ON THE STRUCTURAL REPRESENTATION OF S-HOMOGENIZED OPTIMAL CONTROL PROBLEMS

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The aim of this paper is an application of variational S-convergence [1-3] to homogenization theory of optimal control problems and to explore the structure of homogenized problems. We have proved the existence of strongly S-homogenized optimal control problems for a family of nonlinear systems with distributed parameters, have derived some important properties which will be use in future and give the formula for representation of homogenized problems.

Let us consider the following family of optimal control problems

$$\inf I_\varepsilon(u,y), \tag{1}$$

$$A_\varepsilon(u,y)=f_\varepsilon,\; F_\varepsilon(u,y)\geq 0, \quad u\in U_\partial, \quad y\in K_\varepsilon, \tag{2}$$

where $U=V^\star$ – control space, which is dual of a separable Banach space V, U_∂ – admissible class of control with U, K_ε – weakly closed subsets in a separable Banach space Y, $Y\subset X$ with continuous and dense injection, where X is a reflexive Banach space, Z – semi-ordered by reproducing cone L a Banach space, ε denotes a "small" multiparameter with a set E, partialy ordered by decreasing ($0\leq\varepsilon$ for every $\varepsilon\in E$ and 0 is the minimal element in E), f_ε is a fixed element with Y^*, $A_\varepsilon : U\times(D(A_\varepsilon)\subset X)\to Y^*$, $F_\varepsilon : U\times Y\to Z$ – are nonlinear operators, which may arbitrarily depend on ε, $I_\varepsilon : U\times X\to\overline{R}$ – is a cost function. In so doing, we will assume that $K_\varepsilon\subset(D(A_\varepsilon)\cap Y)$.

Definition 1. The pair "control-state" $(u,y)\in U\times Y$ will be called an admissible if (u,y) satisfies restrictions (2). We denote by Ξ_ε the set of all admissible pairs for the fixed ε.

Definition 2. The admissible pair $\left(u_\varepsilon^0,y_\varepsilon^0\right)\in\Xi_\varepsilon$, which gives the least possible value to the functional I_ε we shall call an optimal pair, i.e. $I_\varepsilon\left(u_\varepsilon^0,y_\varepsilon^0\right)=\inf\limits_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)$.

In the future we assume that: **(a)** the cost function I_ε is sequentially lower semicontinuous in the $*$-weak topology of U and weak topology of Y, i.e. from $u_n\to u$ $*$-weakly in U and $y_n\to y$ weakly in Y it follows that $\liminf\limits_{n\to\infty} I_\varepsilon(u_n,y_n)\geq I_\varepsilon(u,y)$; **(b)** A_ε is a $*$-demicontinuous operator, i.e. A_ε is a continuous operator from $U\times(D(A_\varepsilon)\subset X)$ with the $*$-weak topology of U and the weak topology of Y in Y^* with the $*$-weak topology; **(c)** for every $\varepsilon\in E$ the operator A_ε is coercive on Y, i.e. for every bounded set G of U we have $\inf\limits_{u\in G}\frac{\langle A_\varepsilon(u,y),y\rangle_Y}{\|y\|_Y}$ as $\|y\|_Y\to\infty$; **(d)** the operator $F_\varepsilon : U\times Y\to Z$ is a weakly continuous; **(e)** U_∂ is a nonempty, bounded and $*$-weakly closed subset of U; **(f)** the injection $Y\subset X$ is a compact; **(g)** $K_\varepsilon\subset(D(A_\varepsilon)\cap Y)$ is nonempty and weakly closed subsets of Y.

By analogy with [4] it is possible to prove the following result.

Theorem 1. *Assume that conditions (a)-(g) are satisfyed. Then for fixed ε the optimal control problem* (1)-(2) *has a solution if and only if this problem is a regular (i.e. the set $\Xi_\varepsilon \subset U_\partial \times K_\varepsilon$ is nonempty).*

Let τ be the topology of $U \times Y$ wich equal to the product of $*$-weak topology of U and weak topology of Y. For any $x = (u, y)$ belonging to $U \times Y$, let us denote by $\mathfrak{N}_\tau(x)$ the filter of neighbourhoods of x with respect to topology τ. Let $\tau{-}Li\,\Xi_\varepsilon$, $\tau{-}Ls\,\Xi_\varepsilon$ are lower and upper topological limits of the generalized sequence $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ (see [5,6]). If $\Xi = \tau{-}Li\,\Xi_\varepsilon = \tau{-}Ls\,\Xi_\varepsilon$ then the sequence $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ is said to be topological convergent to Ξ and the limit set Ξ will be denoted by $\tau{-}Lm\,\Xi_\varepsilon$.

Before introducing the formal axiomatics for homogenization process we rewrite the problems (1)-(2) in another form:

$$\left\{\left\langle \inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)\right\rangle, \varepsilon \in E\right\}, \tag{3}$$

Thus the homoginization of family problems (1)-(2) consists in studing of limitary and variational properties of generalized sequence (3).

Definition 3. The lower S-limit of the generalized sequence $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{R}\right\}_{\varepsilon\in E}$, denoted by I_s (or $\tau{-}li_s I_\varepsilon$), is the functional from $\tau{-}Ls\,\Xi_\varepsilon$ to $\overline{R}$ defined by

$$I_s(u,y) = \sup_{W\in\mathfrak{N}_\tau(u,y)} \liminf_{(\varepsilon\in E, W\cap\Xi_\varepsilon\neq\emptyset)} \inf_{(v,p)\in W\cap\Xi_\varepsilon} I_\varepsilon(v,p).$$

Definition 4. The upper S-limit of the generalized sequence $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{R}\right\}_{\varepsilon\in E}$, denoted by I^s (or $\tau{-}ls_s I_\varepsilon$), is the functional from $\tau{-}Li\,\Xi_\varepsilon$ to $\overline{R}$ defined by

$$I^s(u,y) = \sup_{W\in\mathfrak{N}_\tau(u,y)} \limsup_{\varepsilon\in E} \inf_{(v,p)\in W\cap\Xi_\varepsilon} I_\varepsilon(v,p).$$

Definition 5. The generalized sequence $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{R}\right\}_{\varepsilon\in E}$ is said to be S-convergent if the equality $I_s(u,y) = I^s(u,y)$ holds for every $(u,y) \in \tau{-}Li\,\Xi_\varepsilon$. This common value is then denoted as $\tau{-}lm_s I_\varepsilon$. If the foregoing identity is true for every $(u,y) \in \tau{-}Ls\,\Xi_\varepsilon$, then this common value, denoted by $\tau{-}lm^a_s I_\varepsilon$, will be called the absolute S-limit of such sequence.

The techniques of S-convergence and basic variational and topological properties of S-limits are discussed in more detail in [1-3].

Definition 6. The lower homogenized (respectively, upper homogenized) optimal control problem for the family problems (1)-(2) is defined by $\inf_{(u,y)\in\tau-Ls\,\Xi_\varepsilon} I_s(u,y)$, (respectively, $\inf_{(u,y)\in\tau-Li\,\Xi_\varepsilon} I^s(u,y)$). If directedness $\left\{I_\varepsilon : \Xi_\varepsilon \to \overline{R}\right\}_{\varepsilon\in E}$ S-converges to $\tau{-}lm_s I_\varepsilon(u,y)$ (respectively, absolutely S-convergence to $\tau{-}lm^a_s I_\varepsilon(u,y)$), then the minimization problem $\inf_{(u,y)\in\tau-Li\,\Xi_\varepsilon} (\tau{-}lm_s I_\varepsilon)(u,y)$ (respectively, $\inf_{(u,y)\in\tau-Lm\,\Xi_\varepsilon} (\tau{-}lm^a_s I_\varepsilon)$ (u,y)) is called S-homogenized (respec- tively, strong S-homogenized) optimal control problem.

Let us denote by

$$M\left(I_\varepsilon, \Xi_\varepsilon\right)=\left\{\left(u_\varepsilon^0, y_\varepsilon^0\right) \in \Xi_\varepsilon \,\middle|\, I_\varepsilon\left(u_\varepsilon^0, y_\varepsilon^0\right)=\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)\right\},$$

$$M^\alpha\left(I_\varepsilon, \Xi_\varepsilon\right)=\left\{\left(u_\varepsilon, y_\varepsilon\right) \in \Xi_\varepsilon \,\middle|\, I_\varepsilon\left(u_\varepsilon, y_\varepsilon\right) \leq \sup\left(\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)+\alpha, -1/\alpha\right)\right\}, \forall\alpha>0$$

the sets of all optimal and α-optimal solution of (1)-(2).

Definition 7. The generalized sequence $\left\{I_\varepsilon : \Xi_\varepsilon \rightarrow \overline{R}\right\}_{\varepsilon\in E}$ will be called an equi-coercive if exists a τ-lower semicontinuous and τ-lower semicompactness function $\Psi : U \times Y \rightarrow \overline{R}$ such that $I_\varepsilon(u,y) \geq \Psi(u,y) \quad \forall(u,y)\in\Xi_\varepsilon \quad \forall\varepsilon\in E$.

Let us assume that hypotheses (a)–(g) are true. Then the main results of [1-3] can be extended on the case of homogenized optimal control problem.

Proposition 1. *The functionals cost $I_s : \tau{-}Ls\,\Xi_\varepsilon \rightarrow \overline{R}$ and $I^s : \tau{-}Li\,\Xi_\varepsilon \rightarrow \overline{R}$ are τ-lower semicontinuous on $\tau{-}Ls\,\Xi_\varepsilon$ and $\tau{-}Li\,\Xi_\varepsilon$. Furthermore, the domaines of its functionals are τ-closed sets.*

Proposition 2. *The following inequalities and inclutions hold:*

$$\tau{-}Li\,\Xi_\varepsilon \subseteq \tau{-}Ls\,\Xi_\varepsilon \subseteq U_\partial \times \tau{-}Ls\,K_\varepsilon;$$

$$I^s(u,y) \geq I_s(u,y) \text{ for every } (u,y)\in\tau{-}Li\,\Xi_\varepsilon.$$

Proposition 3. *Suppose that $\tau{-}Li\,\Xi_\varepsilon \neq \emptyset$ and $\left\{I_\varepsilon : \Xi_\varepsilon \rightarrow \overline{R}\right\}_{\varepsilon\in E}$ is equi-coercive. Then the S-limits I_s and I^s are τ-lower semicompactness and the sets of solutions for lower and upper S-homogenized optimal control problems are nonempty and τ-compact.*

Proposition 4. *Let $\left\{\left(u_\varepsilon^\alpha, y_\varepsilon^\alpha\right) \in M^\alpha\left(I_\varepsilon, \Xi_\varepsilon\right) \subset U_\partial \times K_\varepsilon\right\}_{\varepsilon\in E}$ be a generalized sequence of α-optimal pairs of (1)-(2) and $\left(u_\varepsilon^\alpha, y_\varepsilon^\alpha\right) \xrightarrow{\tau} \left(u^0, y^0\right)$. Then*
$I_s\left(u^0,y^0\right)=\liminf_{\varepsilon\in E}\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)$; $\left(u^0,y^0\right)\in M\left(I_s, \tau{-}Ls\,\Xi_\varepsilon\right)$;
$I^s\left(u^0,y^0\right)=\limsup_{\varepsilon\in E}\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)$; $\left(u^0,y^0\right)\in M\left(I^s, \tau{-}Li\,\Xi_\varepsilon\right)$.

Proposition 5. *Assume that $\left\{I_\varepsilon : \Xi_\varepsilon \rightarrow \overline{R}\right\}_{\varepsilon\in E}$ S-converges to a functional $I : \tau{-}Li\,\Xi_\varepsilon \rightarrow \overline{R}$ and that I is a not identicaly $+\infty$ on $\tau{-}Li\,\Xi_\varepsilon \neq \emptyset$. Then: (i) for the family problems (1)-(2) there exists of a strong S-homogenized optimal control problem;*
ii) $M\left(I, \tau{-}li\,\Xi_\varepsilon\right)=\bigcap_{\alpha>0} \tau{-}Li\,M^\alpha\left(I_\varepsilon, \Xi_\varepsilon\right) \subset U_\partial \times \tau{-}Li\,K_\varepsilon$;
iii) $\min_{(u,y)\in\tau-Li\,\Xi_\varepsilon} I(u,y)=\lim_{\varepsilon\in E}\inf_{(u,y)\in\Xi_\varepsilon} I_\varepsilon(u,y)$.
Morever, if $\left\{I_\varepsilon : \Xi_\varepsilon \rightarrow \overline{R}\right\}_{\varepsilon\in E}$ is equi-coercive, then $M\left(I, \tau{-}Li\,\Xi_\varepsilon\right)$ is nonempty and τ-compact.

We now turn to studing the compactness properties of the class of optimal control problems (1)-(2) with respect to S-homogenization. The main compactness theorem is founded on the following abstract result (7): from each directedness of functions $\left\{G^{\varepsilon}:W_{\varepsilon}\rightarrow\overline{R}\right\}_{\varepsilon\in E}$ defined on the subsets of a second countable topological space (W,τ) and for which , one $\tau - Li\, W_{\varepsilon}\neq\emptyset$, can extract an absolute S-convergence sequence $\left\{G^{n}:W_{n}\rightarrow\overline{R}\right\}_{n\in N}$.

Remark 1. It may be noted that the τ-topology on $U\times Y$ is a separable since the control space U is separable in $*$-weak topology and Y is weakly separable by initial assumptions. Let us assume that the hypothesis (a)-(g) are true. Consider the subset $U_{\partial}\times B$ of $U\times Y$, where B is bounded and weakly closed subset of K_{ε}. Since the space Y is nonreflexive, the product of sets $U_{\partial}\times B$, generally is not τ-compact. Hence τ-topology on $U_{\partial}\times B$ is nonmetrizable. Let us now make use of the fact that any weakly closed subset B of a weakly compact set K_{ε} is also weakly compact. As it appears from the above mentioned, the τ-topology on $U_{\partial}\times B$ is metrizable if the sets K_{ε} are compact in weak topology on Y.

THEOREM 2. *Assume that: (1) the generalized sequence of the sets of all admissible pairs* $\{\Xi_{\varepsilon}\}_{\varepsilon\in E}$ *satisfies the condition* $\tau - Li\,\Xi_{\varepsilon}\neq\emptyset$; *(2) the sets* K_{ε} *are compact in weak topology of* Y; *(3) the initial assumptions (a)-(g) are true. Then: (i) from each directedness of optimal control problems* (1)-(2) *one can extract the sequence*

$$\inf I_{\varepsilon_n}(u,y), \tag{4}$$

$$A_{\varepsilon_n}(u,y)=f_{\varepsilon_n},\; F_{\varepsilon_n}(u,y)\geq 0,\; u\in U_{\partial},\; y\in K_{\varepsilon_n},\; \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0, \tag{5}$$

for wich a strong homogenized optimal control problem in sense of definition 6 exists;
(ii) any sequence of optimal pairs $\left\{\left(u^{0}_{\varepsilon_n},y^{0}_{\varepsilon_n}\right)\right\}_{n\in N}$ *for the family of problems* (4)-(5) *is compact in* τ*-topology;*
(iii) if $(\overline{u},\overline{y})$ *is* τ*-limit of sequence* $\left\{\left(u^{0}_{\varepsilon_n},y^{0}_{\varepsilon_n}\right)\right\}_{n\in N}$ *then*
$(\overline{u},\overline{y})\in U_{\partial}\times\tau\left(Y_w\right)-Li\,K_{\varepsilon}$, $\lim\limits_{n\to\infty} I_{\varepsilon_n}\left(u^{0}_{\varepsilon_n},y^{0}_{\varepsilon_n}\right)=\left(\tau - lm^{a}_{s} I_{\varepsilon}\right)(\overline{u},\overline{y})$, *where*
$\left(\tau - lm^{a}_{s} I_{\varepsilon}\right)(\overline{u},\overline{y}) = \inf\limits_{(u,y)\in\tau - Lm\,\Xi_{\varepsilon}}\left(\tau - lm^{a}_{s} I_{\varepsilon}\right)(u,y)$, *i.e.* $(\overline{u},\overline{y})$ *is optimal solution in strong S-homogenized optimal control problem. (Here we denote by* $\tau\left(Y_w\right)$ *a weak topology on* Y *).*

In order to study a structure of S-homogenized problems we consider the family of μ-approximatical minimum problems

$$\left\{\left\langle\inf_{(u,y)\in U_{\partial}\times K_{\varepsilon}} I^{\mu}_{\varepsilon}(u,y)\right\rangle,\; \varepsilon\in E,\; \mu>0\right\}, \tag{6}$$

where

$$I^{\mu}_{\varepsilon}(u,y)=I_{\varepsilon}(u,y)+\mu^{-1}\left(\left\|A_{\varepsilon}(u,y)-f_{\varepsilon}\right\|_{Y^{*}}\right.$$
$$\left.+\sup_{\phi\in S^{*}_{1}\cap L^{*}}\left[\nu\left(\langle\phi,F_{\varepsilon}(u,y)\rangle_{Z}\right)\right]^{2}\right),$$

S^{*}_{1} is a unit sphere in Z^{*}, L^{*} is a dual cone of L, i.e.

$$L=\left\{\xi\,|\xi\in Z\;,\langle\phi,\xi\rangle_{Z}\geq 0\quad\forall\phi\in L^{*}\right\}.$$

Let $G_1^\varepsilon, G_2^\varepsilon$ be are functions difined by $G_1^\varepsilon(u,y) = \|A_\varepsilon(u,y) - f_\varepsilon\|_{Y^*}$, $G_2^\varepsilon(u,y) = \sup_{\phi\in S_1^*\cap L^*} \left[\nu\left(\langle\phi, F_\varepsilon(u,y)\rangle_Z\right)\right]^2$. By analogy with [4] we can prove that under initial assumptions of Theorem 1 the problem (6) has solutions for every $\varepsilon \in E$ and $\mu > 0$. Furthermore, if $\{(u_\varepsilon^\mu, y_\varepsilon^\mu) \in U_\partial \times K_\varepsilon\}_{\mu>0}$ be an arbitrary sequence of "μ-optimal pair", i.e. solutions of problems (6), then we can extract a subsequence which τ-convergence to some optimal pair in problem (1)-(2).

In the future by $[F \,|\, E]$ we denote the restriction of any function F on a set E.

Proposition 6. *Assume that conditions (a)-(g) are satisfied and the family of optimal control problems* (1)-(2) *are uniformly regular (i.e.* $\Xi_\varepsilon \neq \emptyset \quad \forall \varepsilon \in E$*). Then*

$$\tau{-}Li\,\Xi_\varepsilon = \mathbf{A}, \tau{-}Ls\,\Xi_\varepsilon = \mathbf{B},$$

where τ-closed sets $\mathbf{A} \subset U \times Y$ and $\mathbf{B} \subset U \times Y$ are defined by

$$\mathbf{A} = \left\{ (u,y) \left| \begin{array}{l} \sup\limits_{\mu>0} \left(\tau{-}ls_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right]\right) \\ \equiv \tau{-}ls_s \left[\lim\limits_{\mu\downarrow 0} \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] < +\infty \end{array} \right. \right\},$$

$$\mathbf{B} = \left\{ (u,y) \left| \begin{array}{l} \sup\limits_{\mu>0} \left(\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right]\right) \\ \equiv \tau{-}li_s \left[\lim\limits_{\mu\downarrow 0} \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] < +\infty \end{array} \right. \right\}.$$

Proof. We shall prove only the second equality, the proof of the other one being analogous. Define

$$\Omega_\varepsilon = \left\{(\overline{u}, \overline{y}) \in U \times Y \,|A_\varepsilon\left(\overline{u}, \overline{y}\right) = f_\varepsilon, \quad F_\varepsilon\left(\overline{u}, \overline{y}\right) \geq 0\right\}.$$

Given $\varepsilon \in E$, by the definition of Ω_ε we can write $\lim\limits_{\mu\downarrow 0}\left\{\mu^{-1}\left(G_1^\varepsilon + \;\; G_2^\varepsilon\right)\right\} =$ $\left[\chi_{cl_\tau\Omega_\varepsilon}\,|D\left(A_\varepsilon\right) \cap Y\right]$, where $\chi_{cl_\tau\Omega_\varepsilon}$ denotes the indicator function of τ-closure of the set Ω_ε. Taking into account that $\chi_{cl_\tau\Omega_\varepsilon} = \chi_{\Omega_\varepsilon}$, we have

$$\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] = \tau{-}li_s\left[\chi_{\Omega_\varepsilon}\,|U_\partial \times K_\varepsilon\right].$$

Note that, by definition 3, the function $\tau{-}li_s\left[\chi_{\Omega_\varepsilon}\,|U_\partial \times K_\varepsilon\right]$ there exists on the set $\tau{-}Ls\,\left(U_\partial \times K_\varepsilon\right)$. But it is ibvious that $\tau{-}Ls\,\left(U_\partial \times K_\varepsilon\right) = U_\partial \times \tau\left(Y_w\right){-}Ls\,K_\varepsilon$, where $\tau\left(Y_w\right)$ is a weak topology on Y. Then, using the equality

$$\left[\chi_{\Omega_\varepsilon}\,|U_\partial \times K_\varepsilon\right] = \left[\chi_{\Omega_\varepsilon \cap (U_\partial \times K_\varepsilon)}\,|U_\partial \times K_\varepsilon\right] = \left[\chi_{\Xi_\varepsilon}\,|U_\partial \times K_\varepsilon\right]$$

and the standard properties of S-limits, we can conclude that

$$\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] = \left[\chi_{\tau-Ls\,\Xi_\varepsilon}\,|U_\partial \times \tau\left(Y_w\right){-}Ls\,K_\varepsilon\right].$$

Since $\Xi_\varepsilon \subseteq \left(U_\varepsilon \times K_\varepsilon\right)$ with the above equality we have

$$\tau{-}Ls\,\Xi_\varepsilon = \left\{(u,y) \left| \tau{-}li_s\left[\left(\lim\limits_{\mu\downarrow 0} \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)\right)|U_\partial \times K_\varepsilon\right] < +\infty \right. \right\}.$$

Now we need to show that

$$\tau{-}Ls\,\Xi_\varepsilon = \left\{ (u,y) \middle| \sup_{\mu>0} \left(\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right]\right) < +\infty \right\}.$$

For this purpose we fix a arbitrary pair $(u,y) \in \tau{-}Ls\,\Xi_\varepsilon$. Then for every $V \in \mathfrak{N}_\tau(u,y)$ there exists subdirectedness $\{\Theta_\beta\}_{\beta\in B}$, $\{L_\beta\}_{\beta\in B}$, $\left\{Q_1^\beta + Q_2^\beta\right\}_{\beta\in B}$ of respectively directedness $\{\Xi_\varepsilon\}_{\varepsilon\in E}$, $\{K_\varepsilon\}_{\varepsilon\in E}$, $\{G_1^\varepsilon + G_2^\varepsilon\}_{\varepsilon\in E}$ such that $V \cap \Theta_\beta \neq \emptyset$ for every $\beta \in B$. Hence $\inf\limits_{(v,x)\in V\cap(U_\partial\times L_\beta)} \mu^{-1}\left(Q_1^\beta(v,x) + Q_2^\beta(v,x)\right) = 0, \beta \in B$. Since $G_1^\varepsilon + G_2^\varepsilon \geq 0$ on the set $U \times (D(A_\varepsilon) \cap Y)$, we have (for every $V \in \mathfrak{N}_\tau(u,y)$)

$$0 \leq \inf_{(\varepsilon\in E, V\cap(U_\partial\times K_\varepsilon)\neq\emptyset)} \inf_{(v,x)\in V\cap(U_\partial\times K_\varepsilon)} \mu^{-1}\left(G_1^\varepsilon(v,x) + G_2^\varepsilon(v,x)\right)$$
$$\leq \liminf_{\beta\in B} \inf_{(v,x)\in V\cap(U_\partial\times L_\beta)} \mu^{-1}\left(Q_1^\beta(v,x) + Q_2^\beta(v,x)\right) = 0.$$

On the taking the supremum over all $V \in \mathfrak{N}_\tau(u,y)$, we see that

$$\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right](u,y) = 0 \quad \forall (u,y) \in \tau{-}Ls\,\Xi_\varepsilon$$

for every $\mu > 0$. Thus

$$\sup_{\mu>0} \tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right]$$
$$\leq \left[\chi_{\tau-Ls\,\Xi_\varepsilon} | U_\partial \times \tau\left(Y_w\right) {-}Ls\,K_\varepsilon\right]. \quad (7)$$

To prove the opposite inequality, we fix $(\overline{u},\overline{y})$ with the set $(U_\partial \times\ \tau\left(Y_w\right) {-}Ls\,K_\varepsilon) \setminus \tau{-}Ls\,\Xi_\varepsilon$. Then there exists $V \in \mathfrak{N}_\tau\left(\overline{u},\overline{y}\right)$ and $\varepsilon^0 \in E$ such that $V \cap \Xi_\varepsilon \neq \emptyset$ for every $\varepsilon \leq \varepsilon^0$. Since $\Xi_\varepsilon = (U_\partial \times K_\varepsilon) \cap \Omega_\varepsilon$, there exists $\gamma > 0$ such that, for every $\varepsilon \leq \varepsilon^0$, $\inf\limits_{(v,x)\in V\cap(U_\partial\times K_\varepsilon)} \mu^{-1}\left(G_1^\varepsilon(v,x) + G_2^\varepsilon(v,x)\right) \geq \mu^{-1}\gamma$. Then we have

$$\inf_{\varepsilon\in E} \inf_{(v,x)\in V\cap(U_\partial\times K_\varepsilon)} \mu^{-1}\left(G_1^\varepsilon(v,x) + G_2^\varepsilon(v,x)\right) \geq \mu^{-1}\gamma,$$

Hence $\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right](\overline{u},\overline{y}) \geq \mu^{-1}\gamma$. Therefore, for every $(\overline{u},\overline{y}) \in (U_\partial \times \tau\left(Y_w\right) {-}Ls\,K_\varepsilon) \setminus \tau{-}Ls\,\Xi_\varepsilon$, we have

$$\sup_{\mu>0} \left(\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right]\right)(\overline{u},\overline{y}) \geq +\infty.$$

Thus, on the set $U_\partial \times \tau\left(Y_w\right) {-}Ls\,K_\varepsilon$, will be true the next inequality

$$\sup_{\mu>0} \left(\tau{-}li_s\left[\mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right) | U_\partial \times K_\varepsilon\right]\right)$$
$$\geq \left[\chi_{\tau-Ls\,\Xi_\varepsilon} | U_\partial \times \tau\left(Y_w\right) {-}Ls\,K_\varepsilon\right],$$

which, together with (7), concludes the prof of the proposition.

COROLLARY. *The directedness of the sets of admissible pair* $\{\Xi_\varepsilon\}_{\varepsilon\in E}$ *convergence in topological sense to* $\Xi \neq \emptyset$ *iff* $\mathbf{B} \subseteq \Xi \subseteq \mathbf{A} \neq \emptyset$.

By analogy with provious it may be proved the following results.

PROPOSITION 7. *Under initial conditions of proposition 6, suppose in addition that* $\left\{I_\varepsilon : U \times X \to \overline{R}\right\}_{\varepsilon\in E}$ *is lower equi-bounded and there exists a constant* $\gamma > 0$ *such that* $I_\varepsilon(u, y) < \gamma$ *for every* $(u, y) \in U_\partial \times K_\varepsilon$ *and* $\varepsilon \in E$*.Then*

$$\mathrm{Dom}\left(\sup_{\mu>0}\left(\tau{-}ls_s\left[I_\varepsilon + \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\,\right]\right)\right) = \tau{-}Li\,\Xi_\varepsilon$$

$$= \mathrm{Dom}\left(\tau{-}ls_s\left[\lim_{\mu\downarrow 0}\left(I_\varepsilon + \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)\right)|U_\partial \times K_\varepsilon\right]\right),$$

$$\mathrm{Dom}\left(\sup_{\mu>0}\left(\tau{-}li_s\left[I_\varepsilon + \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\,\right]\right)\right) = \tau{-}Ls\,\Xi_\varepsilon$$

$$= \mathrm{Dom}\left(\tau{-}li_s\left[\lim_{\mu\downarrow 0}\left(I_\varepsilon + \mu^{-1}\left(G_1^\varepsilon + G_2^\varepsilon\right)\right)|U_\partial \times K_\varepsilon\right]\right).$$

PROPOSITION 8. *Suppose that initial conditions of proposition 6 will be true and functionals* $\left\{I_\varepsilon : U \times X \to \overline{R}\right\}_{\varepsilon\in E}$ *are lower equi-bounded. Then*

$$\sup_{\mu>0}\left(\tau{-}li_s\left[I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\,|U_\partial \times K_\varepsilon\,\right]\right) = \tau{-}li_s\left[I_\varepsilon\,|\Xi_\varepsilon\,\right]$$

$$= \tau{-}li_s\left[\lim_{\mu\downarrow 0}\left(I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] \text{ on } \tau{-}Ls\,\Xi_\varepsilon,$$

$$\sup_{\mu>0}\left(\tau{-}ls_s\left[I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\,|U_\partial \times K_\varepsilon\,\right]\right) = \tau{-}ls_s\left[I_\varepsilon\,|\Xi_\varepsilon\,\right]$$

$$= \tau{-}ls_s\left[\lim_{\mu\downarrow 0}\left(I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\right)|U_\partial \times K_\varepsilon\right] \text{ on } \tau{-}Li\,\Xi_\varepsilon.$$

THEOREM 3. (MAIN RESULT). *Let us assume that initial conditions of proposition 7 will be true. Then for the family of optimal control problems* (1)-(2) *there exists a strong S-homogenized problem iff the quality*

$$\sup_{\mu>0}\left(\tau{-}li_s\left[I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\,|U_\partial \times K_\varepsilon\,\right]\right)$$

$$= \sup_{\mu>0}\left(\tau{-}ls_s\left[I_\varepsilon + \mu^{-1}G_1^\varepsilon + \mu^{-1}G_2^\varepsilon\,|U_\partial \times K_\varepsilon\,\right]\right) \not\equiv +\infty$$

holds on the set $U_\partial \times \tau\left(Y_w\right){-}Ls\,K_\varepsilon$.

COROLLARY. *Under initial conditions of main theorem for S-homogenized optimal control problem* $\left\langle \inf\limits_{(u,y)\in\tau-Lm\,\Xi_\varepsilon}\left(\tau{-}lm_s^a I_\varepsilon\right)(u, y)\right\rangle$ *will be holds the next representation*

$$\left\langle \inf_{(u,y)\in\left\{\sup\limits_{\mu>0}\left(\tau-ls_s\left[I_\varepsilon^\mu|U_\partial\times K_\varepsilon\right]\right)(u,y)<+\infty\right\}} \sup_{\mu>0}\left(\tau{-}ls_s\left[I_\varepsilon^\mu\,|U_\partial \times K_\varepsilon\,\right]\right)\right\rangle.$$

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