_\varepsilon \in \mathfrak{U}(D;\partial D)$. Moreover, $\mathfrak{A}_\varepsilon(y) \not\ni 0$ for each $y \in \partial D$. In fact, $\|\mathfrak{A}_\varepsilon(y)\|_- \geq \|A(y)\|_- - \varepsilon\|A_0(y)\|_+ \geq \delta_0 - \varepsilon M$. If $0 < \varepsilon < \delta_0 M^{-1}$, then $\|\mathfrak{A}_\varepsilon(y)\|_- > 0$ for each $y \in \partial D$. Thus, the degree $\mathrm{Deg}(\mathfrak{A}_\varepsilon, \overline{D}, 0)$ is defined as $0 < \varepsilon < \delta_0 M^{-1}$. Let us show that the defined degree is independent on ε. Let $0 < \varepsilon_i < \delta_0 M^{-1}$, $i = 1, 2$ and we consider corresponding $\mathfrak{A}_{\varepsilon_i}$. Let us assume that $\mathfrak{A}(t, y) = (t\varepsilon_2 + (1-t)\varepsilon_1)A_0(y) + A(y)$. Obviously, $\mathfrak{A}(0, y) = \mathfrak{A}_{\varepsilon_1}(y)$, $\mathfrak{A}(1, y) = \mathfrak{A}_{\varepsilon_2}(y)$ and in additional $\mathfrak{A}(t, y) \not\ni 0$ for each $t \in [0,1]$ and for each $y \in \partial D$.

Lemma 4.1. *The mapping $A : [0,1] \times \overline{D} \to 2^{X^*}$ satisfies the condition $\alpha_{0,t}(\partial D)$.*

The conditions of Theorem 3.1 are satisfied, thus,

$$\mathrm{Deg}(\mathfrak{A}_{\varepsilon_1}, \overline{D}, 0) = \mathrm{Deg}(\mathfrak{A}_{\varepsilon_2}, \overline{D}, 0).$$

Hence, there exists the limit $\lim_{\varepsilon \to 0} \mathrm{Deg}(\mathfrak{A}_{\varepsilon}, \overline{D}, 0)$ which we will call the degree $\mathrm{Deg}(A, \overline{D}, 0)$ of mapping A and set D with respect to the point $0 \in X^*$.

Using the constructions given above we can prove that this limit is not depended on the mapping A_0, i.e. the degree of pseudomonotone map is correct.

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