GREEN'S FORMULA AND THEOREMS ON COMPLETE COLLECTION OF ISOMORPHISMS FOR GENERAL ELLIPTIC BOUNDARY VALUE PROBLEMS FOR DOUGLIS-NIRENBERG SYSTEMS

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1 Green's formula.

In the bounded domain $G \in R^n$ with the boundary $\partial G \in C^\infty$ we consider the elliptic boundary value problem for Douglis-Nirenberg elliptic (T,S)-system of order $(T,S) = (t_1,\dots,t_N,s_1,\dots s_N)$:

$$l(x,D)u(x) = (l_{rj}(x,D))_{r,j=1,\dots N}u(x) = f(x) \quad (x \in G), \tag{1}$$

$$b(x,D)u(x) = (b_{hj}(x,D))_{\substack{h=1,\dots,m\\ j=1,\dots,N}}u(x) = \varphi(x) \quad (x \in \partial G) \tag{2}$$

Here $\operatorname{ord} l_{rj} \le s_r + t_j$ ($t_1,\dots,t_N,s_1,\dots,s_N$ are given integer), $s_1 + \dots + s_N + t_1 + \dots + t_N = 2m$, $t_1 \ge \dots \ge t_N \ge 0 = s_1 \ge \dots \ge s_N$, $l_{rj}(x,D) \equiv 0$ for $s_r + t_j < 0$; $\operatorname{ord} b_{hj} \le \sigma_h + t_j$, $b_{hj}(x,D) \equiv 0$ for $\sigma_h + t_j < 0$, $\sigma_1,\dots,\sigma_m$ are given integer.

In the case of normal boundary conditions Green's formula was deduced in [1] for Petrovskii elliptic systems. Without assumption of normality Green's formula was obtained in [2] for one equation and in [3] for Petrovskii elliptic systems. For general systems of equations Green's formula was obtained in [4]; but there the scalar product was considered in Sobolev space W_2^k with suffisiently large k in place of the scalar product in L_2.

In the present work Green's formula is obtained for the general elliptic boundary valye problem (1)-(2) for the system of Douglis-Nirenberg structure; furthermore, the formally adjoint problem with respect to the Green's formula is studied too, and the solvability conditions for the problem (1)-(2) are written more presisely. Green's formula holds also for parameter elliptic problems and, therefore, for general parabolic problems for general systems of equations. For general parabolic boundary value problems for one equation Green's formula was deduced in [5].

Let $\kappa = \max\{0, \sigma_1 + 1, \dots, \sigma_m + 1\}$.

The function u_j is differentiated $\tau_j := t_j + \kappa (j = 1,\dots,N)$ times in the system and in the boundary conditions. Thus, if $u_j \in H^{t_j+\kappa,p}(G)$ then $l_r u = \sum_{j=1}^N l_{rj}u_j \in H^{-s_r+\kappa,p}(G)$. Therefore there exist the traces

$$D_\nu^{k-1} l_r u|_{\partial G} \qquad (k = 1,\dots,-s_r+\kappa)$$

of the function $l_r u$ on ∂G.

Hence, if $l_r u = f_r$, $bu = \varphi$, then the equalities

$$\begin{cases} l_r u = f_r(x) \;\; (x \in G, r = 1,\dots,N); \\ D_\nu^{k-1} l_r u(x)|_{\partial G} = f_{rk}(x) \;\; (k = 1,\dots,-s_r+\kappa; f_{rk} = D_n u^{k-1} f_r|_{\partial G}); \\ b_h u(x)|_{\partial G} = \varphi_h \;\; (h = 1,\dots,m) \end{cases}$$

must be valid automatically.

It turns out that the boundary conditions (2) may be complemented by the matrix $c(x, D) = (c_{hj}(x, D))_{\substack{h=1,\dots m \\ j=1,\dots,N}}$ of boundary differential expressions of orders $\operatorname{ord} c_{hj} \le t_j + \sigma'_h$, $(\sigma'_h < 0)$ so that the system

$$\begin{cases} D_\nu^{k-1} l_r u(x)|_{\partial G} = f_{rk}(x) \;\; (r = 1, \dots N, k = 1, \dots, -s_r + \kappa); \\ bu|_{\partial G} = \varphi \\ cu|_{\partial G} = \psi \end{cases}$$

that is equivalent to the system

$$\sum_{j=1}^{N} \sum_{s=1}^{\tau_j} \Lambda_{hjs}(x, D')\eta_{js} = \Phi_h$$

$$(h = 1, \dots, |\tau|; \tau_j = t_j + \kappa; |\tau| = \tau_1 + \dots, \tau_N; \eta_{js} = D_\nu^{s-1} u_j|_{\partial G})$$

or to the system

$$\Lambda\eta = \Phi$$

is a Douglis-Nirenberg elliptic system on ∂G.

For simplicity, now let the defect of Λ be equal to $\{0\}$. Then

$$\eta = \Lambda^{-1}\Phi \quad \text{or} \quad D_\nu^{s-1} u_j|_{\partial G} = \sum_{h=1}^{|\tau|} \Gamma_{sj,h}(x, D')\Phi_h.$$

It permit us to obtain Green's formula.

If some additional condition holds (this condition will be given later) then the following theorem is true.

Theorem 1. *Green's formula*

$$(lu, v) + \sum_{h=1}^{m} \langle b_h u, c'_h v\rangle + \sum_{r=1}^{N} \sum_{k=1}^{-s_r+\kappa} \langle D_\nu^{k-1} l_r u, e'_{rk}(x, D)v\rangle = (u, l^+ v) + \sum_{h=1}^{m} \langle c_h u, b'_h v\rangle \qquad (3)$$

$$(u, v \in (C^\infty(\overline{G}))^N)$$

holds.
Here $D_\nu = i\partial/\partial\nu$, ν is an interior normal to ∂G; $(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle$ denote the scalar products in $(L_2(G))^N$ and $L_2(\partial G)$, respectively.

Definition. The problem

$$l^+ v = g \ (in\ G), \;\; b'_h v|_{\partial G} = \psi_h \ (h = 1, \dots, m) \qquad (4)$$

is called a formally adjoint to the problem (1)-(2) with respect to Green's formula (3).

They are proved the following theorems.

Theorem 2. *The problem (4) is elliptic if the problem (1), (2) is elliptic.*

Theorem 3. *The problem (1), (2) is solvable if and only if the equality*

$$(f_0, v) + \sum_{h=1}^{m} \langle \varphi_h, c'_h v\rangle + \sum_{r=1}^{N} \sum_{k=1}^{-s_r+\kappa} \langle f_{rk}, e'_{kr}(x, D)v\rangle = 0 \quad (\forall v \in \mathcal{N}^+) \qquad (5)$$

holds.
Here $\mathcal{N}^+ = \{\sqsubseteq \in (\mathcal{C}^\infty(\overline{\mathcal{G}}))^{\mathcal{N}} : \updownarrow^+ \sqsubseteq = \prime, \lfloor'_\langle \sqsubseteq|_{\partial\mathcal{G}} = \prime(\langle = \infty, \ldots, \Updownarrow)\}$ *denotes the kernel of the problem (4);* $f_0 = f|_{\overline{G}},\ f_{rk} D_\nu^{k-1} f_r|_{\partial G}\quad (r = 1, \ldots, N,\ \ k = 1, \ldots, -s_r + \kappa)$.

In order to precise Theorem 3 we introduce corresponding functional spaces.

For every $s \geq 0$ and $p \in (1, \infty)$ we denote by $H^{s,p}(G)$ the space of Bessel potentials (Liouville classes); $H^{-s,p}(G) = (H^{s,p'}(G))^*$ $(1/p + 1/p' = 1)$; $||\cdot||_{s,p}$ denotes the norm in $H^{s,p}(G)$ $(s \in R)$.

By $B^{s,p}(\partial G)$ $(s \in R)$ denote the Besov space with the norm $\langle\langle \cdot \rangle\rangle_{s,p}$; $B^{-s,p}(\partial G) = (B^{s,p'}(\partial G))^*$.

For every positive integer r and $s \in R$ $(s \neq k + 1/p,\ k = 0, \ldots, r-1)$ denote by $\widetilde{H}^{s,p,(r)}$ the completion of $C^\infty(\overline{G})$ in the norm

$$|||u|||_{s,p,(r)} := (||u||^p_{s,p} + \sum_{j=1}^{r} \langle\langle D_/ n u^{j-1} u\rangle\rangle^p_{s-j+1-1/p,p})^{1/p}. \tag{6}$$

For $s = k + 1/p$ $(k = 0, \ldots, r-1)$ the space $\widetilde{H}^{s,p,(r)}$ and the norm $|||u|||_{s,p,(r)}$ (6) are defined by method of complex interpolation. Finally, if $r = 0$ then $\widetilde{H}^{s,p,(r)} = H^{s,p}(G)$; $|||\cdot|||_{s,p,(0)} = ||\cdot||_{s,p}$.

Let $\tau = (\tau_1, \ldots \tau_N)$, $|\tau| = \tau_1 + \ldots \tau_N$, $\tau_j = t_j + \kappa$ $(j = 1, \ldots, N)$.

For any $s \in R$, $p \in (1, \infty)$ the closure $A = A_{s,p}$ of the mapping

$$u \mapsto ((l_j u|_{\overline{G}} : j = 1, \ldots, N),\ (D_\nu^{k-1} l_j u|_{\partial G} : j = 1, \ldots, N,\ k = 1, \ldots, -s_J + \kappa),$$
$$(b_j u|_{\partial G} : j = 1, \ldots m)) =: Au$$

acts continuously in the pair of spaces

$$\widetilde{H}^{T+s,p,(\tau)} := \prod_{j=1}^{N} \widetilde{H}^{t_j+s,p,(\tau_j)} \to K_{s,p} := \prod_{j=1}^{N} \widetilde{H}^{s-s_j,p,(\kappa-s_j)} \times \prod_{h=1}^{m} B^{s-\sigma_h-1/p,p}(\partial G).$$

Definition. The element $u \in \widetilde{H}^{T+s,p,(\tau)}$ such that

$$Au = F = (f, \varphi_1, \ldots, \varphi_m),\ f = (f_1, \ldots, f_N),\ f_j = (f_{j0}, \ldots, f_{j,\kappa-s_j}) \tag{7}$$

is called a generalized solution of the problem (1), (2).

It turns out (see, for example, [6]) that *for any* $s \in R$, $p \in (1, +\infty)$ *the operator* $A = A_{s,p}$ *is Noetherian, the kernel* $\mathcal{N}$ *and the cokernel* $\mathcal{N}^*$ *are finitedimentional and do not depend on s and p and consist of infinitely smooth functions.*
This **theorem on complete collection of isomorphisms** gives us the more precise formulation of Theorem 3.

Theorem 3'. *The problem (6)* $A_{s,p}u = F = (f, \varphi) \in K_{s,p}$ *is solvable in* $\widetilde{H}^{T+s,p,(\tau)}$ *if and only if relation (5) holds.*

It was above mentioned that the Theorem 1 is true if the some additional condition is valid. Now we write this condition.

Consider the equalities

$$b_h u|_{\partial G} = \varphi_h \in C^\infty(\partial G)\ (h = 1, \ldots, m)$$

$$D_\nu^{k-1} l_j u|_{\partial G} = f_{jk} \in C^\infty(\partial G)\ (j = 1, \ldots, N,\ k = 1, \ldots, -s + \kappa) \tag{8}$$

$$c_h u|_{\partial G} = \chi_h \in C^\infty(\partial G)\ (h = 1, \ldots, m)$$

We will find the elements $D_\nu^{k-1} u_j|_{\partial G} = \eta_{jk} \in C^\infty(\partial G)$ $(j = 1, \dots, N,\ k = 1, \dots, \tau_j)$ from these relations (8). Thus, we construct the system of $|\tau| = \tau_1 + \dots + \tau_N$ equations with $|\tau|$ variables η_{jk} (we will call this system as (8')).

This system (8') is elliptic in the sense of Douglis-Nirenberg on the manifold ∂G without border.

The Theorem 1 is true if the next condition is valid: *the problem (8') must be uniquely solvable for any smooth right-hand sides; in orther words, the defect of this problem must be equal to* $\{0\}$.

If the problem (1), (2) is parameter elliptic or if it is a Dirichlet system then this condition is always taking place.

If this condition does not hold then Green's formula has additional terms with projective operators. Newetheless, the statements of the Theorem 1 (And Theorem 2, and Theorem 3) remain true.

2 Theorems on complete collection of isomorphisms.

For simplisity, again let $\mathcal{N} = \mathcal{N}^* = \{\mathcal{0}\}$. Recall that in these case the operator $A = A_{s,p}$ realizes an isomorphism

$$A : \widetilde{H}^{T+s,p,(\tau)} \to K^{s,p}. \tag{9}$$

Green's formula permit us to obtain a number of different theorems on complete collection of isomorphisms from the theorem on isomorphisms (9).

To obtain them we need the following too simple statements.

"Pasting" (or factorization) method. Let B_1 and B_2 be Banach spaces, and let T be a linear operator isomorphically mapping the space B_1 onto the space B_2. Let E_1 be a subspace of the space B_1, and let $E_2 = TE_1$. Then it is clear that the operator T by the natural way defines the linear operator T_1 isomorphically mapping the factor space B_1/E_1 onto the factor space B_2/E_2.

Graph method. Let Q_2 be a Banach space, and let $Q_2 \subset B_2$ (this imbedding is algebraic and topological). Then $Q_1 = T^{-1}Q_2$ is a linear (generally speaking, nonclosed) subset of B_1. However, the space Q_1 becams the Banach space denoted by Q_{1T} with respect to the graph norm

$$||x||_{Q_{1T}} = ||x||_{B_1} + ||Tx||_{Q_2} \quad (x \in Q_1).$$

The restriction of the operator T onto Q_1 establishes an isomorphism $Q_{1T} \to Q_2$.

Different elements $u \in \widetilde{H}^{T+s,p,(\tau)}$ may have the same components

$$(u_0, c_1 u|_{\partial G}, \dots, c_m u|_{\partial G}).$$

"Pasting" tham and doing the corresponding factorization in the space of images, we obtain new statement on isomorphisms from isomorphism (9).

Theorem 4. *The closure* $A_1 = A_{1,s,p}$ *of the mapping*

$$u \mapsto (lu, \{b_h u|_{\partial G} : s - \sigma - 1/p > 0\}) \ \ (u \in (C^\infty(\overline{G}))^N)$$

which is considered acting in the pair of spaces

$$\begin{gathered} \prod_{j=1}^N H^{t_j+s,p}(G) \times \prod_{h: s-\sigma_h^c - 1/p < 0} B^{s-\sigma_h^c - 1/p,p}(\partial G) \to \\ \Big(\prod_{j=1}^N H^{s-s_j,p}(G) \times \prod B^{s-\sigma_h - 1/p,p}(\partial G)\Big)/M_{s,p}^1 \end{gathered} \tag{10}$$

(here the subspace $M^1_{s,p}$ dascribed by Greens formula), realizes an isomorphism (10).

Corollary. There holds the next estimate

$$\sum_{j=1}^{N} ||u_j||_{H^{t_j+s,p}(G)} + \sum_{j=1}^{m} \langle\langle c_j u\rangle\rangle_{B^{s-\sigma^c_j-1/p,p}(\partial G)} \leq c_1(||lu||_{\prod_{j=1}^{N} H^{s-s_j,p}(G)} +$$

$$+ \sum_{h:s-\sigma_h-1/p>0} \langle\langle b_h u\rangle\rangle_{B^{s-\sigma_h-1/p,p}(\partial G)} + \sum_{j=1}^{N} ||u_j||_{H^{t_j+s_k,p}(G)}) \quad (u \in (C^\infty(\overline{G}))^N),$$

where $k > 0$ may be choosen arbitrary large.

Different elements $u \in \widetilde{H}^{T+s,p,(\tau)}$ may have the same components

$$(u_0, c_1 u|_{\partial G}, \ldots, c_m u|_{\partial G}, b_1 u|_{\partial G}, \ldots, b_m u|_{\partial G}).$$

We "paste" them and make a corresponding factorization in the space of images. The obtained space of preimages, denoted by $\widetilde{H}^{T+s,p,(\tau)}_{c,b}$ is the completion of $(C^\infty(\overline{G}))^N$ in the morm

$$|||u|||_{T+s,p,(c,b)} := (||u||^p_{T+s,p} + \sum_{j=1}^{m} \langle\langle c_j u\rangle\rangle^p_{s-\sigma^c_h-1/p,p} + \sum_{j=1}^{m} \langle\langle b_j u\rangle\rangle^p_{s-\sigma_j-1/p,p})^{1/p}.$$

The obtained space of images is:

$$(\prod_{j=1}^{N} H^{s-s_j,p}(G) \times \prod_{h=1}^{m} B^{s-\sigma_h-1/p,p}(\partial G))/M^2_{s,p} := K^2_{s,p},$$

where $M^2_{s,p}$ described by means of the Green's formula (3).

Theorem 5. *The closure $A_2 = A_{2,s,p}$ of the mapping*

$$u \mapsto (lu, bu) \;\; (u \in (C^\infty(\overline{G}))^N)$$

which is considered acting in the pair of spaces

$$\widetilde{H}^{T+s,p,(\tau)}_{c,b} \to K^2_{s,p} \tag{11}$$

is an isomorphism between the spaces (13).

Corollary. The following estimate is true:

$$|||u|||_{T+s,p,(c,b)} \leq c_1(||lu||_{s-S,p} + \sum \langle\langle b_j u\rangle\rangle_{s-\sigma_j-1/p,p} + ||u||_{T+s-k,p})$$

$$(u \in (C^\infty(\overline{G}))^N),$$

where $k > 0$ may be choosen arbitrary large.

Different elements $u \in \widetilde{H}^{T+s,p,(\tau)}$ may have the same components

$$(u_0, b_1 u|_{\partial G}, \ldots, b_m u|_{\partial G}).$$

Using the "pasting" method we obtain new **Theorem 6** on complete collection of isomorphisms:

$$\widetilde{H}^{T+s,p,(\tau)}_{b} \to K^3_{s,p}.$$

Here $\widetilde{H}_b^{T+s,p,(\tau)}$ denotes a completion of of $(C^\infty(\overline{G}))^N$ in the morm

$$|||u|||_{T+s,p,(b)} := (||u||^p_{T+s,p} + \sum_{j=1}^m \langle\langle b_j u\rangle\rangle^p_{s-\sigma_j-1/p,p})^{1/p},$$

and the space of images is

$$(\prod_{j=1}^N H^{s-s_j,p}(G) \times \prod_{h=1}^m B^{s-\sigma_h-1/p,p}(\partial G))/M^3_{s,p} := K^3_{s,p},$$

where $M^3_{s,p}$ described by means of Green's formula (3). The corresponding estimate is also true.

Let

$$C_B^\infty := \{u \in (C^\infty(\overline{G}))^N : bu|_{\partial G} = 0\}$$

and let

$$H_B^{T+s} = \{u \in \widetilde{H}^{T+s,p,(\tau)} : bu|_{\partial G} = 0\} =$$
$$= \{u \in H^{T+s,p}(G) : b_h u|_{\partial G} = 0/, (\forall h : s - \sigma_h - 1/p > 0)\} \subset \widetilde{H}_b^{T+s,p,(\tau)}.$$

The Theorem 6 implies the following theorem on isomorphisms.

Theorem 7. *The closure $A_{3(B)}$ of the mapping*

$$u \mapsto lu \qquad (u \in C^\infty_{(B)})$$

which we consider acting in the pair of spaces

$$H_B^{T+s} \to H^{s-S,p}/M^4_{s,p}, \tag{12}$$

where

$$M^4_{s,p} = \{f \in H^{s-S,p} : (f,v) + \sum_{k,r:s-s_r-k+1-1/p>0} \langle D_\nu^{k-1} f_r, e'_{kr} v\rangle = 0 \ \ (\forall v \in$$
$$(C^\infty(\overline{G}))^N : e'_{kr}v|_{\partial G} = 0 (k,r : s - s_r - k + 1 - 1/p < 0), b'v|_{\partial G} = 0)\} \subset H^{s-S,p}$$

realizes an isomorphism between the spaces (12).

In the special case of one equation with normal boundary conditions the isomorphism (12) was obtained by Yu. Berezanskii - S. Krein - Ya. Roitberg (1963).

One can obtain a number of the theorems on complete collection of isomorphisms by means of "graph method". In the special case of one equation with normal boundary conditions by using this method one can obtain the isomorphism of Lions - Magenes.

3 Generalizations.

All these results remain true for elliptic with a parameter systems. But now instead of Noetherity the unique solvability is taking place for sufficiently large parameter. It implies that the results remain true for parabolic problems. In addition, one can consider the multy-times parabolic problems.

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References

1. Roitberg Ya.A., Sheftel Z.G., Mat. Sbornik, 78 (1969), No. 3, p. 446-472.
2. Roitberg Ya.A., Mat. Sbornik, 83 (1970), No. 6, p. 181-213.
3. Roitberg Ya.A., Sheftel Z.G., Math. Investigations, Kishinjov, 1972, 7 (1972), No. 2, p. 143-157.
4. Lvin S.Ja., Green's formula and solvability of elliptic problems with boundary conditions of arbitrary order, VINITI, No. 3318-78, p. 1-30.
5. Roitberg I., Roitberg Ya., Differentsialnye Uravneniya, 31 (1995), No. 8, p. 1437-1444.
6. Roitberg Ya.A., Elliptic Boundary Value Problems in the Spaces of Distributions, Kluwer Akad. Publishers, Dordrecht/Boston/London, 1996, 427 pp.

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