THE HARDY'S INEQUALITY AND POSITIVE INVERTABILITY OF ELLIPTIC OPERATORS

© Pavel E. Sobolevskii

1^0. We will consider the simplest elliptic boundary value problem

$$-\Delta u(x) - a(x)u(x) = f(x)(x \in \Omega); u(x) = 0(x \in \partial\Omega) \tag{1}$$

and correspondent initial boundary value problem

$$\partial v(t,x)/\partial t - \Delta v(t,x) - a(x)v(t,x) = 0(t \geq 0, x \in \Omega); v(t,x) = 0$$
$$(t \geq 0, x \in \partial\Omega); v(0,x) = v^0(x)(x \in \bar{\Omega} = \Omega \cup \partial\Omega). \tag{2}$$

Here Ω is a bounded open set in R^n of points $x = (x_1, \cdots, x_n)$ with boundary $\partial\Omega \in C^2; a(x), f(x)$ and $v^0(x)$ are given (continuous) functions, defined on $\bar{\Omega}$.

Let A be the acting in the space $L_p(\Omega)(1 < p < +\infty)$ linear operator, defined by formula

$$(Au)(x) = -\Delta u(x) - a(x)u(x) \tag{3}$$

on domain

$$D(A) = \overset{0}{W}{}^2_p(\Omega) \equiv \overset{0}{W}{}^1_p(\Omega) \cap W^2_p(\Omega). \tag{4}$$

It is well-known (see, for example, [1]), that $-A$ is the generator of analytic semigroup $\exp\{-tA\}(t \geq 0)$ of bounded in $L_p(\Omega)$ operators, i.e. estimates

$$\| \exp\{-tA\}\|_{L_p \to L_p}, \|tA \cdot \exp\{-tA\}\|_{L_p \to L_p} \leq M(p) \cdot e^{\omega(p)t}(t > 0) \tag{5}$$

hold for some $1 \leq M(p) < +\infty, -\infty < \omega(p) < +\infty$. We will consider boundary value problem (1) as operator problem

$$Au = f \tag{6}$$

in some functional Banach space. We will say, that $u = u(x)$ is the solution in L_p of problem (1), if $u \in D(A)$, and (operator) equation (6) is fulfilled. If there exists such solution, then, evidently,

$$f \in L_p. \tag{7}$$

We will consider initial boundary value problem (2) as (operator) Cauchy problem

$$dv(t)/dt + Av(t) = 0(t \geq 0), v(0) = v^0. \tag{8}$$

We will say, that $v(t) = v(t,x)$ is the solution in L_p of problem (2), if $v(t), dv(t)/dt, Av(t) \in C([0,T], L_p)$ for any number $T < 0$, and (operator) ordinary differential equation and initial condition (8) are fulfilled. If there exists such solution $v(t) = v(t,x)$, then, evidently,

$$v^0 \in D(A). \tag{9}$$

It is well-known [1], that property (9) is not only necessary, but also sufficient condition for the existence of unique solution $v(t)$ in L_p of problem (2), which is defined by formula

$$v(t) = \exp\{-tA\}v^0. \tag{10}$$

Let following condition be satisfied:

$$a(x) \leq 0 \ (x \in \bar{\Omega}). \tag{11}$$

Then (see [2]), from maximum principle it follows, that estimates (5) are true for some

$$\omega(p) < 0. \tag{12}$$

Therefore there exists the bounded inverse A^{-1}, which is defined by formula

$$A^{-1} = \int_0^\infty \exp\{-tA\}dt. \tag{13}$$

From maximum principle also it follows, that

$$\exp\{-tA\} \cdot v^0(x) \geq 0, \quad \text{if } v^0(x) \geq 0 (x \in \bar{\Omega}). \tag{14}$$

and therefore, in virtue of formula (13),

$$(A^{-1}f)(x) \geq 0, \ \text{if } f(x) \geq 0 \ (x \in \bar{\Omega}). \tag{15}$$

It is well known [2], that these statements are valid under less restriction

$$a(x) < \lambda_1 \quad (x \in \bar{\Omega}) \tag{16}$$

on coefficient $a(x)$. Here $\lambda_1 > 0$ is the miminal eigenvalue of operator $-\Delta$.

It turns out, that these statements take place for some functions $a(x)$, which can have very large positive values near boundary $\partial\Omega$ and small (positive) values inside of Ω. Usually the statements for equations with variable coefficients are true, in particular, for equations with constant coefficients. In our case it is not true. In fact for $a(x) = \text{const.} = \lambda_1$ operator $A = -\Delta - a(x)I$ does not have inverse.

Such statements were discovered in [3] for general elliptic operators of the second order and for the correspondent parabolic operators. In this paper, for partial cases of elliptic operators, some amplification of results from [3] is obtained. This amplification is based on exact values of generalized Hardy's constants.

2^0. Let us define generalized Hardy's functional

$$\ell_{p,\Omega}(u) = [\int_\Omega |\nabla u|^p dx]^{1/p} \cdot [\int_\Omega |u|^p \cdot \rho_{\partial\Omega}^{-p} dx]^{-1/p} \tag{17}$$

on the set $\{u \in \overset{0}{W}{}^1_p(\Omega), u(x) \not\equiv 0\}$. Here $|\nabla u|^2 = \sum_{i=1}^n |\partial u/\partial x_i|^2, \rho_{\partial\Omega}(x)$ is the Euclidean distance from point $x \in \Omega$ to boundary $\partial\Omega$. It turns out, that value

$$H_{p,\Omega} = \inf_{u \in \overset{0}{W}{}^1_p(\Omega), u \not\equiv 0} \ell_{p,\Omega}(u) \tag{18}$$

is positive. The proof of this statement is based on partition of unity and classical one-dimensional Hardy's inequality [4].

Theorem 1. *Let*

$$H_{2,\Omega}^{-1} \cdot \max_{x\in\bar{\Omega}}\{[a(x)]_+^{1/2} \cdot \rho_{\partial\Omega}(x)\} < 1 \tag{19}$$

Then, for any $p \in (1, +\infty)$ there exists the bounded positive inverse A^{-1} in L_p.

Here and in what follows

$$[\psi(x)]_+ = \max[0, \psi(x)]. \tag{20}$$

Scheme of proof. Let us consider boundary value problem with parameter $\lambda \geq 0$

$$-\Delta u(x) - a(x)u(x) + \lambda u(x) = f(x)(x \in \Omega);\ u(x) = 0(x \in \partial\Omega) \tag{21}$$

as operator problem in L_2. From maximum principle it follows, that operator $A + \lambda I$ has bounded inverse, when

$$\lambda \geq \max[a(x)]_+. \tag{22}$$

Further, in virtue of Friedrichs inequality

$$M_{F(\Omega)} \cdot \|u\|_{L_2(\Omega)} \leq \||\nabla u|\|_{L_2(\Omega)}, u \in \overset{0}{W}{}_2^1(\Omega) \tag{23}$$

for some $0 < M_F(\Omega) < +\infty(\bar{\Omega}$ is bounded set in $R^n)$, functional $\||\nabla u|\|_{L-2(\Omega)}$ defines the equivalent norm in the space $\overset{0}{W}{}_2^1(\Omega)$. Therefore the norm of the space $W_2^{-1}(\Omega)$ [of the space of bounded linear functionals on $\overset{0}{W}{}_2^1(\Omega)$] can be define by formula

$$\|\varphi\|_{W_2^{-1}(\Omega)} = \sup_{\psi\in \overset{0}{W}{}_2^1(\Omega), \psi\not\equiv 0} |\int_\Omega \varphi(x) \cdot \psi(x)dx| \cdot \|\psi\|^{-1}_{\overset{0}{W}{}_2^1(\Omega)}. \tag{24}$$

Then from (21) it follows, that $(\lambda \geq 0)$

$$\||\nabla u\|^2_{L-2(\Omega)} - \int_\Omega [a(x)]_+ u^2(x)dx \leq \|f\|_{W_2^{-1}(\Omega)} \cdot \||\nabla u|\|_{L_2(\Omega)}. \tag{25}$$

Further, in virtue of condition (19),

$$H_{2,\Omega}^{-2} \cdot \max_{x\in\bar{\Omega}}\{[a(x)]_+ \cdot \rho^2_{\partial\Omega}(x)\} = 1 - \varepsilon \tag{26}$$

for some $\varepsilon \in (0, 1]$. It leads us to inequality

$$\varepsilon \cdot \||\nabla|\|_{L_2(\Omega)} \leq \|f\|_{W_2^{-1}(\Omega)}, \tag{27}$$

in virtue of definitions (17) and (18). Finally, we apply the Friedrichs inequality (23) and obtain the first a priory estimate

$$\|u\|_{W_2^1(\Omega)} \leq M_1 \cdot \|f\|_{L_2(\Omega)}, M_1 = [1 + \varepsilon^{-2} \cdot M^{-2}_{F(\Omega)}]^{1/2}, \tag{28}$$

uniformly with respect to $\lambda \geq 0$.

Further, from identity (21) (under integration by parts) it follows, that

$$\| - \Delta u\|^2_{L-2(\Omega)} + \lambda \||\nabla u|\|^2_{L_2(\Omega)} \leq \max_{x \in \bar{\Omega}} |a(x)| \cdot \|u\|_{L_2(\Omega)} \cdot \| - \Delta u\|_{L_2(\Omega)} + \\ + \|f\|_{L_2(\Omega)} \cdot \| - \Delta u\|_{L_2(\Omega)}. \tag{29}$$

We use estimate (28) and coercive inequality [2] for operator $-\Delta$ and obtain the second a priory estimate

$$\|u\|_{W_2^2(\Omega)} \leq M_2 \cdot \|f\|_{L_2(\Omega)}, \tag{30}$$

uniformly with respect to $\lambda \geq 0$. It permits us to establish, that for any $\lambda \geq 0$ operator $A + \lambda I$ has bounded positive inverse $(A + \lambda I)^{-1}$ in $L_2(\Omega)$.

For $p \geq 2$ we use identity $[u \in \overset{0}{W}{}_2^2(\Omega)]$

$$- \int_\Omega \Delta u(x) \cdot u|u(x)|^{p-2} dx = (p-1) \int_\Omega |\nabla u(x)|^2 \cdot |u(x)|^{p-2} dx, \tag{31}$$

imbedding theorems for the spaces $W_q^2(\Omega)(q > 1)$ (see [1, 2]) and come to a priory estimate

$$\|u\|_{W_p^2(\Omega)} \leq M_p \cdot \|f\|_{L_p(\Omega)}, \tag{32}$$

which permits to show, that operator $A + \lambda I$ has the bounded positive inverse for all $\lambda \geq 0$ in $L_p(\Omega)$. Finally, in the case $1 < p < 2$ we consider operator $\lambda I + A^* = \lambda I + A$, acting in the space $L_q, q = \frac{p}{p-1}$.

The parabolic problem (2) can be investigated in the same manner. Analogously to inequality (27) differential inequality

$$1/2 \cdot d[\|v(t)\|^2_{L_2(\Omega)}/dt + \varepsilon \||\nabla v(t)|\|^2_{L_2(\Omega)} \leq 0 \tag{33}$$

for solutions of Cauchy problem (8) is established.

Since, evidently,

$$\||\nabla|\|_{L_2(\Omega)} \geq \lambda_1^{1/2} \cdot \|\psi\|_{L_2(\Omega)}, \psi \in \overset{0}{W}{}_2^1(\Omega), \tag{34}$$

then from (33), it follows, that

$$\|v(t)\|_{L_2(\Omega)} \leq e^{-\varepsilon \lambda_1 t} \cdot \|v^0\|_{L_2(\Omega)}. \tag{35}$$

It means, that following estimate

$$\| \exp\{-tA\}\|_{L_2(\Omega) \to L_2(\Omega)} \leq e^{-\varepsilon \lambda_1 t} (t \geq 0) \tag{36}$$

is true. Further the application of semigroup property and imbedding theorems leads us to following result:

THEOREM 2. *Under condition of Theorem 1 for any $1 < p < +\infty$ estimate*

$$\|\exp\{-tA\}\|_{L_p(\Omega)\to L_p(\Omega)} \leq M_p \cdot e^{-\varepsilon\lambda_1 t}(t \geq 0) \tag{37}$$

is true for some $1 \leq M_p < +\infty$ *and*

$$\varepsilon = 1 - H_{2,\Omega}^{-2} \cdot \max_{x\in\bar{\Omega}}\{[a(x)]_+ \cdot \rho_{\partial\Omega}^2(x)\}. \qquad (38.)$$

It is easy to see [see formula (13)], that Theorem 1 is the consequence of Theorem 2.

3^0. Let us consider general elliptic boundary value problem

$$-\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x)\frac{\partial u(x)}{\partial x_i} - a_0(x)u(x) = f(x)(x \in \Omega),$$
$$u(x) = 0(x \in \partial\Omega); \tag{39}$$

and correspondent parabolic initial boundary value problem

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2 v(t,x)}{\partial x_i \partial x_j} = \sum_{i=1}^n a_i(x)\frac{\partial v(t,x)}{\partial x_i} - a_0(x)v(t,x) = 0 \ (t \geq 0, x \in \Omega);$$
$$v(t,x) = 0(t \geq 0, x \in \partial\Omega)' v(0,x) = v^0(x) \ (x \in \bar{\Omega} = \Omega \cup \partial\Omega). \tag{40}$$

We will suppose, that functions

$$a_{i,j}(x), \partial a_{i,j}(x)/\partial x_i, \partial^2 a_{i,j}(x)/\partial x_i \partial x_j, a_i(x), \partial a_i(x)/\partial x_i, a_0(x) \tag{41}$$

are continuous on $\bar{\Omega}$, and ellipticity condition

$$\sum_{i,j=1}^n a_{i,j}(x)\gamma_i\gamma_j \geq \lambda_0 \sum_{i=1}^n \gamma_1^2 \quad (x \in \bar{\Omega}) \tag{42}$$

is fulfilled for all real numbers $\gamma_i(i = \overline{1,n})$ and some $0 < \lambda_0 + \infty$. It is possible to show, that in this general nonselfadjoint case the analogous of Theorems 1 and 2 statements are true, if condition

$$\max_{x\in\bar{\Omega}}\{[\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 a_{i,j}(x)}{\partial x_i \partial x_j} + \frac{1}{2}\sum_{i=1}^n \frac{\partial a_i(x)}{\partial x_i} + a_0(x)]_+ \cdot \rho_{\partial\Omega}(x)\} < \lambda_0^{1/2} \cdot H_{2,\Omega} \tag{43}$$

is fulfilled.

4^0. Condition (19) of Theorems 1 and 2 depends on value $H_{2,\Omega}$, but the partition of unity is not explicit approach for the calculation of this value. However there exists the wide class of domains $\Omega \subset R^n$, for which value $H_{p,\Omega}$ is defined by explicit formula. Namely [5], if Ω is (bounded or unbounded) convex domain, then

$$H_{p,\Omega} = \frac{p-1}{p}. \tag{44}$$

In this case condition (19) has form

$$\max_{x\in\bar{\Omega}}\{[a(x)]_+^{1/2}\cdot\rho_{\partial\Omega}(x)\}<1/2. \quad (45)$$

The proofs of Theorems 1 and 2 are based also on the Friedrichs inequality (23). There are convex unbounded domains in R^n, for which inequality (23) is valid. For such domains the statements of Theorems 1 and 2 are true under condition (45). For example, in R^2 for stripe

$$\Omega=\{x=(x_1,x_2)/0\le x_2\le 1,-\infty<x_1<+\infty\} \quad (46)$$

the Friedrichs inequality is true.

For any convex unbounded domain $\Omega\in R^n$, for example for $\Omega=R^n$, operator $-\Delta$ does not have the bounded inverse, and therefore the Friedrichs inequality is not valid. Only for any $\gamma_0>0$ operator

$$-\Delta+\gamma_0 I \quad (47)$$

has the bounded inverse. It leads us to condition

$$\sup_{x\in\bar{\Omega}}\{[a(x)+\gamma_0]_+^{1/2}\cdot\rho_{\partial\Omega}(x)\}<1/2 \quad (48)$$

for any convex domain $\Omega\in R^n$ Condition (48) means, that function $a(x)$ must tend to $-\gamma_0$, when $\rho_{\partial\Omega}(x)$ tends to $+\infty$, i.e. we suppose, that maximum principle is fulfilled on the infinity.

5^0. The condition of type

$$\sup_{x\in\bar{\Omega}}\{[a(x)+\gamma_0]^{1/2}\cdot\rho_{\partial\Omega}(x)<H_{2,\Omega} \quad (49)$$

has the sense, if

$$H_{2,\Omega}>0. \quad (50)$$

It is true, when Ω is bounded domain, or Ω is unbounded convex domain. However there exists the wide class of unbounded domains $\Omega\in R^n$, for which

$$H_{2,\Omega}=0. \quad (51)$$

Namely, let

$$\Omega=R^n\setminus\omega, \quad (52)$$

and ω is bounded convex domain. Then [6]

$$H_{p,\Omega}=\min\{|\frac{p-1}{p}|,|\frac{p-2}{p}|,\cdots,|\frac{p-n}{p}|\}(1<p<+\infty). \quad (53)$$

It means, that such domains have property (51).

Now we will write problem (2) in form

$$A_0u-[a(x)+\gamma_0]u=f \quad (54)$$

where

$$A_0 u = -\Delta + \gamma_0 I, \gamma_0 > 0. \tag{55}$$

It is evident, that for any $p \in (1, +\infty)$ and $\lambda \geq 0$ operator $A_0 + \lambda I$ has the bounded (positive) inverse in $L_p(\Omega)$, and following estimate

$$\|(A_0 + \lambda I)^{-1}\|_{L_p(\Omega) - L_p(\Omega)} \leq (\lambda + \gamma_0)^{-1} \tag{56}$$

is true. It means (see [1]), that A_0 is positive operator in Banach space $L_p(\Omega)$ and therefore any powers $A_0^\alpha(-\infty < \alpha < +\infty)$ are defined, $A_0^{-\alpha}(\alpha > 0)$ are bounded operators and $A_0^\alpha(\alpha > 0)$ are unbounded operators with dense domains $D(A_0^\alpha)$. Further [7]

$$D(A_0^\alpha) = \overset{0}{W}{}_p^{2\alpha}(\Omega) \ (0 < \alpha < 1) \tag{57}$$

and following estimates

$$\|A_0^{-\alpha}\|_{L_p(\Omega) \to \overset{0}{W}{}_p^{2\alpha}(\Omega} \leq M_s(\alpha, p) \tag{58}$$

are true with some $1 \leq M_s(\alpha, p) < +\infty$. Equation (21) with parameter $\lambda \geq 0$ we can write in form

$$A_0 u - [a(x) + \gamma_0]u + \lambda u = f. \tag{59}$$

It, evidently, leads us to inequality

$$\begin{aligned} &\|A_0^{1/2} u\|^2_{L_2(\Omega)} - \sup_{x \in \bar{\Omega}} \{[a(x) + \gamma_0]_+ \rho^2_{\partial\Omega}(x)\} \cdot \|u \cdot \rho^{-1}_{\partial\Omega}\|^2_{L_2(\Omega)} \leq \\ &\leq \|A_0^{-1/2} f\|_{L_2(\Omega)} \cdot \|A_0^{1/2} u\|_{L_2(\Omega)} \end{aligned} \tag{60}$$

for its solutions. Therefore substitution

$$z = A_0^{1/2} u \tag{61}$$

gives

$$\begin{aligned} &\|z\|_{L_2(\Omega)} - \sup_{x \in \bar{\Omega}} \{[a(x) + \gamma_0]_+ \cdot \rho^2_{\partial\Omega}(x)\} \cdot \|\rho^{-1}_{\partial\Omega} \cdot A_0^{-1/2}\|_{L_2(\Omega) \to L_2(\Omega)} \cdot \|z\|_{L_2(\Omega)} \leq \\ &\leq \|A_0^{-1/2} f\|_{L_2(\Omega)}. \end{aligned} \tag{62}$$

So, we must estimate value

$$\|\rho^{-1}_{\partial\Omega} \cdot A_0^{-1/2}\|_{L_2(\Omega) \to L_2(\Omega)}. \tag{63}$$

Of course, we can apply here the implicit approach, which is based on the partition of unity, but we will use here other method. Let $1 < p_1 < 2 < p_2 < +\infty$. From M. Riesz interpolation theorem [8] it follows, that

$$\begin{aligned} \|\rho^{-1}_{\partial\Omega} \cdot A_0^{-1/2}\|_{L_2(\Omega) \to L_2(\Omega)} \leq M_R(p_1, 2, p_2) \cdot \|\rho^{-1}_{\partial\Omega} \cdot A_0^{-1/2}\|^{\frac{p_1 \cdot (p_2 - 2)}{2 \cdot (p_2 - p_1)}}_{L_{p_1}(\Omega) \to L_{p_1}(\Omega)} \times \\ \times \|\rho^{-1}_{\partial\Omega} \cdot A_0^{-1/2}\|^{\frac{p_2 \cdot (2 - p_1)}{2 \cdot (p_2 - p_1)}}_{L_{p_2}(\Omega) \to L_{p_2}(\Omega)}. \end{aligned} \tag{64}$$

Further we will suppose, that numbers p_1 and p_2 do not coincide with integers $2, 3, \cdots, n$. Then from formula (53) it follows, that

$$H_{p_i,\Omega} > 0 (i = 1, 2). \tag{65}$$

Therefore from definition (18) it follows, that

$$\|\rho_{\partial\Omega}^{-1} \cdot A_0^{-1/2}\|_{L_{p_i}(\Omega)\to L_{p_i}(\Omega)} \leq H_{p_i,\Omega}^{-1} \cdot \||\nabla A_0^{-1/2}|\|_{L_{p_i}(\Omega)\to L_{p_i}(\Omega)} (i = 1, 2). \tag{66}$$

Finally, we apply estimate (58) and obtain

$$\|\rho_{\partial\Omega}^{-1} \cdot A_0^{-1/2}\|_{L_{p_i}(\Omega)\to L_{p_i}(\Omega)} \leq H_{p_i}^{-1} \cdot M_s(1/2, p_i) \ (i = 1, 2). \tag{67}$$

Then from (64) and from (59)-(62) it follows, that operator $A + \lambda I$ for all $\lambda \geq 0$ has bounded positive inverse in $L_2(\Omega)$, if

$$\sup_{x\in\bar{\Omega}} \{[a(x) + \gamma_0] \cdot \rho_{\partial\Omega}^2(x)\} \cdot M_R^2(p_1, 2, p_2) \times$$
$$\times [H_{p_1}^{-1} \cdot M_s(1/2, p_1)]^{\frac{p_1\cdot(p_2-2)}{p_2-p_1}} \cdot [H_{p_2}^{-1} \cdot M_s(1/2, p_2)]^{\frac{p_2(2-p_1)}{p_2-p_1}} < 1. \tag{68}$$

It turns out, that condition (68) permit also to establish the exponential decreasing of norm $\| \exp\{-ta\}\|_{L_2(\Omega)\to L_2(\Omega)}$. Finally, the embedding theorems permit to prove the analogous statements in the space $L_p(\Omega)$ for any $p \in (1, +\infty)$.

References

[1] M. A. Krasnoselskii et al; Integral operators in spaces of summable functions, Noordhoff international publishing Leyden, 1976; p. 520.

[2] O. A. Ladyzenskaya, V. A. Solonnikov, N. N. Uralceva; Linear and quasilinear equations of parabolic type; Nauka-Moskow; 1967; p. 736 (Russia).

[3] I. F. Lezhenina, P. E. Sobolevskii; Elliptic and Parabolic boundary value problems with singular estimate of coefficient; Dokl. Acad. Nauk Ukrain, SSR, Ser A, 1989, No. 3, pp. 27–31 (Russia).

[4] G. H. Hardy; Note on a theorem of Hilbert; Math. Zeitsch. 6 (1920); pp. 314–317.

[5] T. Matskewich, P. E. Sobolevskii; The best possible constant in generalized, Hardy's inequality for convex domain in R^n; Elliptic and Parabolic P. D. E.'s and Applications, Capri, September 19-23, 1994, Summaries.

[6] T. Matskewich, P. E.. Sobolevskii; The sharp constant in the Hardy's inequality for complement of bounded domain; American Mathematical Society-Israel Mathematical Union; Joint Meeting May 24-26, 1995; Jerusalem, Israel; Summaries.

[7] Seeley, R; Fractional powers of boundary problems; Actes Congres Intern. Math., t. 2 (Nice). Paris, Ganhier-Villars, 1970, pp. 203–205.

[8] M. Riesz; Sur les maxima des forms bilineaires et sur les fonctionelles linéaires; Acta Math. 49 (1927); pp.465–497.

Institute of Mathematics
Hebrew University of Jerusale
Givat Ram Campus,
91904 Jerusalem, Israel