AVERAGING OF THE PERIODIC BY TIME BOUNDARY VALUE PROBLEM FOR THE NONLINEAR WAVE EQUATION IN A PERFORATED DOMAIN

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ABSTRACT. In a bounded domain with infinitely increasing quantity of infinitesimal holes whose asymptotic behaviour is described formally, the homogeneous Dirichlet problem for the equation with additional power nonlinearity on the time derivative of the solution is averaged.

1. Setting of the problem. Let Ω be a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$. We denote Ω_ε a domain obtained by removing from Ω a number $N(\varepsilon)$ of closed subsets S_ε^i, i.e. $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$, $S_\varepsilon = \cup S_\varepsilon^i$ $(i = \overline{1, N(\varepsilon)})$. Here $\varepsilon > 0$ is a parameter wich tends to zero on some subsequence, herewith $N(\varepsilon) \to \infty$. Let $T = \text{const} > 0$. It being investigated a behaviour when $\varepsilon \to 0$ of a solution $u_\varepsilon(t,x)$ of the problem

$$\begin{gathered} u_\varepsilon'' - \Delta u_\varepsilon + |u_\varepsilon'|^{p-2} u_\varepsilon' = f_\varepsilon(t,x) \ \text{ in } Q_\varepsilon, \\ u_\varepsilon(t,\cdot)|_{\partial\Omega_\varepsilon} = 0, \ u_\varepsilon(0,x) = u_\varepsilon(T,x), \\ u_\varepsilon'(0,x) = u_\varepsilon'(T,x), \ x \in \Omega_\varepsilon, \end{gathered} \tag{1}$$

where $u_\varepsilon' = \partial u_\varepsilon / \partial t$, $u_\varepsilon'' = \partial^2 u_\varepsilon / \partial t^2$, $Q_\varepsilon = \{(t,x) \in (0,T) \times \Omega_\varepsilon\}$, $f_\varepsilon(t,x) \in L^{p'}(Q_\varepsilon)$, $p' = p/(p-1), 2 < p < \infty$. For each value ε it has the unique solution [1]

$$\begin{gathered} u_\varepsilon(t,x) = \bar{u}_\varepsilon(x) + \tilde{u}_\varepsilon(t,x), \langle \tilde{u}_\varepsilon(\cdot,x) \rangle \equiv \frac{1}{T} \int\limits_0^T \tilde{u}_\varepsilon(t,x)dt = 0, \\ \bar{u}_\varepsilon \in \overset{\circ}{W}{}^{1,p'}(\Omega_\varepsilon), \ \tilde{u}_\varepsilon \in L^2(0,T; \overset{\circ}{H}{}^1(\Omega_\varepsilon)) \cap W^{1,p}(0,T; L^p(\Omega_\varepsilon)). \end{gathered} \tag{2}$$

The behaviour of a closed set $S_\varepsilon \subset \bar{\Omega}$ when $\varepsilon \to 0$ being described by the following basic hypothesis [2]:

there exists such a sequence of functions $w_\varepsilon(x)$ that
1) $w_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), ||w_\varepsilon||_{L^\infty(\Omega)} \leq M_0$, 2) $w_\varepsilon|_{S_\varepsilon} = 0$,
3) $w_\varepsilon \to 1$ weakly in $H^1(\Omega)$ and almost everywhere in Ω,
4) $-\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon$ where $\mu_\varepsilon, \gamma_\varepsilon \in H^{-1}(\Omega), \mu_\varepsilon \to \mu$ strongly in $H^{-1}(\Omega)$, $(\gamma_\varepsilon, v)_\Omega = 0$ for each $v \in \overset{\circ}{H}{}^1(\Omega)$ if $v|_{S_\varepsilon} = 0$. (A)

Brackets $(\cdot,\cdot)_\Omega$ denote the scalar product in $L^2(\Omega)$ with respect to the Lebesgue measure dx and correspondent duality relations. Designations $(\cdot,\cdot)_{\Omega_\varepsilon}$, $(\cdot,\cdot)_{Q_\varepsilon}$, $(\cdot,\cdot)_Q$ are analogous.

The function $\tilde{u}_\varepsilon(t,x)$ is defined by the integral identity [1]

$$\int\limits_0^T [-(\tilde{u}'_\varepsilon, v'')_{\Omega_\varepsilon} + (\nabla \tilde{u}_\varepsilon, \nabla v')_{\Omega_\varepsilon} + (F(\tilde{u}'_\varepsilon) - f_\varepsilon, v')_{\Omega_\varepsilon}\,]dt = 0, \tag{3}$$

where $F(u) = |u|^{p-2}u$, $v(t,x)$ is an arbitrary function such that

$$\begin{gathered} v \in W^{2,p'}(0,T; L^{p'}(\Omega_\varepsilon)) \cap W^{1,p}(0,T; L^p(\Omega_\varepsilon)) \cap \\ \cap H^1(0,T; \overset{\circ}{H}{}^1(\Omega_\varepsilon)), \\ v(0,x) = v(T,x), \;\; v'(0,x) = v'(T,x), \;\; \langle v(\cdot,x)\rangle = 0, \; x \in \Omega_\varepsilon. \end{gathered} \tag{4}$$

2. The estimates of solutions. Periodicity of a function φ by t with the period T is denoted by inclusion $\varphi \in \mathcal{T}_t[0,T]$. Let us choose a sequence of such $\theta_n(t) \in C^\infty(\mathbf{R}) \cap \mathcal{T}_t[0,T]$, $\mathbf{N} \ni n > T^{-1}$, that $\theta_n(-t) \equiv \theta_n(t)$, supp $\theta_n = \cup[kT-(2n)^{-1}, kT+(2n)^{-1}]$ $(k \in \mathbf{Z})$, $\int_0^{(2n)^{-1}} \theta_n(t)dt = \frac{1}{2}$. We put $(\varphi * \psi)(t) = \int_0^T \varphi(t-\tau)\psi(\tau)d\tau = (\psi * \varphi)(t)$ for any $\varphi, \psi \in \mathcal{T}_t[0,T] \cap L^1(0,T)$. Since $\tilde{u}_\varepsilon * \theta_n \in C^\infty(\mathbf{R}; \overset{\circ}{H}{}^1(\Omega_\varepsilon) \cap L^p(\Omega_\varepsilon)) \cap \mathcal{T}_t[0,T]$, $\langle(\tilde{u}_\varepsilon * \theta_n)(\cdot,x)\rangle = 0$, in (3) we can take $v = \tilde{u}_\varepsilon * \theta_n * \theta_n$. Herewith $(F(\tilde{u}'_\varepsilon) - f_\varepsilon, \tilde{u}'_\varepsilon * \theta_n * \theta_n)_{Q_\varepsilon} = 0$. As a result of passing to the limit by $n \to \infty$ and in view of (2), we obtain

$$\begin{gathered} ||\tilde{u}'_\varepsilon||^p_{L^p(Q_\varepsilon)} = (f_\varepsilon, \tilde{u}'_\varepsilon)_{Q_\varepsilon}, \;\; ||\tilde{u}'_\varepsilon||_{L^p(Q_\varepsilon)} \le ||f_\varepsilon||^{1/(p-1)}_{L^{p'}(Q_\varepsilon)}, \\ ||\tilde{u}_\varepsilon||_{L^p(Q_\varepsilon)} \le C_0 ||\tilde{u}'_\varepsilon||_{L^p(Q_\varepsilon)} \le C_0 ||f_\varepsilon||^{1/(p-1)}_{L^{p'}(Q_\varepsilon)}, \; C_0 = \text{const.} \end{gathered} \tag{5}$$

If $\sup ||f_\varepsilon||_{L^{p'}(Q_\varepsilon)} = M < \infty$, then the following estimates are valid:

$$||\tilde{u}'_\varepsilon||_{L^p(Q_\varepsilon)} \le M^{1/(p-1)}, \;\; ||\tilde{u}_\varepsilon||_{L^p(Q_\varepsilon)} \le C_0 M^{1/(p-1)} \; \forall \varepsilon. \tag{6}$$

Putting further

$$v(t,x) = \int\limits_0^t \tilde{u}_\varepsilon(\tau,x)d\tau - \frac{1}{T}\int\limits_0^T (T-\tau)\tilde{u}_\varepsilon(\tau,x)d\tau,$$

we have the function satisfying all the conditions (4). Substituting it into (3), we obtain $||\nabla \tilde{u}_\varepsilon||^2_{L^2(Q_\varepsilon)} = ||\tilde{u}'_\varepsilon||_{L^2(Q_\varepsilon)} + (f_\varepsilon - F(\tilde{u}'_\varepsilon), \tilde{u}_\varepsilon)_{Q_\varepsilon} \le |Q|^{(p-2)/p} ||\tilde{u}'_\varepsilon||^2_{L^p(Q_\varepsilon)} + ||\tilde{u}'_\varepsilon||^{p-1}_{L^p(Q_\varepsilon)} ||\tilde{u}_\varepsilon||_{L^p(Q_\varepsilon)} + ||f_\varepsilon||_{L^{p'}(Q_\varepsilon)} ||\tilde{u}_\varepsilon||_{L^p(Q_\varepsilon)} \le |Q|^{(p-2)/p} M^{2/(p-1)} + 2C_0 M^{p'} = C_1^2$, where $Q = \{(t,x) \in (0,T) \times \Omega\}$, i.e. the estimate is valid

$$||\tilde{u}_\varepsilon||_{L^2(0,T; \overset{\circ}{H}{}^1(\Omega_\varepsilon))} \le C_1 \; \forall \varepsilon. \tag{7}$$

3. Passing to the limit. For an arbitrary function v given on Ω_ε, we denote $\hat{v}$ its propagation on Ω by definition by zero on S_ε. In view of (2), we have

$$\begin{gathered} \hat{\tilde{u}}_\varepsilon \in L^2(0,T; \overset{\circ}{H}{}^1(\Omega)) \cap W^{1,p}(0,T; L^p(\Omega)) \cap \mathcal{T}_t[0,T], \\ \langle \hat{\tilde{u}}_\varepsilon(\cdot,x)\rangle = 0, \end{gathered} \tag{8}$$

and from (6),(7) the estimates follow

$$||\hat{\tilde{u}}'_\varepsilon||_{L^p(Q)} \le M^{1/(p-1)}, \;\; ||\hat{\tilde{u}}_\varepsilon||_{L^p(Q)} \le C_0 M^{1/(p-1)}, \\ ||\hat{\tilde{u}}_\varepsilon||_{L^2(0,T;\overset{\circ}{H}{}^1(\Omega))} \le C_1. \tag{9}$$

The last estimates give a possibility to extract a subsequence (again denoted by index ε) for which the convergences are valid

$$\hat{\tilde{u}}_\varepsilon \to \tilde{u} \text{ weakly in } L^2(0,T;\overset{\circ}{H}{}^1(\Omega)) \cap L^p(Q), \\ \hat{\tilde{u}}'_\varepsilon \to \tilde{u}' \text{ weakly in } L^p(Q), \;\; F(\hat{\tilde{u}}'_\varepsilon) \to \chi \text{ weakly in } L^{p'}(Q). \tag{10}$$

For any function $\psi(t,x)$ such that

$$\psi \in W^{1,p'}(0,T;L^{p'}(\Omega_\varepsilon)) \cap L^2(0,T;\overset{\circ}{H}{}^1(\Omega_\varepsilon)) \cap \\ \cap L^p(Q_\varepsilon) \cap \mathcal{T}_t[0,T], \;\; \langle\psi(\cdot,x)\rangle = 0, \tag{11}$$

we define the test function for the identity (3)

$$v(t,x) = \int_0^t \psi(\tau,x)d\tau - \frac{1}{T}\int_0^T (T-\tau)\psi(\tau,x)d\tau,$$

which satisfies the conditions (4). Substituting it into (3), we obtain

$$-(\hat{\tilde{u}}'_\varepsilon, \hat{\psi}')_Q + (\nabla\hat{\tilde{u}}_\varepsilon, \nabla\hat{\psi})_Q + (F(\hat{\tilde{u}}'_\varepsilon) - \hat{f}_\varepsilon, \hat{\psi})_Q = 0, \tag{12}$$

where $\hat{\psi}(t,x)$ is an arbitrary function defined on $\mathbf{R}\times\Omega$, whose properties on Q are analogous to (11), $\hat{\psi}(t,\cdot)|_{S_\varepsilon} = 0$. We put $\hat{\psi}(t,x) = w_\varepsilon(x)v(t,x)$, where w_ε are the same functions as in the hypothesis (A) and $v(t,x) \in C^{1,2}(\mathbf{R}\times\bar{\Omega}) \cap \mathcal{T}_t[0,T] : v(t,\cdot)|_{\partial\Omega} = 0$, $\langle v(\cdot,x)\rangle = 0$. After substitution of this function $\hat{\psi}$ into (12), we have

$$0 = -(\hat{\tilde{u}}'_\varepsilon, w_\varepsilon v')_Q + (\hat{\tilde{u}}_\varepsilon, -\Delta(w_\varepsilon v))_Q + (F(\hat{\tilde{u}}'_\varepsilon) - \hat{f}_\varepsilon, w_\varepsilon v)_Q = \\ = -(\hat{\tilde{u}}'_\varepsilon, w_\varepsilon v')_Q + (\mu_\varepsilon, v\hat{\tilde{u}}_\varepsilon)_Q - 2(\hat{\tilde{u}}_\varepsilon, \nabla w_\varepsilon \cdot \nabla v)_Q - \\ -(\hat{\tilde{u}}_\varepsilon, w_\varepsilon \Delta v)_Q + (F(\hat{\tilde{u}}'_\varepsilon) - \hat{f}_\varepsilon, w_\varepsilon v)_Q.$$

Here it has been taken into account that $(\gamma_\varepsilon, \langle v\hat{\tilde{u}}_\varepsilon\rangle)_\Omega = 0$. We consider this expression on the extracted subsequence $\varepsilon \to 0$ taking into account (10) and the hypothesis (A) and supposing that the entire sequence $\hat{f}_\varepsilon \to f$ weakly in $L^{p'}(Q)$. In the limit we come to the identity

$$-(\tilde{u}', v')_Q + (\mu\tilde{u}, v)_Q + (\nabla\tilde{u}, \nabla v)_Q + (\chi - f, v)_Q = 0, \tag{13}$$

which is valid for all $v(t,x) \in L^2(0,T;\overset{\circ}{H}{}^1(\Omega)) \cap L^p(Q) \cap \mathcal{T}_t[0,T]$ such that $v' \in L^{p'}(Q)$, $\langle v(\cdot,x)\rangle = 0$.

Further, we take into consideration that the generalized function $\mu(x) \in H^{-1}(\Omega)$, introduced in the hypothesis (A), generates simultaneously the positive finite Radon measure $d\mu(x) = \mu(x)dx$ on Ω [2].

It follows from (8) and (10) that

$$\begin{gathered}\tilde{u}(0,x) = \tilde{u}(T,x) \\ \text{as a function from } C^0([0,T];L^p(\Omega)), \langle \tilde{u}(\cdot,x)\rangle = 0.\end{gathered} \tag{14}$$

Substituting $v = \tilde{u}$ into (13), we obtain

$$\int_0^T [||\tilde{u}||^2_{L^2(\Omega,d\mu)} + || \nabla \tilde{u}||^2_{L^2(\Omega)}]dt \leq$$
$$\leq |Q|^{(p-2)/p}||\tilde{u}'||^2_{L^p(Q)} + ||f-\chi||_{L^{p'}(Q)}||\tilde{u}||_{L^p(Q)},$$

i.e. we have additionally to (14)

$$\begin{gathered}\tilde{u}(t,x) \in L^2(0,T;V) \cap W^{1,p}(0,T;L^p(\Omega)), \\ V = \overset{\circ}{H^1}(\Omega) \cap L^2(\Omega, d\mu).\end{gathered} \tag{15}$$

For the identification of a function χ in (10), we use the equality (5) and consider on the extracted subsequence the following inequality with an arbitrary $v \in L^p(Q)$

$$\begin{gathered}0 \leq (F(\hat{\tilde{u}}'_\varepsilon) - F(v), \hat{\tilde{u}}'_\varepsilon - v)_Q = \\ = (\hat{f}_\varepsilon, \hat{\tilde{u}}'_\varepsilon)_Q - (F(\hat{\tilde{u}}'_\varepsilon), v)_Q - (F(v), \hat{\tilde{u}}'_\varepsilon - v)_Q.\end{gathered} \tag{16}$$

In supposition that the entire sequence $\hat{f}_\varepsilon \to f$ strongly in $L^{p'}(Q)$, after passing to the limit in (16) we get

$$0 \leq (f, \tilde{u}')_Q - (\chi, v)_Q - (F(v), \tilde{u}' - v)_Q. \tag{17}$$

Taking into account (14),(15), we take in (13) the test function $v = \tilde{u}' * \theta_n * \theta_n \in C^\infty(\mathbf{R}; V \cap L^p(\Omega)) \cap \mathcal{T}_t[0,t]$. As the result we obtain $0 = (f-\chi, \tilde{u}' * \theta_n * \theta_n)_Q = (f-\chi, \tilde{u}')_Q$. From this and (17) it follows $(\chi - F(v), \tilde{u}' - v)_Q \geq 0 \forall v \in L^p(Q)$ signifying that $\chi = F(\tilde{u}')$.

From (13) $\forall v \in D(Q)$ such that $\langle v(\cdot,x)\rangle = 0$, we have

$$(\tilde{u}'' + \mu\tilde{u} - \Delta\tilde{u} + F(\tilde{u}') - f, v)_Q = 0, \tag{18}$$

that is

$$\tilde{u}'' + \mu\tilde{u} - \Delta\tilde{u} + F(\tilde{u}') - f = g(x) \text{ in } D'(Q). \tag{19}$$

It is easy to see that in (19) $g(x) \in V' + L^{p'}(\Omega)$ and $\tilde{u}'' \in L^2(0,T;V') + L^{p'}(Q) \subset L^{p'}(0,T;V' + L^{p'}(\Omega))$, so in view of (15)

$$\tilde{u}' \in W^{1,p'}(0,T;V' + L^{p'}(\Omega)) \subset C^0([0,T];V' + L^{p'}(\Omega)). \tag{20}$$

Then we can write the identity (13) $\forall v \in H^1(0,T;V) \cap W^{1,p}(0,T;L^p(\Omega)) \cap \mathcal{T}_t[0,T] : \langle v(\cdot,x)\rangle = 0$ as follows

$$\begin{gathered}(\tilde{u}'(0,\cdot) - \tilde{u}'(T,\cdot), v(0,\cdot))_\Omega + \\ (\tilde{u}'' + \mu\tilde{u} - \Delta\tilde{u} + F(\tilde{u}') - f, v)_Q = 0.\end{gathered}$$

Herewith, for the functions v appointed (18) is valid as before now, therefore $(\tilde{u}'(0,\cdot)-\tilde{u}'(T,\cdot), v(0,\cdot))_\Omega = 0$ where $v(0,x)$ may be by any function from $V\cap L^p(\Omega)$. In view of (20), from this it follows that

$$\tilde{u}'(0,x) \;=\; \tilde{u}'(T,x). \tag{21}$$

On the base of (14),(20),(21) we make more precise an indefinite function $g(x)$ in (19)

$$g(x) = \langle \tilde{u}'' + \mu\tilde{u} - \Delta\tilde{u} + F(\tilde{u}') - f\rangle = \langle F(\tilde{u}') - f\rangle \in L^{p'}(\Omega). \tag{22}$$

4. The convergence of the mean by time value. The mean value $\bar{u}_\varepsilon(x) = \langle u_\varepsilon(\cdot,x)\rangle$ is defined by the following from (1),(2) linear problem

$$-\Delta\bar{u}_\varepsilon(x) = g_\varepsilon(x) = \bar{f}_\varepsilon(x) - \overline{F(\tilde{u}_\varepsilon')}(x),\;\; x\in\Omega_\varepsilon,\; \bar{u}_\varepsilon \in \overset{\circ}{W}{}^{1,p'}(\Omega_\varepsilon), \tag{23}$$

in which, if $\varepsilon\to 0$ on the extracted subsequence, $\hat{g}_\varepsilon \to -g$ weakly in $L^{p'}(\Omega)$. Starting with this place let us assume that in (1)

$$2 \;<\; p \;\le\; \frac{2n}{n-2}\;(n>2),\;\; 2<p<\infty\;(n=2). \tag{24}$$

Herewith $L^{p'}(\Omega)\subset H^{-1}(\Omega)$ and

$$||\hat{\bar{u}}_\varepsilon||_{\overset{\circ}{H}{}^1(\Omega)} \le ||\hat{g}_\varepsilon||_{H^{-1}(\Omega)} \le C_2||\hat{g}_\varepsilon||_{L^{p'}(\Omega)} \le C_2M_1\;\;\forall\varepsilon. \tag{25}$$

Having extracted a subsequence

$$\hat{\bar{u}}_\varepsilon \to \bar{u} \text{ weakly in } \overset{\circ}{H}{}^1(\Omega), \tag{26}$$

we get as in [2]

$$-\Delta\bar{u} + \mu\bar{u} = -g,\;\; \bar{u}\in V, \tag{27}$$

and $\bar{u}$ defined by the problem (27) is unique. Then the convergence (26) is valid for the entire sequence $\{\hat{\bar{u}}_\varepsilon\}$.

5. Formulation of the result. Substituting the value of g from (27) into (19) we obtain for the function $u(t,x) = \bar{u}(x) + \tilde{u}(t,x)$ the problem

$$\begin{gathered} u'' - \Delta u + \mu u + F(u') = f \text{ in } Q,\\ u(0,x) = u(T,x),\; u'(0,x) = u'(T,x),\; x\in\Omega,\;\; u(t,\cdot)|_{\partial\Omega} = 0, \end{gathered} \tag{28}$$

in which

$$\begin{gathered} u\in L^2(0,T;V)\cap W^{1,p}(0,T;L^p(\Omega)),\\ u''\in L^{p'}(0,T;V'),\;\; u'\in C^0([0,T];V'). \end{gathered} \tag{29}$$

It being easily proved that the solution of the problem (28),(29) is unique. Thus the following result has been established:

Theorem. *Let Ω be a bounded domain in $\mathbf{R}^n$ $(n\ge 2)$, and S_ε be a sequence of its closed subsets for which the hypothesis* (A) *is valid. Let p satisfies inequalities* (24), *the sequence $\hat{f}_\varepsilon \to f$ strongly in $L^{p'}(Q)$. Then for the sequence of solutions of problems* (1) *the following convergences take place:*

$$\hat{u}_\varepsilon \to u \textit{ weakly in } L^2(0,T;\overset{\circ}{H}{}^1(\Omega))\cap L^p(Q),$$

$\hat{u}_\varepsilon' \to u'$ *weakly in* $L^p(Q)$, $|\hat{u}_\varepsilon'|^{p-2}\hat{u}_\varepsilon' \to |u'|^{p-2}u'$ *weakly in* $L^{p'}(Q)$, *where u is the unique solution of the problem* (28) *satisfying the conditions* (29).

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