# DIRECT AND INVERSE THEOREMS ON APPROXIMATION OF SOLUTIONS OF OPERATOR EQUATIONS

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**1.** In the theory of approximation of functions, the so-called direct and inverse theorems give the relation of smoothness of a continuous function to the behavior of its best approximation by more elementary objects. For example, in the periodic case, trigonometric polynomials play the role of such objects. If we put for a continuous $2\pi$-periodic function $f(x)$ $(x \in \mathbb{R}^1)$

$$E_n(f) = \inf_{T \in \mathcal{T}_n} \sup_{x \in [0,2\pi]} |f(x) - T(x)|$$

($\mathcal{T}_n$ is the set of all $(\leq n)$-order polynomials), then the S. N. Bernshtein result asserts:

$$\begin{aligned} f \in \widetilde{C}^\infty[0,2\pi] \quad &\Longleftrightarrow \quad \forall k \in \mathbb{N} \quad n^k E_n(f) \to 0 \ as \ n \to \infty; \\ f \in \widetilde{\mathfrak{A}}[0,2\pi] \quad &\Longleftrightarrow \quad \exists q : 0 < q < 1, \ \exists c > 0 : E_n(f) < cq^n; \\ f \in \widetilde{\mathfrak{A}}_c[0,2\pi] \quad &\Longleftrightarrow \quad \forall q : 0 < q < 1, \exists c = c(q) > 0 : E_n(f) < cq^n. \end{aligned}$$

Here $\widetilde{C}^\infty[0,2\pi], \widetilde{\mathfrak{A}}[0,2\pi]$, and $\widetilde{\mathfrak{A}}_c[0,2\pi]$ denote the space of infinitely differentiable on $\mathbb{R}^1$, analytic on $\mathbb{R}^1$, and entire $2\pi$-periodic functions respectively. There are a number of other direct and inverse theorems (see [1]). They belong mainly to D. Jackson and S. N. Bernshtein. The similar situation holds in the case where a function continuous on a closed interval is approximated by algebraic polynomials, or a continuous function given on $\mathbb{R}^1$ is approximated by entire functions of exponential type.

Some of the direct theorems were often applied to finding the error estimation in approximation of the exact solutions of various equations (differential, integral, operator) by their approximate ones. As for inverse theorems (they make possible to predetermine the smoothness degree of a solution on the basis of the available estimates), the number of them is considerably lesser. As far as we know, the results of such kind were obtained for the first time by A. V. Babin [2, 3] (1976 - 1984). We formulate one of them which concerns an abstract parabolic equation.

Consider the Cauchy problem

$$\frac{dy(t)}{dt} + Ay(t) = 0 \quad (t \geq 0), \quad y(0) = f, \quad f \in \mathfrak{H},$$

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where $A$ is a nonnegative selfadjoint operator in a complex separable Hilbert space $\mathfrak{H}$. As is known, the solution of this problem is of the form

$$y(t) = e^{-At} f.$$

It was shown that if $f \in \mathcal{D}(e^{\alpha\sqrt{A}})$ for some $\alpha > 0$, then

$$y = \lim_{n\to\infty} P_n(A) f,$$

where $P_n(A)$ is a certain $n$-order polynomial of $A$, and there exist the constants $\sigma > 0$ and $c > 0$ such that the following exact estimate is valid:

$$\|y - P_n(A)f\| \leq c \exp\left(-\sigma(\ln n)^2\right). \tag{1}$$

Taking a partial differential operator with polynomial coefficients as $A$ and a polynomial as $f$ (then $P_n(A)f$ are also polynomials), A. V. Babin found, by using in this concrete situation the S. N. Bernshtein weighted inverse theorems and the estimate (1), the smoothness degree of the solution of a corresponding equation. But the process itself of constructing the polynomials $P_n$ was rather complicated.

M. L. Gorbachuk and V. V. Gorodetskii [4] revealed (1984) that for the functions associated with some evolution equations (for the above equation, for example, this function is $e^{-\lambda t}$), the partial sums of their Fourier-Laguerre series give (up to the factor $e^{-\lambda}$) the polynomials $L_n(\lambda)$ such that

$$\lim_{n\to\infty} L_n(A) f = y.$$

Moreover, if $f$ is an analytic vector of the operator $A$, then for every closed interval $[0, b]$

$$\exists \rho = \rho(b) : 0 < \rho < 1,\ \exists c > 0 : \|y - L_n(A)f\| \leq c\rho^{n+1} \tag{2}$$

uniformly. The polynomials $L_n(\lambda)$ are of the form

$$L_n(\lambda) = \mu^{1/2}(t+\mu)^{-1} \sum_{k=0}^{n} \left(-\frac{t}{t+\mu}\right)^k l_{0,\mu,k}(\lambda),$$

where

$$l_{0,\mu,k}(\lambda) = (-1)^k \mu^{1/2} e^{\mu\lambda} \frac{\left((\mu\lambda)^k e^{-\mu\lambda}\right)^{(k)}}{k!}, \quad \mu = \mu(f) = const.$$

Conversely, if the relation (2) is fulfilled for a vector $f$ infinitely differentiable for the operator $A$, then $f$ is an analytic vector of $A$.

As we can see, for the polynomial approximation method of solving the Cauchy problem considered we have both direct and inverse theorems. The analogous theorems were proved for some other classes of differential equations in a Hilbert space.

**2.** Let us turn attention to the direct and inverse theorems in the approximation of solutions of operator equations by projective not polynomial methods. To do this we need a brief survey of the results on approximation of vectors from a Banach space by entire vectors of exponential type of a closed operator acting in this space.

Let $A$ be a closed linear operator in a Banach space $\mathfrak{B}$, $\overline{\mathcal{D}(A)} = \mathfrak{B}$, $\{m_n\}_{n\in\mathbb{N}_0}$ ($\mathbb{N}_0 = \{0, 1, 2, \dots\}$) a non-decreasing sequence of positive numbers (for the sake of simplicity we consider $m_0 = 1$). Denote by $C^\infty(A)$ the set of infinitely differentiable vectors of $A$:

$$C^\infty(A) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n).$$

For $\alpha > 0$ we put

$$C_{\{m_n\}}(A) = \underset{\alpha\to\infty}{ind\,lim}\, C_\alpha\langle m_n\rangle(A),\ C_{(m_n)}(A) = \underset{\alpha\to 0}{proj\,lim}\, C_\alpha\langle m_n\rangle(A),$$

where

$$C_\alpha\langle m_n\rangle(A) = \big\{f \in C^\infty(A) \big| \exists c > 0 : \|A^k f\| \le c\alpha^k m_k,\ \forall k \in \mathbb{N}_0\big\}$$

($\|\cdot\|$ is the norm in $\mathfrak{B}$; everywhere $c$ means a certain constant) is a Banach space with respect to the norm

$$\|f\|_{C_\alpha\langle m_n\rangle(A)} = \sup_{n\in\mathbb{N}_0} \frac{\|A^n f\|}{\alpha^n m_n}.$$

The convergence in $C_{\{m_n\}}(A)$ ($C_{(m_n)}(A)$) is that in some (any) $C_\alpha\langle m_n\rangle(A)$. If $m_n = n!$ or $m_n \equiv 1$, then we get the spaces $C_{\{n!\}}(A)$, $C_{(n!)}(A)$, and $C_{\{1\}}(A)$ known as the spaces of analytic, entire, and entire of exponential type vectors of the operator $A$ respectively. The spaces $C_{\{n^{n\beta}\}}(A)$ and $C_{(n^{n\beta})}(A)$ are called the Gevrey classes of Roumieu and Beurling type.

E x a m p l e 1. If $\mathfrak{B} = C[a, b]$ ($-\infty < a < b < \infty$), $A = \frac{d}{dx}$, $\mathcal{D}(A) = C^1[a, b]$, then $C^\infty(A)$, $C_{\{n!\}}(A)$, $C_{(n!)}(A)$, $C_{\{1\}}(A)$ coincide with the spaces of usual infinitely differentiable on $[a, b]$, analytic on $[a, b]$, entire and entire of exponential type functions; $C_{(1)}(A)$ is the set of all polynomials; $C_{\{n^{n\beta}\}}(A)$ and $C_{(n^{n\beta})}(A)$ are the usual Gevrey classes.

E x a m p l e 2. If $\mathfrak{B} = L_2(\mathbb{R}^1)$, $A$ is the closure of the operator $A_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right)$, $\mathcal{D}(A_0) = C_0^\infty(\mathbb{R}^1)$ (the set of infinitely differentiable functions with compact support), then $C^\infty(A) = S$, $C_{\{n^{n\beta}\}}(A) = S_{\beta/2}^{\beta/2}$, where $S$ is the well-known Schwartz space of slowly decreasing functions and

$$S_\alpha^\beta = \big\{f \big| \exists h > 0 : \sup_{m,n\in\mathbb{N}_0, x\in\mathbb{R}^1} \frac{|x^m f^{(n)}(x)|}{h^{m+n} m^{m\alpha} n^{n\beta}} < \infty\big\}.$$

So, by entire vectors of exponential type of the operator $A$ we mean the vectors

$$g \in C_{\{1\}}(A) = \bigcup_{\alpha>0} C_\alpha\langle 1\rangle(A).$$

The type $\sigma(g)$ of a vector $g \in C_{\{1\}}(A)$ is defined as follows:

$$\sigma(g) = \inf_{\alpha>0: g\in C_\alpha\langle 1\rangle(A)} \alpha.$$

For every $f \in \mathfrak{B}$ we set

$$\mathcal{E}_r(f) = \inf_{g \in C_{\{1\}}(A): \sigma(g) \le r} \|f - g\|.$$

The function $\mathcal{E}_r(f)$ is monotonically non-increasing, and

$$\forall f \in \mathfrak{B} \quad \mathcal{E}_r(f) \to 0 \text{ as } r \to \infty \Longleftrightarrow \overline{C_{\{1\}}(A)} = \mathfrak{B}. \tag{3}$$

If $A$ is a normal operator in a Hilbert space $\mathfrak{H}$, the equality in the right hand side of (3) is fulfilled because

$$C_{\{1\}}(A) = \{y \in \mathfrak{H} \big| y = E_\Delta x\},$$

where $x$ runs through the whole space $\mathfrak{H}$, $\Delta$ is any compact set in $\mathbb{R}^2$, $E_\Delta$ is the spectral measure of the operator $A$.

Theorem 1. *Let $A$ be a normal operator in $\mathfrak{H}$, $\gamma(t)$ $(t \in [0, \infty))$ a monotonically non-increasing positive function, and $\gamma(t) \to 0$ as $t \to \infty$. Then there exists a vector $f \in \mathfrak{H}$ such that*

$$\mathcal{E}_r(f) = \gamma(r).$$

Theorem 1 shows that in the case of a normal $A$ the rate of the convergence $\mathcal{E}_r(f) \to 0$ $(r \to \infty)$ may be arbitrary. It depends on the smoothness degree of $f$ with respect to the operator $A$.

Denote by $\mathfrak{H}^\alpha(A)$ the $|A|$-scale of Hilbert spaces that is,

$$\mathfrak{H}^\alpha(A) = \mathcal{D}(C^\alpha), \quad C = I + (A^*A)^{1/2},$$

$$(f, g)_{\mathfrak{H}^\alpha(A)} = (C^\alpha f, C^\alpha g)$$

($(\cdot, \cdot)$ is the inner product in $\mathfrak{H}$), and $\mathfrak{H}^{-\alpha}(A)$ is the dual of $\mathfrak{H}^\alpha(A)$, $(\alpha \ge 0)$ with respect to $(\cdot, \cdot)$.

Theorem 2. *Let $A$ be a normal operator in $\mathfrak{H}$ and $\mathcal{B}$ a Banach space such that the continuous embeddings*

$$\mathfrak{H}^{k_1}(A) \subseteq \mathcal{B} \subseteq \mathfrak{H}^{-k_2}(A)$$

*hold for some $k_i \in \mathbb{N}_0$ $(i = 1, 2)$. Assume also that the sequence $\{m_n\}_{n \in \mathbb{N}_0}$ satisfies the condition*

$$\exists c > 0, \ \exists h > 1 : m_{n+1} \le c h^n m_n.$$

*Then the following equivalences are true:*

$$f \in C^\infty(A) \Longleftrightarrow \lim_{r \to \infty} r^\alpha \mathcal{E}_r^{\mathcal{B}}(f) = 0, \ \forall \alpha > 0,$$

$$f \in C_{\{m_n\}}(A) \Longleftrightarrow \exists \alpha > 0, \exists c > 0 : \mathcal{E}_r^{\mathcal{B}}(f) \le c G^{-1}(\alpha r),$$

$$f \in C_{(m_n)}(A) \Longleftrightarrow \forall \alpha > 0, \exists c > 0 : \mathcal{E}_r^{\mathcal{B}}(f) \le c G^{-1}(\alpha r),$$

*where* $G(\lambda) = \sup_{n \in \mathbb{N}_0} (\lambda^n / m_n)$, $\quad \mathcal{E}_r^{\mathcal{B}}(f) = \inf_{g \in C_{\{1\}}(A): \sigma(g) \le r} \|f - g\|_{\mathcal{B}}.$

(The proofs of Theorems 1 and 2 under stronger assumptions are contained in [5]).

Theorem 2 implies many well-known results of the theory of approximation of functions. For instance, put

$$\mathfrak{H} = L_2(0, 2\pi),\ Af = i\frac{d}{dx},\ \mathcal{D}(A) = \{f \in W_2^1[0, 2\pi] : f(0) = f(2\pi)\}.$$

In this case the spectrum of $A$ $\sigma(A) = \mathbb{Z}$ and $\left\{\frac{1}{\sqrt{2\pi}}e^{ikx}\right\}_{k\in\mathbb{Z}}$ is the orthonormal basis of eigenvectors of the operator $A$, $C^\infty(A) = \widetilde{C}^\infty[0, 2\pi]$, $C_{\{1\}}(A)$ coincides with the set of trigonometric polynomials, $\sigma(g) \le n$ means $g \in \mathcal{T}_n$. In view of the estimate

$$c_1\|f\|_{L_2(0,2\pi)} \le \|f\|_{\widetilde{C}[0,2\pi]} \le c_2\|f\|_{W_2^1[0,2\pi]},$$

the space $\mathcal{B} = \widetilde{C}[0, 2\pi]$ of continuous $2\pi$-periodic functions with the norm $\|f\|_{\widetilde{C}[0,2\pi]} = \max\limits_{x\in[0,2\pi]} |f(x)|$ satisfies the conditions of Theorem 2. So, the next assertion is valid.

Theorem 3. *The following equivalences hold:*

$$f \in \widetilde{C}^\infty[0, 2\pi] \Longleftrightarrow \forall\alpha > 0 \quad \lim_{r\to\infty} n^\alpha E_n(f) = 0 \tag{4}$$

*(S. N. Bernshtein),*

$$f \in \widetilde{C}_{\{m_n\}}[0, 2\pi] \Longleftrightarrow \exists\alpha > 0,\ \exists c > 0 : E_n(f) \le cG^{-1}(\alpha n), \tag{5}$$

$$f \in \widetilde{C}_{(m_n)}[0, 2\pi] \Longleftrightarrow \forall\alpha > 0,\ \exists c > 0 : E_n(f) \le cG^{-1}(\alpha n), \tag{6}$$

*where*

$$\widetilde{C}_{\{m_n\}}[0, 2\pi] = \{f \in \widetilde{C}^\infty[0, 2\pi] : \exists\alpha > 0,\ \exists c > 0 : |f^{(k)}(x) \le c\alpha^k m_k\}$$

$$\widetilde{C}_{(m_n)}[0, 2\pi] = \{f \in \widetilde{C}^\infty[0, 2\pi] : \forall\alpha > 0,\ \exists c > 0 : |f^{(k)}(x) \le c\alpha^k m_k\}$$

*($\forall k \in \mathbb{N}_0$), $G(\lambda) = \sup\limits_{n\in\mathbb{N}_0} \frac{\lambda^n}{m_n}$, $E_n(f) = \mathcal{E}_n^{\widetilde{C}[0,2\pi]}(f)$ is the best uniform approximation of $f(x)$ by trigonometric polynomials from $\mathcal{T}_n$.*

The equivalences (5), (6) with $m_n = n!$ lead to the S. N. Bernshtein results from the subsection 1 on behavior of the best approximation $E_n(f)$ of an analytic or entire function by trigonometric polynomials. Indeed, in this case $\widetilde{C}_{\{m_n\}}[0, 2\pi] = \widetilde{\mathfrak{A}}[0, 2\pi]$, $\widetilde{C}_{(m_n)}[0, 2\pi] = \widetilde{\mathfrak{A}}_c[0, 2\pi]$, $G(\lambda) = e^\lambda$. Thus, the mentioned S. N. Bernshtein results are a consequence of (5) and (6).

Taking the spaces $L_2\big([-1, 1], \frac{1}{\sqrt{1-x^2}}\big)$, $L_2(R^1)$, $L_2\big(R^1, e^{-x^2}\big)$ as $\mathfrak{H}$, and the operators generated by the expressions

$$l_1 = \sqrt{1 - x^2}\frac{d}{dx}(\sqrt{1 - x^2}\frac{d}{dx}),\ l_2 = i\frac{d}{dx},\ l_3 = e^{x^2}\big(-\frac{d^2}{dx^2} + x^2\big)$$

respectively and the corresponding boundary conditions as $A$, we arrive at some of known and unknown results on the approximation of smooth functions by algebraic polynomials on a finite interval, entire functions of exponential type and algebraic polynomials on the whole real axis in the corresponding metrics.

**3.** Now consider the equation

$$Au = f, \tag{7}$$

where $A = A^*$ is a positive definite operator in $\mathfrak{H}$ with a discrete spectrum. The equation (7) has a unique solution. According to the Dirichlet principle (see, e. g. [6]), finding the solution of (7) is equivalent to that of the vector $u \in \mathcal{D}(A)$ such that the functional

$$F(v) = (Av, v) - 2\Re(f, v) \tag{8}$$

given on $\mathcal{D}(A)$ attains its minimum at $u$.

Let $\{e_k\}_{k\in\mathbb{N}}$ be a complete system of linearly independent vectors in $\mathfrak{H}^1(A)$ and $L_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$. Denote by $u_n \in L_n$ the vector on which the functional (8) considered on $L_n$ has minimum. The vector $u_n$ is called the Rietz approximation of the solution $u$. If $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of the eigenvectors of a selfadjoint positive definite operator $B$ related to $A$ (that is, $\mathcal{D}(B) = \mathcal{D}(A)$), then as is well-known,

$$u_n \to u \text{ in } \mathfrak{H}^{1/2}, \quad Au_n \to f \text{ in } \mathfrak{H}. \tag{9}$$

It turns out to be that the convergence $Au_n \to f$ in (8) may be arbitrarily slow.

Theorem 4. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers convergent monotonically to 0. Under the above conditions on the operator A and the system $\{e_k\}_{k\in\mathbb{N}}$, there exists a vector $f \in \mathfrak{H}$ such that*

$$\|Au_n - f\| \geq \alpha_n.$$

The question arises under what conditions on $f \in \mathfrak{H}$, the value $Au_n - f$ has a preassigned order of decrease. The predecessors' results (see, e. g. [7]) concerned only the power decrease , and they dealt only with direct theorems. The following statement reduces this restriction.

Theorem 5. *Let $A = A^* > \gamma I$ $(\gamma > 0)$ be an operator with a discrete spectrum, and $\{e_k\}_{k\in\mathbb{N}}$ be the orthonormal basis of eigenvectors of an operator $B$ related to $A$. Then*

$$u \in C^\infty(B) \Longleftrightarrow \forall k \in \mathbb{N} \quad \|Au_n - f\| = o(\lambda_n^{-k}(B)), \tag{10}$$

$$u \in C_{\{m_n\}}(B) \Longleftrightarrow \exists \alpha > 0, \exists c > 0 : \|Au_k - f\| \leq cG^{-1}(\alpha\lambda_k) \tag{11}$$

*($k \in \mathbb{N}$ is arbitrary),*

$$u \in C_{(m_n)}(B) \Longleftrightarrow \forall \alpha > 0, \exists c > 0 : \|Au_k - f\| \leq cG^{-1}(\alpha\lambda_k), \tag{12}$$

*where $\lambda_n$ is the eigenvalue of the operator $B$ corresponding to $e_n$, $G(\lambda) = \sup\limits_{n\in\mathbb{N}_0} \frac{\lambda^n}{m_n}$.*

If $m_n = n^{n\beta}$ $(\beta > 0)$, then $G(\lambda) = \exp(\lambda^{1/\beta})$, and the estimate in (11) and (12) takes the form

$$\|Au_n - f\| \leq c\exp(-\lambda_n^{1/\beta}).$$

Even in the elementary case where $\mathfrak{H} = L_2(0, \pi)$, $A = -\frac{d^2}{dx^2} + q(x)$ ($q \geq 0$ is a continuous function on $[0, \pi]$, $\mathcal{D}(A) = W_2^2[0, \pi] \cap \overset{\circ}{W_2^1}[0, \pi]$, $B = -\frac{d^2}{dx^2}$, $\mathcal{D}(B) = \mathcal{D}(A)$, we obtain new results of both direct and inverse character.

Theorem 6. *Let the function $q(x)$ be infinitely differentiable, and*

$$q^{(2k-1)}(0) = q^{(2k-1)}(\pi) = 0 \quad (k \in \mathbb{N}). \tag{13}$$

*Then*

$$\forall \alpha > 0 \quad n^{\alpha}\|u - u_n\| \to 0 \ (n \to \infty)$$

$$\Longleftrightarrow f \text{ is infinitely differentiable on } [0, \pi] \text{ and}$$

$$f^{(2k)}(0) = f^{(2k)}(\pi) = 0 \quad (k \in \mathbb{N}). \tag{14}$$

*If $q(x)$ is analytic and satisfies (13), then*

$$\exists \alpha > 0 : e^{\alpha n}\|u - u_n\| \to 0 \ (n \to \infty)$$

$$\Longleftrightarrow f \text{ is analytic on } [0, \pi] \text{ and satisfies } (14)$$

*If $q(x)$ is entire and satisfies (13), then*

$$\forall \alpha > 0 : e^{\alpha n}\|u - u_n\| \to 0 (n \to \infty) \Longleftrightarrow f \text{ is entire and satisfies } (14).$$

We have formulated the results for the Rietz method. The similar assertions are true for the moment method (in this case the operator $A$ in (7) is invertible and $K$-positive definite in the W.V. Petryshin sense [8]). If $K = I$, the latter method becomes the Rietz one. If $K = A$, we have the method of least squares.

## References


1. Gutler R.S., Kudryavtzev L.D., Levitan B.M., *Elements of Function Theory*, Moscow: Nauka, 1963.
2. Babin A.V., *Solving the Cauchy problem with the help of weighted approximation of exponents by polynomials*, Funkts. Anal. and Pril. **17** (1983), no. 4, 75-76.
3. Babin A.V., *Construction and investigation of solutions of differential equations by approximation theory methods*, Mat. Sb. **123** (1984), no. 2, 147-173.
4. Gorbachuk M.L., Gorodetskii V.V., *On polynomial approximation of solutions of operator differential equations in a Hilbert space*, Ukr. Mat. Zh. **36** (1984), no. 4, 500-502.
5. Gorbachuk M.L., Gorbachuk V.I., *On approximation of smooth vectors of a closed operator by entire vectors of exponential type*, Ukr. Mat. Zh. **47** (1995), no. 5, 616-628.
6. Mikhlin S.G., *Variation Methods in Mathematical Physics*, Moscow: Nauka, 1970.
7. Luchka A.Yu., Luchka T.F., *Origin and Development of direct methods of Mathematical Physics*, Kiev: Naukova Dumka, 1985.
8. Petryshin W.V., *On a class of K.-p.d. and non-K.-p.d. operators and operator equations*, J. Math. Anal. and Appl. **10** (1965), 1-24.



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