ENTROPY SOLUTIONS OF DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR ELLIPTIC FOURTH ORDER EQUATIONS WITH L^1-DATA

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1. Introduction

Last years some interesting works were published on solvability of nonlinear elliptic second order equations with data which are not kept within usual scheme of the theory of monotone operators [1]. First of all it concerns equations with L^1-right-hand sides [2]–[5], equations with measures [6]–[8] and problems leading in weakened statement to so-called renormalized solutions [9]–[11]. Among the mentioned works the paper [4] is especially distinguished. In that paper a theory of existence and uniqueness of entropy solutions of Dirichlet problem for nonlinear elliptic second order equations with L^1-data was constructed. Note that the proof of solvability of the problem considered in [4] was based on special estimates of solutions of Dirichlet problem for approximating equations with bounded right-hand sides. The main tool to obtain those estimates consisted in the use of test functions which are superpositions of solutions of the approximating problems and of standard truncated functions $T_k(s) = \max\{\min\{s, k\}, -k\}$, $k > 0$, $s \in \mathbb{R}$. Moreover superpositions of entropy solutions and of the standard truncated functions played a significant part in the proof of uniqueness of an entropy solution. On the whole the use of the functions T_k was in the base of the definition of new functional classes which are wider than Sobolev spaces. Namely in these classes single-valued entropy solvability of the problem under consideration was established.

All above-stated concerns second order equations and as far as the author knows any results were not before now obtained on solvability of higher order equations with L^1-data, although the development of corresponding theory has a great interests. This circumstance was an inducement for the author to carry out some research. The main its results are given in this paper.

We consider the question on solvability and uniqueness of entropy solutions of Dirichlet problem for a class of nonlinear elliptic fourth order equations with L^1-right-hand sides. We restrict ourselves with equations of the fourth order, but it is not so significant. As a matter of fact, the passage from second order equations to higher order ones is more principal for us. We follow the approach of [4], however its realization meets a series of difficulties. First of all it is connected with the fact that in the case of higher order equations to obtain needful estimates for solutions of the approximating problems, the functions T_k can not be used in the same way as it holds for second order equations. In particular, in our case arises necessity to construct some functions substituting the truncations T_k and to choose an energy space for the approximating problems. In so doing the following requirements should be satisfied:

1) superpositions of new functions with solutions of the approximating problems are twice differentiable and belong to chosen energy space;

2) the use of these superpositions as test functions in corresponding integral identities allows to estimate in a suitable way the terms connected with the second order derivatives of the superpositions.

Some appropriate substitutions of the standard truncations and an energy space for the approximating problems are established in the paper. By their means needful estimates for solutions of the approximating problems are obtained and theorems on existence and uniqueness of the solutions of the problem under consideration are proved.

Now we pass to the statement of initial assumptions of the paper.

Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

We denote by Λ the set of all n-dimensional multiindices α such that $|\alpha| = 1$ or $|\alpha| = 2$. We shall also use the following notations: $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \to \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \to \mathbb{R}^{n,2}$ and for every $x \in \Omega$ and $\alpha \in \Lambda$, $(\nabla_2 u(x))_\alpha = D^\alpha u(x)$.

Let p, q be real numbers such that

$$1 < p < \frac{n}{2} \, , \tag{1.1}$$

$$2p < q < n \, . \tag{1.2}$$

Let $c_1, c_2 > 0$, let g_1, g_2 be non-negative functions in Ω, $g_1, g_2 \in L^1(\Omega)$, and let for every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be a Carathéodory function. We shall assume that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{n,2}$,

$$\begin{aligned} &\sum_{|\alpha|=1} |A_\alpha(x,\xi)|^{q/(q-1)} + \sum_{|\alpha|=2} |A_\alpha(x,\xi)|^{p/(p-1)} \\ &\qquad \leqslant c_1 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_1(x) \, , \end{aligned} \tag{1.3}$$

$$\sum_{\alpha \in \Lambda} A_\alpha(x,\xi)\xi_\alpha \geqslant c_2 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} - g_2(x) \, . \tag{1.4}$$

Moreover we shall assume that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$,

$$\sum_{\alpha \in \Lambda} \big[A_\alpha(x,\xi) - A_\alpha(x,\xi') \big] (\xi_\alpha - \xi'_\alpha) > 0 \, . \tag{1.5}$$

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We shall study the following problem:

$$\sum_{\alpha \in \Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u) = F(x,u) \quad \text{in} \quad \Omega \, , \tag{1.6}$$

$$D^\alpha u = 0 \, , |\alpha| = 0, 1 \, , \quad \text{on} \quad \partial\Omega \, . \tag{1.7}$$

Precise definitions of solutions of this problem will be given in the main content of the paper. Some additional conditions on the function F (such as in [4]) will be formulated in theorems of existence and uniqueness of the solutions.

Let us give some remarks in connection with the conditions on the numbers p and q. The condition (1.1) is one of possible for p. Introduction of the number q with the restriction (1.2) is explained with the fact that under the condition (1.1) the space $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ is a suitable energy space for problems approximating the problem (1.6), (1.7). By this circumstance the conditions (1.3), (1.4) is also dictated.

In the case where $p > \frac{n}{2}$ solvability of the problem (1.6), (1.7) under corresponding conditions on the coefficients A_α and under the same conditions on F as in given paper can be established in the space $\overset{\circ}{W}{}^{2,p}(\Omega)$ on the base of known results of the theory of monotone operators [1]. It follows from boundedness of the embedding of $\overset{\circ}{W}{}^{2,p}(\Omega)$ into $C(\overline{\Omega})$, if $n < 2p$. The case where $p = \frac{n}{2}$ requires a separate consideration (a paper with corresponding results is in preparation now).

Note that a class of nonlinear elliptic higher order equations under conditions on coefficients of type (1.3), (1.4), but with right-hand sides which are kept within usual framework of the theory of monotone operators, was introduced in the paper [12] devoted to the study of regularity of solutions.

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2. Functional class $\overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$

We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions of $W^{1,q}(\Omega)$ having weak derivatives of the second order from $L^p(\Omega)$. $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \Big(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p} .$$

We denote by $\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ the closure in $W^{1,q}_{2,p}(\Omega)$ of the set $C_0^\infty(\Omega)$.

We shall also use the following notations: if $t \in [1, +\infty]$, then $|\cdot|_t$ is the norm in $L^t(\Omega)$; if $t \in [1, n)$, then $t^* = \frac{nt}{n-t}$.

It is well known that $\overset{\circ}{W}{}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ and there exists a positive constant c' depending only on n, q and such that for every $u \in \overset{\circ}{W}{}^{1,q}(\Omega)$

$$|u|_{q^*} \leqslant c' \sum_{|\alpha|=1} |D^\alpha u|_q \ . \tag{2.1}$$

Let for every$k \in \mathbb{N}$, ψ_k be the function on $\mathbb{R}$ such that

$$\psi_k(s) = s - s^{k+2} + \frac{k+1}{k+3} s^{k+3} \ , \quad s \in \mathbb{R} \ .$$

We define for every $k \in \mathbb{N}$ the function $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(s) = \begin{cases} s\ , & \text{if} \quad |s| \leqslant k\ , \\ \Big[\psi_k\Big(\dfrac{|s|-k}{k}\Big) + 1\Big] k \operatorname{sign} s\ , & \text{if} \quad k < |s| < 2k\ , \\ 2k\dfrac{k+2}{k+3}\operatorname{sign} s\ , & \text{if} \quad |s| \geqslant 2k\ . \end{cases}$$

For every $k \in \mathbb{N}$ we have $h_k \in C^2(\mathbb{R})$, $|h_k| \leqslant 2k$, $0 \leqslant h_k' \leqslant 1$, $|h_k''| \leqslant 3$ on $\mathbb{R}$. Moreover, if $k, j \in \mathbb{N}$ and $j \geqslant 2k$, then for every $s \in \mathbb{R}$,

$$h_k(h_j(s)) = h_k(s)\ . \tag{2.2}$$

We denote by $\overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ satisfying the condition: $\forall k \in \mathbb{N}$, $h_k(u) \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$.

The following properties hold:

$$\overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \subset \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)\ ,$$
$$\overset{\circ}{H}{}^{1,q}_{2,p}(\Omega) \setminus L^1_{loc}(\Omega) \neq \emptyset\ .$$

Definition 2.1. If $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$ and $\alpha \in \Lambda$, then $\delta^\alpha u$ is the function in Ω such that

$$\delta^\alpha u = D^\alpha h_1(u) \quad \text{in} \quad \{|u| \leqslant 1\}$$

and $\forall k \in \mathbb{N}$,

$$\delta^\alpha u = D^\alpha h_{2^k}(u) \quad \text{in} \quad \{2^{k-1} < |u| \leqslant 2^k\}.$$

Lemma 2.2. Let $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$. Then for every $\alpha \in \Lambda$ and $k \in \mathbb{N}$,

$$\delta^\alpha u = D^\alpha h_k(u) \quad \text{a.e. in} \quad \{|u| \leqslant k\}\ .$$

The proof of the lemma is based on the use of the relation (2.2).

Due to Lemma 2.2 for every $u \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega)$ and $\alpha \in \Lambda$, $\delta^\alpha u = D^\alpha u$ a.e. in Ω.

We set

$$r = \frac{n(q-1)}{n-1}\ ,$$
$$c'' = 2^{q^*/q}(c'n)^{q^*} + 2n^q\ .$$

Lemma 2.3. Let $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$, $M \geqslant 1$ and let for every $k \in \mathbb{N}$,

$$\int_{\{|u|<k\}} \Big\{ \sum_{|\alpha|=1} |\delta^\alpha u|^q + \sum_{|\alpha|=2} |\delta^\alpha u|^p \Big\} dx \leqslant Mk\ .$$

Then for every $k \in \mathbb{N}$,

$$\mathrm{meas}\{|u| \geqslant k\} \leqslant c'' M^{n/(n-q)} k^{-r^*} ,$$
$$\mathrm{meas}\Big\{ \sum_{|\alpha|=1} |\delta^\alpha u| \geqslant k \Big\} \leqslant c'' M^{n/(n-1)} k^{-r} ,$$
$$\mathrm{meas}\Big\{ \sum_{|\alpha|=2} |\delta^\alpha u| \geqslant k \Big\} \leqslant c'' M^{n/(n-1)} k^{-pr/q} .$$

This result is proved by means of (2.1).

3. Entropy solutions

Introduce the notation: if $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$, then $\delta_2 u : \Omega \to \mathbb{R}^{n,2}$ and for every $x \in \Omega$ and $\alpha \in \Lambda$, $(\delta_2 u(x))_\alpha = \delta^\alpha u(x)$.

Definition 3.1. An entropy solution of the problem (1.6), (1.7) is a function $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$ satisfying the following conditions:

1) $F(x,u) \in L^1(\Omega)$;
2) there exist $c > 0$, $b \in (1,r)$ and $\gamma > 0$ such that for every $\varphi \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$,

$$\int_{\{|u-\varphi|<2k\}} \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,\delta_2 u)(\delta^\alpha u - \delta^\alpha \varphi)\Big\} h_k'(u-\varphi)dx \leqslant \int_\Omega F(x,u)h_k(u-\varphi)dx + c\big[1+\|\varphi\|_{W^{1,b}(\Omega)}\big]^b k^{-\gamma} . \tag{3.1}$$

In the further considerations we shall denote by c_i, $i = 3,4,\dots$, positive constants which depend only on $n,p,q,c_1,c_2,|g_1|_1,|g_2|_1$ and on $\operatorname{meas}\Omega$.

Lemma 3.2. Let u be an entropy solution of the problem (1.6), (1.7). Then there exists $c > 0$ such that for every $k \in \mathbb{N}$,

$$\int_{\{|u|\leqslant k\}} \Big\{ \sum_{|\alpha|=1} |\delta^\alpha u|^q + \sum_{|\alpha|=2} |\delta^\alpha u|^p \Big\} dx \leqslant c_3[\,|F(x,u)|_1 + c + 1\,]k .$$

This result is proved with the use of (1.4) and of the properties of the functions h_k. From Lemmas 3.2, 2.3 and (1.3) we deduce the following result.

Lemma 3.3. Let u be an entropy solution of the problem (1.6), (1.7). Then

1) for every $\lambda \in (0,r^*)$, $\quad u \in L^\lambda(\Omega)$;
2) for every α, $|\alpha| = 1$, and $\lambda \in (0,r)$, $\ \delta^\alpha u \in L^\lambda(\Omega)$;
3) for every α, $|\alpha| = 2$, and $\lambda \in \big(0,\frac{pr}{q}\big)$, $\ \delta^\alpha u \in L^\lambda(\Omega)$;
4) for every α, $|\alpha| = 1$, and $\lambda \in \big(0,\frac{r}{q-1}\big)$, $\ A_\alpha(x,\delta_2 u) \in L^\lambda(\Omega)$;
5) for every α, $|\alpha| = 2$, and $\lambda \in \big(0,\frac{pr}{q(p-1)}\big)$, $\ A_\alpha(x,\delta_2 u) \in L^\lambda(\Omega)$;
6) for every $\alpha \in \Lambda$, $\ A_\alpha(x,\delta_2 u) \in L^1(\Omega)$.

Proposition 3.4. Let u be an entropy solution of the problem (1.6), (1.7). Then for every function $\varphi \in C_0^\infty(\Omega)$,

$$\limsup_{k\to\infty}\Big[\int_{\{|u-\varphi|<k\}} \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,\delta_2 u)(\delta^\alpha u - \delta^\alpha\varphi)\Big\}dx - \int_\Omega F(x,u)h_k(u-\varphi)dx\Big] \leqslant 0 .$$

This result is proved by means of (1.4), Lemmas 3.2, 2.3, and of the assertion 6) of Lemma 3.3.

Lemma 3.5. Let u be an entropy solution of the problem (1.6), (1.7). Then there exist $c > 0$, $b \in (1, r)$ and $\gamma > 0$ such that for every $\varphi \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and $k \in \mathbb{N}$ the inequality (3.1) holds.

The proof of this fact is based on the use of an approximation of a function $\varphi \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ by an uniformly bounded sequence of functions of the class $C_0^\infty(\Omega)$.

Lemma 3.6. Let u be an entropy solution of the problem (1.6), (1.7). Then there exist $c > 0$, $b \in (1, r)$ and $\gamma > 0$ such that for every $k \in \mathbb{N}$, $k \geqslant 2$, and $m \in \mathbb{N}$ the following inequality holds:

$$c_4 \int_{\{k \leqslant |u| \leqslant k+m\}} \Big\{ \sum_{|\alpha|=1} |\delta^\alpha u|^q + \sum_{|\alpha|=2} |\delta^\alpha u|^p \Big\} dx$$

$$\leqslant m \int_{\{|u| \geqslant k/4\}} \{ |F(x,u)| + g_1 + g_2 \} dx + \big[|F(x,u)|_1 + c + 1 \big]^{n/(n-q)} k^{-1}$$

$$+ c \Big[1 + |u|_b + \sum_{|\alpha|=1} |\delta^\alpha u|_b \Big]^b m^{-\gamma} \; .$$

The proof of this lemma takes into account the special behaviour of the functions h_k in $(-\infty, k) \cup (k, +\infty)$ and the equality

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1$$

which allows to use in a suitable way Hölder and Young inequalities in estimates.

4. H-solutions

Definition 4.1. An H-solution of the problem (1.6), (1.7) is a function $u \in \overset{\circ}{H}{}^{1,q}_{2,p}(\Omega)$ satisfying the following conditions:

1) $F(x, u) \in L^1(\Omega)$;
2) for every $\alpha \in \Lambda$, $\quad A_\alpha(x, \delta_2 u) \in L^1(\Omega)$;
3) for every function $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \delta_2 u) \delta^\alpha \varphi \Big\} dx = \int_\Omega F(x,u) \varphi dx \; .$$

Theorem 4.2. Let u be an entropy solution of the problem (1.6), (1.7). Then u is an H-solution of the same problem.

As far as the proof of this theorem is concerned we emphasize the importance of the same things which are mentioned after Lemma 3.6.

5. Uniqueness and existence

Theorem 5.1. Let for almost every $x \in \Omega$ the function $F(x, \cdot)$ be nonincreasing on $\mathbb{R}$. Let u, v be entropy solutions of the problem (1.6), (1.7). Then $u = v$ a.e. in Ω.

The proof of this result is based on the use of the inequalities (1.5), (2.1), of the properties of the functions h_k and of Lemmas 3.5, 3.6.

Theorem 5.2. Let the following conditions be satisfied:
1) for almost every $x \in \Omega$ the function $F(x, \cdot)$ is nonincreasing on $\mathbb{R}$;
2) for every $s \in \mathbb{R}$ the function $F(\cdot, s)$ belongs to $L^1(\Omega)$.
Then there exists an H-solution of the problem (1.6), (1.7).

We set
$$p_1 = \frac{3n-2}{n+p-1} p \ .$$
In virtue of (1.1) we have $p_1 \in (2p, n)$.

Theorem 5.3. Let the conditions 1), 2) of Theorem 5.2 be satisfied and let $q > p_1$. Then there exists an entropy solution of the problem (1.6), (1.7).

A common part of the proof of these theorems is connected with obtaining of some estimates for solutions of problems approximating the problem (1.6), (1.7). We state below only first two of them. We shall assume that the conditions 1), 2) of Theorem 5.2 are satisfied.

We set $f = F(\cdot, 0)$ and define for every $l \in \mathbb{N}$ the function $F_l : \Omega \times \mathbb{R} \to \mathbb{R}$ by
$$F_l(x, s) = h_l(f(x) - F(x, s)) \ , \quad (x, s) \in \Omega \times \mathbb{R} \ .$$
By the condition 1) of Theorem 5.2 we have: if $l \in \mathbb{N}$, then
$$\text{for almost every } x \in \Omega \text{ the function } F_l(x, \cdot) \text{ is nondecreasing on } \mathbb{R} \ . \tag{5.1}$$

In virtue of the condition 2) of the same theorem $f \in L^1(\Omega)$. Therefore there exists $\{f_l\} \subset C_0^\infty(\Omega)$ such that $|f_l - f|_1 \to 0$ and $\forall l \in \mathbb{N}, \quad |f_l|_1 \leqslant |f|_1 + 1$.

From (1.3)–(1.5), (5.1) and from results of the theory of monotone operators we obtain: if $l \in \mathbb{N}$, then there exists a function $u_l \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that $\forall v \in \mathring{W}^{1,q}_{2,p}(\Omega)$,
$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, \nabla_2 u_l) D^\alpha v + F_l(x, u_l) v \Big\} dx = \int_\Omega f_l v dx \ . \tag{5.2}$$

Lemma 5.4. For every $k, l \in \mathbb{N}$ the following inequalities hold:
$$\int_{\{|u_l| \leqslant k\}} \Big\{ \sum_{|\alpha|=1} |D^\alpha u_l|^q + \sum_{|\alpha|=2} |D^\alpha u_l|^p \Big\} dx \leqslant c_5 [|f|_1 + 1] k \ , \tag{5.3}$$
$$c_6 \int_{\{|u_l| \geqslant 2k\}} |F_l(x, u_l)| dx \leqslant \int_{\{|u_l| \geqslant k\}} |f| dx + [|f|_1 + 1] k^{-1} + |f_l - f|_1 \ . \tag{5.4}$$

To prove (5.3) we first define a sequence of functions $\chi_k \in C^2(\mathbb{R})$ which have the properties: $\chi_k(s) = s$ if $|s| \leqslant k$; $|\chi_k| \leqslant 3k$, $0 < \chi_k' \leqslant 1$ and $|\chi_k''| \leqslant (8/k)\chi_k'$ on $\mathbb{R}$. Then we put the function $\chi_k(u_l)$ in (5.2) instead of v, and using (1.3), (1.4), (3.2), Young inequality, (5.1) and the properties of the functions χ_k, we obtain the required estimate. To prove (5.4) we utilize some other test function in (5.2).

Remark 5.5. Some modifications in the definition of entropy solution allow to prove the result of existence of (modified) entropy solution without the condition $q > p_1$.

6. W-solutions

Definition 6.1. A W-solution of the problem (1.6), (1.7) is a function $u \in \overset{\circ}{W}{}^{2,1}(\Omega)$ satisfying the following conditions:
1) $F(x, u) \in L^1(\Omega)$;
2) for every $\alpha \in \Lambda$, $\;A_\alpha(x, \nabla_2 u) \in L^1(\Omega)$;
3) for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha \varphi \Big\} dx = \int_\Omega F(x, u)\varphi dx \; .$$

We set

$$p_2 = \frac{np}{n(p-1)+1} \; .$$

In virtue of (1.1) we have $p_2 \in (1, n)$.

Theorem 6.2. Let the conditions 1), 2) of Theorem 5.2 be satisfied and let $q > p_2$. Then there exists a W-solution of the problem (1.6), (1.7).

This result is established simultaneously with the proof of Theorem 5.2. We only note that under the condition $q > p_2$ a limit function of the solutions u_l of the approximating problems belongs to $\overset{\circ}{W}{}^{2,1}(\Omega)$.

Remark 6.3. We have $p_2 > 2p$ if and only if $p < \frac{3}{2} - \frac{1}{n}$. Therefore if $p \geqslant \frac{3}{2} - \frac{1}{n}$, due to (1.2) we obtain $q > p_2$.

From Theorem 6.2 and Remark 6.3 we get the following result.

Corollary 6.4. Let the conditions 1), 2) of Theorem 5.2 be satisfied and let $p \geqslant \frac{3}{2} - \frac{1}{n}$. Then there exists a W-solution of the problem (1.6), (1.7).

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