ABOUT BOUNDED PROPERTIES OF SMOOTH SOLUTIONS OF SOME DIFFERENTIAL-OPERATOR EQUATIONS

© V.V.Gorodetsky, I.I.Drin

1. **Fractional differentiation and integration in space D'_+**

Recall that symbol $D = D(\mathbb{R})$ denotes the set of all finite unlimited differentiable on $\mathbb{R}$ functions. Convergence in D is defined as below: sequence $\{\varphi_n, n \geq 1\} \subset D$ is called the converged sequence to function $\varphi \in D$ (maps as follows: $\varphi_n \to \varphi$ when $n \to \infty$ in D) if:

a) exists such $R > 0$, that $supp\,\varphi_n \subset (-R, R), \quad \forall n \in \mathbb{N}$;

b) $\varphi_n^{(k)} \rightrightarrows \varphi^{(k)}$ when $n \to \infty$ on $\mathbb{R}, \forall k \in \mathbb{Z}_+$.

Totality of all linear continuous functionals on D with weak convergence is mapped with symbol $D' \equiv D'(\mathbb{R})$. Elements D' are named the generalized functions. Totality of generalized functions from D', which are equal zero on the half-axis $(-\infty, 0)$, is mapped by D'_+. It is known from [1] that D'_+ creates associative and commutative algebra on folding operation, and $\delta * f = f * \delta = f, \forall f \in D'_+$. The δ-function of Dirac is the one in this algebra.

Let the generalized function f_α from D'_+ depend from parameter $\alpha, -\infty < \alpha < +\infty$, and be denoted by formula

$$f_\alpha(t) = \theta(t)t^{\alpha-1}/\Gamma(\alpha), \quad \alpha > 0,$$

$$f_\alpha(t) = f^{(m)}_{\alpha+m}(t), \quad \alpha \leq 0,$$

where m is the smallest from natural numbers and $m+\alpha > 0$, θ is the Heaviside function.

The following assertions are valid:

1) $\forall \{\alpha, \beta\} \subset \mathbb{R} : f_\alpha * f_\beta = f_{\alpha+\beta}$;

2) Let $I(\alpha)f = f * f_\alpha, \forall f \in D'_+$. Then

a) $\forall f \in D'_+ : \quad I(0)f = f$;

b) $\forall f \in D'_+ \quad \forall n \in \mathbb{N} : \quad I(-n)f = f^{(n)}$;

c) $\forall f \in D'_+ \quad \forall n \in \mathbb{N} : \quad (I(n)f)^{(n)} = f$;

d) $\forall f \in D'_+ \quad \forall \{\alpha, \beta\} \subset \mathbb{R} : \quad I(\alpha)I(\beta)f = I(\alpha+\beta)f$.

2. **Spaces of based and generalized functions**

Let $H = L_2(\mathbb{R})$,

$$\Phi = \lim_{m\to\infty} ind \Phi_m, \quad \Phi_m = \{\varphi \,|\, \varphi = \sum_{k=0}^{m} c_k h_k(x), c_k \in \mathbb{C}\},$$

where

$$h_k(x) = (2^k k!)^{-1/2}(-1)^k \pi^{-1/4} e^{x^2/2}(e^{-x^2/2})^{(k)}, \quad k \in \mathbb{Z}_+$$

are Hermite functions which create the ortonormous basis in H. It is evident that Φ lies densily in H. In space Φ the differentiation operation is defined and continuous.

Let symbol Φ' map the space of all antilinear continuous functionals on Φ with weak convergence. Elements of Φ' also are named the generalized functions. Each element f from space Φ' is unlimited differentiable and

$$< f^{(n)}, \varphi >= (-1)^n < f, \varphi_n >, \quad \forall \varphi \in \Phi, \quad \forall n \in \mathbb{N}$$

(here $< f, \cdot >$ maps the action of functional f on the based element).

Series $\sum\limits_{k=0}^{\infty} c_k h_k$, where $c_k =< f, h_k >, k \in \mathbb{Z}_+, f \in \Phi'$, is named Fourier-Hermite series of generalized function f. For any generalized function f her Fourier-Hermite series converges in Φ'. Otherwise, any series of type $\sum\limits_{k=0}^{\infty} c_k h_k$ converges in Φ' to some function $f \in \Phi'$ and this series is Fourier-Hermite series for f [2]. Then, Φ' can be interpreted as the space of formal series of type $\sum\limits_{k=0}^{\infty} c_k h_k$.

I.M.Gelfand and G.E.Shylov described in [3] the collection of spa- ces, which are called the spaces of type S. These spaces consist of unlimitly differentiable functions, which are defined on $\mathbb{R}$ and satisfy some decreasing conditions on the infinity and conditions of increasing of derivatives. Denote some of them.

For arbitrary $\alpha, \beta > 0$ let

$$S_\alpha^\beta(\mathbb{R}) \equiv S_\alpha^\beta := \{\varphi \in C^\infty(\mathbb{R}) | \exists c, B, A, > 0 \quad \forall \{k, m\} \subset \mathbb{Z}_+$$

$$\forall x \in \mathbb{R} : \quad |x^k \varphi^{(m)}(x)| \leq c A^k B^m k^{k\alpha} m^{m\beta} \}.$$

Spaces S_α^β are nontrivial for $\alpha + \beta \geq 1$ and create dense sets in $L_2(\mathbb{R})$. If $0 < \beta < 1$ and $\alpha > 1 - \beta$ then S_α^β consists only of functions $\varphi : \mathbb{R} \to \mathbb{C}$ which let analytical continuation into whole complex plane and for which

$$|\varphi(x + iy)| \leq c \exp\{-a|x|^{1/a} + b|y|^{1/(1-\beta)}\}, \quad c, a, b > 0.$$

Note, that spaces S_α^β create topological algebras on simple operations of multiplication and folding. In S_α^β the operations of shear of argument and differentiation are defined and continuous. This operations translate S_α^β into itself [3].

Space of all antilinear continuous functionals on S_α^β with weak convergence is mapped by symbol $(S_\alpha^\beta)'$. Elements $(S_\alpha^\beta)'$ are called Gevrey ultradistributions of order β.

For abovementioned spaces the following continuous and dense implications are valid:

$$\Phi \subset S_\alpha^\beta \subset L_2(\mathbb{R}) \subset (S_\alpha^\beta)' \subset \Phi', \quad \alpha + \beta \geq 1.$$

3. About smooth solutions of parabolic equations with increasing coefficients

Consider in space Φ' operator $\hat{A}$ with such action:

$$\Phi' \ni \sum_{k=0}^{\infty} c_k h_k = f \mapsto \hat{A} f = \sum_{k=0}^{\infty} (2k+1)^{\nu} c_k h_k \in \Phi',$$

where $\nu > 0$ is the fixed parameter. It is evident, that operator $\hat{A}$ is linear and continuous in Φ'. The following assertion is valid.

Theorem 1. *Let A be the contraction of operator $\hat{A}$ on $L_2(\mathbb{R})$. Then A is nonnegative selfadjoint operator in $L_2(\mathbb{R})$ with dense range of definition $D(A)$ and $\Phi \subset D(A)$.*

Corollary 1. *Spectrum of operator A is clear discrete with unique limited point on infinity. Hermite functions $\{h_k, k \in \mathbb{Z}_+\}$ are eigenfunctios for operator A. These functions have eigenvalues $\mu_k = (2k+1)^{\nu}, k \in \mathbb{Z}_+$. Each eigenvalue μ_k is prime.*

Remark 1. If $\nu = 1$ then operator A converges with operator which is created in $L_2(\mathbb{R})$ by the differential expression $-d^2/dx^2 + x^2$, that is in this case A is harmonic oscilliator (see [2]).

Consider equation

$$D_t^{\beta} u(t,x) + (-1)^{-[\beta]+1} D_t^{\{\beta\}} A^{\alpha} u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R} \equiv \Omega, \tag{1}$$

where $\beta \in [-3,0), \alpha > 0$ are fixed numbers, $[\beta]$ is whole party and $\{\beta\}$ is fractional party of number β, $D_t^{\beta} \equiv I(\beta)$ is operator of fractional differentiation which acts on variable t in space D'_+, A^{α} is degree of operator A and

$$D(A^{\alpha}) = \{\varphi \in L_2(\mathbb{R}) | \sum_{k=0}^{\infty} (2k+1)^{2\nu\alpha} |c_k(\varphi)|^2 < \infty,$$

$$c_k(\varphi) = (\varphi, h_k), \quad k \in \mathbb{Z}_+\}.$$

The solution of equation (1) we called the function u which sastisfies the equations:

1) $u(\cdot, x) \in D'_+ \cap C^{-[\beta]}((0,\infty))$ for all $x \in \mathbb{R}$;

2) $u(t,\cdot) \in D(A^{\alpha}) \subset L_2(\mathbb{R})$ for all $t > 0; u(t,\cdot) = 0$ for $t < 0$;

3) u satisfies equation (1).

If $\beta \in [-3,-1)$, we assume that u satisfies also the condition:

4) for arbitrary fixed interval $[\delta, +\infty) \subset (0,+\infty)$ constant $c = c(\delta) > 0$ exists such that

$$\sup_{t \in [\delta, +\infty)} \| D_t^{\{\beta\}} u(t,\cdot) \|_{L_2(\mathbb{R})} \leq c.$$

Theorem 2. *Function u is the solution of equation (1) if and only if it can be represented in following type*

$$u(t,x) = \sum_{k=0}^{\infty} (\theta(t) \exp\{-t(2k+1)^{\nu\alpha/(-[\beta])}\} * f_{\{\beta\}}(t)) c_k h_k(x), \tag{2}$$

$$t \in \mathbb{R} \backslash \{0\}, \quad x \in \mathbb{R},$$

where

$$f=\sum_{k=0}^{\infty} c_k h_k \in (S_\omega^\omega)',$$

$\omega=1/2$, *if* $\nu_\alpha/(-[\beta]) \equiv \gamma \geq 1$ *and* $\omega = 1/(2\gamma)$, *if* $0<\gamma<1$. *And* $u(t,\cdot)\in S_\omega^\omega$ *for all* $t>0$.

Remark 2. From theorem 2 follows that when $t>0$ formula (2) describes **all** unlimited differentiable on x solutions of equation (1).

Corollary 2. *Bounded value* $D_t^{\{\beta\}}u(t,\cdot)$ *when* $t\to+0$ *exists in space* $(S_\omega^\omega)'$, *that is*

$$D_t^{\{\beta\}}u(t,\cdot)\to f=\sum_{k=0}^{\infty} c_k h_k, \quad t\to+0, \quad in \quad (S_\omega^\omega)'.$$

Then, $(S_\omega^\omega)'$ is (in some sence) "maximal" space in which bounded values of function $D_t^{\{\beta\}}u(t,\cdot)$ exist for $t\to+0$. Using the representation of function as formal Fourier-Hermite series we can establish necessary and unique conditions for which the bounded values $D_t^{\{\beta\}}u(t,\cdot)$ when $t\to+0$ exist in narrow (intermediate) spaces. These spaces are situated between $L_2(\mathbb{R})$ and $(S_\omega^\omega)'$. The following assertion are valid.

Theorem 3. *In order for bounded value of function* $D_t^{\{\beta\}}u(t,\cdot)$ *when* $t\to+0$ *to belong to space* $(S_\beta^\beta)'$ $\quad(\beta>1/2$, *if* $\gamma\geq 1$; $\beta>1/(2\gamma)$, *if* $0<\gamma<1; \gamma=\nu\alpha/(-[\beta]))$, *it is necessary and enough that*

$$\forall\mu>0 \quad \exists c=c(\mu)>0: \quad \int_{\mathbb{R}} |D_t^{\{\beta\}}u(t,x)|^2dx \leq ce^{\mu t^{-q}},$$

$$q=1/(2\gamma\beta-1),$$

for small values $t>0$.

Denote, that abovementioned spaces link one to another by following chain:

$$L_2(\mathbb{R})\subset (S_\beta^\beta)'\subset (S_\omega^\omega)'\subset \Phi'.$$

Remark 3. If parameter β has one of values of set $\{-1,-2,-3\}$ then $\{\beta\}=0$ and $D_t^{\{\beta\}}=D_t^0=E$ (E is identity operator), $D_t^\beta u(t,\cdot)\equiv I(\beta)u(t,\cdot)=\partial^p u(t,\cdot)/\partial t^p, p=-\beta$ (see item 1). Then we obtain the equation

$$\partial^p u/\partial t^p+(-1)^{p+1}A^\alpha u=0, \quad (t,x)\in\Omega, \quad p\in\{1,2,3\}. \tag{3}$$

$$f_{-\{\beta\}}=f_0=f_1'=\theta'=\delta$$

(see item 1), then

$$\theta(t)\exp\{-t(2k+1)^\gamma\}*f_{-\{\beta\}}(t)=\theta(t)\exp\{-t(2k+1)^\gamma\}*\delta(t)=$$

$$=\theta(t)\exp\{-t(2k+1)^\gamma\}, \quad \gamma=\nu\alpha/p, \quad p=-[\beta], \quad p\in\{1,2,3\}.$$

That is why the solutions of these equations are represented in following form (when $t > 0$)

$$u(t,x) = \sum_{k=0}^{\infty} \exp\{-t(2k+1)^{\gamma}\} c_k h_k(x) =$$

$$=< f, K_{t,x,\gamma}(\cdot) >, \quad t > 0, x \in \mathbb{R},$$

where

$$K_{t,x,\gamma}(y) = \sum_{k=0}^{\infty} \exp\{-t(2k+1)^{\gamma}\} h_k(x) h_k(y), \quad t > 0, \{x, y\} \subset \mathbb{R}.$$

Denote that in case $\gamma = 1$ the cernel $K_{t,x,1}$ can be written explicitly [4]:

$$K_{t,x,1}(y) = (2\pi sh(2t))^{-1/2} \exp\{sh^{-1}(2t)xy - 0.5cth(2t)(x^2 + y^2)\}.$$

Remark 4. If $\nu = 1, \alpha = m, m \in \mathbb{N}$ then (as known from [2]),

$$A^m u(t,x) = (-\partial^2/\partial x^2 + x^2)^m u(t,x) =$$

$$= \sum_{0 \le p+q \le 2m} c^m_{p,q} x^p (\partial^q u(t,x)/\partial x^q),$$

where $c^m_{p,q}$ are constant coefficients for which following estimations are valid:

$$|c^m_{p,q}| \le 10^m m^{m-(p+q)/2}.$$

Thus we define equation (3) as the equation of parabolic type with increasing coefficients.

Corollary 2 from theorem 3 lets us establish Cauchy problem for equation (1) as described below. For (1) we define initial condition

$$D_t^{\{\beta\}} u(t,\cdot)|_{t=0} = f, \tag{4}$$

where $f \in (S_{\omega}^{\omega})'$. The solution of Cauchy problem (1), (4) is the solution of equation (1) which satisfies the initial condition (4) in sence $D_t^{\{\beta\}} u(t,\cdot) \to f, \quad t \to +0$, in space $(S_{\omega}^{\omega})'$. The following asssertion is valid.

Theorem 4. *Cauchy problem (1),(4) is correctly solved problem in space of initial data $(S_{\omega}^{\omega})'$. It's solution described by formula (2); $u(t,\cdot) \in S_{\omega}^{\omega}$ for all $t > 0$.*

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