WAVELETS AND BOUNDARY VALUE PROBLEMS

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ABSTRACT. Method of determination of an approximate solution of a boundary value problem for the ordinary differential equation, based on an expansion by a system of basis functions, constructed on a multiscale system of basis wavelets and satisfying given boundary conditions is described.

Introduction. There exist a great number of the methods for solving of boundary value problems. They are: fictitions domain method, Schvartz alternating method, domain decomposition method, alternating direction implicit method, the method of fictitions components, the methods for constructing of the adaptive grids, and etc. [1,2]. Each of these methods gives the representation of the approximate solution by some functional space basis. The orthogonal wavelet analysis is interesting for the fact, that its basis elements are well localized not only in space, but also in frequency. Precisely this special form of double localization, by means of wavelets, transforms a large class of functions and operators into so-called sparse one or sparse with a high degree of accuracy, while representating them in terms of wavelets. However, the basis elements of these representations do not satisfy the boundary conditions. This fact leads to a slow convergence of an approximate solution to a precise one. The method of constructing of the approximate solution of a boundary value problem for the ordinary differential equation, satisfying the high order precision boundary conditions and containing a few numbers of basis elements is considered in the present paper. The two-dimensional basis elements can easily be constructed as a direct product of one-dimensional ones.

Notations and definitions of wavelet constructions. The function ψ determined on the numerical axes, with nonzero mean value and rather fast decay at infinity is called wavelet in very general form. The term "wavelet" expresses the gist of the matter, since the abovementioned properties mean that the function ψ is a damping oscillation. The wavelet serieses are very convenient for the approximate calculations since the number of operations for calculating the expansion coefficients as well as the number of operations for reconstruction of the function by means of it's wavelet coefficients, is in proportion with the units in the sample of function.

The multiscaled expansion is the increasing sequence $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces $V_j \subset L_2(\mathbf{R})$, $j \in \mathbf{Z}$, possessing the following properties [3, 4]:

1.$V_j \subset V_{j+1}$,

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2.$F(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,
3.$F(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$,
4. $\bigcup_{j=-\infty}^{+\infty} V_j$ dense in $L_2(\mathbf{R})$ and $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$,
5. There exists such a scaling function $\varphi(x) \in V_0$ with a nonzero integral, that the set of functions $\{\varphi(x-k) \mid k \in \mathbf{Z}\}$ forms the Riesz basis in V_0.

The subspaces V_j we'll call levels. It is often supposed, that the set $\{\varphi(x-k)|k \in \mathbf{Z}\}$ represents the orthonormal basis. In this case the function $\varphi(x)$ is called orthonormal.

Let's note, that $\varphi\left(\frac{x}{2}\right) \in V_{-1} \subset V_0$, thus it may be expanded in basis functions of the closed subspace V_0

$$\varphi\left(\tfrac{x}{2}\right) = 2\sum_k h_k \varphi(x-k),\ h_k = \left\langle \varphi\left(\tfrac{x}{2}\right), \varphi(x-k)\right\rangle,\ k \in \mathbf{Z}. \tag{1}$$

This functional equation is a self-similarity or scaling equation. The function $\varphi(x)$ is called the scaling function.

Let W_j denotes a space, complementing V_j in V_{j+1}, i.e. a space satisfying the following relation

$$V_{j+1} = V_j \oplus W_j.$$

We note that the space W_j is not necessary unique. There may be several waves to complement V_j in V_{j+1}.

Space W_j contains "detailed" information, needed to go from an approximation "at resolution j" to an approximation "at resolution $j+1$". Consequently,

$$\bigoplus_j W_j = \mathbf{L}_2(R).$$

The function ψ is a wavelet, if the collection of functions $\{\psi(x-l) \mid l \in \mathbf{Z}\}$ is the Riesz basis of a subspace W_0. Then a set of functions $\{\psi_{j,l} \mid l, j \in \mathbf{Z}\}$ will form the Riesz basis of the space $\mathbf{L}_2(\mathbf{R})$. The functions $\psi_{j,l}$ are defined the same as $\varphi_{j,l}$ in the previous sections. As the wavelet ψ is the element of subspace V_1, there is the sequence $\{g_k\} \in l_2$, such, that

$$\psi(x) = 2\sum_k g_k \varphi(2x-k), \tag{2}$$

i.e. it is expressed by shifts of function $\varphi(x)$ using the formula, similar (1).

Thus, if the scaling function $\varphi(x)$ possesses the property, that its shifts $\varphi(x-1), \varphi(x-2), \dots$ are orthogonal, the coefficients of wavelet expansion (2) can be expressed by the coefficients of the scaling equation. We can put

$$g_k = (-1)^k \overline{h_{1-k}}. \tag{3}$$

Thus if the coefficients h_k than real, and g_k are also real. It's easily seen, that the functions $\varphi(x)$ and $\psi(x)$ are orthogonal, i.e. their scalar product is equal to zero:

$$\int_{-\infty}^{+\infty} \varphi(x)\overline{\psi(x)}dx = 2\sum_{-\infty}^{+\infty} h_k \overline{g_k} \left\|\varphi(x)\right\|^2 = 0.$$

Besides the function $\psi(x)$ has orthogonal both shifts and all rescaling versions are orthogonal as well. It can be said, that the coefficients h_k in (1) specify so-called smoothing filter, and corresponding coefficients g_k in (3) specify high-frequency filter.

Let we have, according to the general theory of the multiscale analysis of wavelets, the expansion of space $L_2(\mathbf{R})$ in a direct sum of subspaces since some fixed level V_j

$$L_2(\mathbf{R}) = V_j \oplus \left\{ \underset{i \geq j}{\oplus} W_i \right\} , \quad j = 0, 1, \dots , \tag{4}$$

where V_j is the subspace with orthonormalized basis of scaling functions

$$\{ \varphi_{j,k}(x) = 2^{j/2} \varphi\left(2^j x - k\right) , \ k \in \mathbf{Z} \} ,$$

and W_j is the subspace with orthogonal basis of wavelets

$$\{ \psi_{i,k}(x) = 2^{i/2} \psi\left(2^i x - k\right) , \ k \in \mathbf{Z} , \ i \geq j \} .$$

We assume [5], that $\varphi(x)$ is forming scaling function with the compact support $[-M, M]$ and with a zero first moment; $\psi(x)$ is corresponding wavelet with the same compact support and with two zero moments, i.e..

$$\int_{-M}^{M} \varphi(x)\, dx = 1 , \quad \int_{-M}^{M} x\varphi(x)\, dx = \int_{-M}^{M} \psi(x)\, dx , \quad \int_{-M}^{M} x\psi(x)\, dx = 0 . \tag{5}$$

We suppose that the functions $\varphi(x)$ and $\psi(x)$ belong to the space $C_\alpha(\mathbf{R})$. One of the sets of such functions is constructed by Coifman and have titles Coiflets [6].

When the functions involved are defined only on a compact set (for example on an interval), then applying of wavelets requires some modification. For a given function on the unit interval [0, 1] , the most obvious approach is to set its equal to zero outside of a unit interval, and then use the wavelet theory on the line. It is possible also to take advantage of the wavelet theory, developed for periodic functions.

We sketch a construction of orthogonal wavelets on a unit interval [0, 1], recently presented by the Ives Meyer [4]. He extracted from sets orthonormalized on the whole axes basis of wavelets with a compact support three subsets:

1) support intersects the left endpoint 0;
2) support lies in the interior;
3) support intersects the right boundary.

Then we have to ortogonalize separately on the unit interval the first and third subsets with the help of well known Gram - Schmidt procedure.

However using of this technique leads to loose of a major property of the wavelet theory namelly uniformity of representation of basis functions under concrete calculations.

The function $f(x) \in L_2[0,1]$ can be considered as the function from $L_2(\mathbf{R})$ and according to the theory of multiscale analysis be presented as orthogonal expansion of wavelets

$$F(x) = \sum_k f_k \varphi_{j,k}(x) + \sum_{i \geq j} \sum_k f_{i,k} \psi_{i,k}(x) . \quad j = 0, 1, \dots \tag{6}$$

There are only those numbers of k in this expansion for which the supports of corresponding scaling functions and wavelets intersect the interval [0, 1]. We shall mark, that the wavelets are the effective tool for the definition of singularities, therefore the artificial discontinuities on endpoints of an interval are similar to inserting of an essential error.

Statement of a problem. Let we need to find an approximate solution of the equation

$$Lu(x) = \frac{d^2u(x)}{dx^2} + b(x)\frac{du(x)}{dx} + c(x)u(x) = f(x) \tag{7}$$

in domain $0 \le x \le 1$ under the boundary conditions

$$\alpha\frac{du(0)}{dx} - u(0) = 0\,,\ u(1) = 0\,. \tag{8}$$

We shall search for a sequence $\widetilde{u}_i(x)$, $i = 0,1,2,\dots$ of approximate solutions of a problem (7), (8) so that the sequence of functions $f_{i+1}(x) = f(x)L\widetilde{u}_i(x)$ had not projections on the low levels V_j with $j \le i$ and $\|f_i(x)\| \to 0$ at $i \to \infty$.

Conditions on the coefficients of the equation (7) we shall formulate in terms of decreasing velocity of coefficients in expansion (6).

Model problem. A solution of the boundary value problem

$$\frac{d^2u(x)}{dx^2} = f(x) \quad , \quad 0 \le x \le 1 \tag{9}$$

under the boundary conditions (8) is of the form of

$$U(x) = \sum_k f_k v_{j,k}(x) + \sum_{i\ge j}\sum_k f_{i,k}w_{i,k}(x)\,,$$

where

$$v_{j,k}(x) = \int_0^1 G(x,\xi)\varphi_{j,k}(\xi)\,d\xi\,, \quad w_{i,k}(x) = \int_0^1 G(x,\xi)\psi_{i,k}(\xi)\,d\xi\,, \tag{10}$$

the function

$$G(x,\xi) = \frac{1}{\alpha+1}\begin{cases}(x+\alpha)(\xi-1) & \text{for } 0 \le x \le \xi\\ (\xi+\alpha)(x-1) & \text{for } \xi \le x \le 1\end{cases}, \tag{11}$$

is the Green function of the boundary value problem for the equation (9) in the same domain $0 \le x \le 1$ under the same boundary conditions (8).

It is obvious, that all $w_{i,k}(x)$ and $v_{j,k}(x)$ belong to the space $C_{\alpha+2}[0,1]$ and satisfy the boundary conditions (8).

At realization of assumptions (8) the functions $w_{i,k}(x)$ at $M \le k \le 2^i - M$ have the same compact support $[(k-M)/2^i, (k+M)/2^i]$, laying strongly on interval [0, 1], as the corresponding wavelet $\psi_{i,k}(x)$. At $-M < k < M$ and $2^i - M < k < 2^i + M$ the functions $w_{i,k}(x)$ are linear functions outside the support of corresponding wavelet

$\psi_{i,k}(x)$. The functions $v_{j,k}(x)$ are linear functions outside the support of corresponding scaling functions $\varphi_{j,k}(x)$ at all permissible k.

We shall denote

$$\Psi(x)=\int_{-\infty}^{x}\psi(\xi)\,d\xi=\int_{-M}^{x}\psi(\xi)\,d\xi\,,\ \Psi_1(x)=\int_{-\infty}^{x}\xi\psi(\xi)\,d\xi=\int_{-M}^{x}\xi\psi(\xi)\,d\xi\,,$$

$$\Phi(x)=\int_{-\infty}^{x}\varphi(\xi)\,d\xi=\int_{-M}^{x}\varphi(\xi)\,d\xi\,,\ \Phi_1(x)=\int_{-\infty}^{x}\xi\varphi(\xi)\,d\xi=\int_{-M}^{x}\xi\varphi(\xi)\,d\xi\,. \tag{12}$$

In these notations at

$$h_j=2^{-j}\,,\ h=2^{-i}$$

the functions $w_{i,k}(x)$ and $v_{j,k}(x)$ are written in a uniform way

$$w_{i,k}(x)=\int_0^1 G(x,\xi)\psi_{i,k}(\xi)\,d\xi=$$

$$=h^{3/2}\Big[\Psi_1(x/h-k)+\frac{x-1}{\alpha+1}\Psi_1(-k)-\frac{x+\alpha}{\alpha+1}\Psi_1(1/h-k)-$$

$$(x/h-k)\Psi(x/h-k)+\frac{x-1}{\alpha+1}(k+\alpha/h)\Psi(-k)+\frac{x+\alpha}{\alpha+1}(1/h-k)\Psi(1/h-k)\Big], \tag{13}$$

$$v_{j,k}(x)=\int_0^1 G(x,\xi)\varphi_{j,k}(\xi)\,d\xi=$$

$$=h_j^{3/2}\Big[\Phi_1(x/h_j-k)+\frac{x-1}{\alpha+1}\Phi_1(-k)-\frac{x+\alpha}{\alpha+1}\Phi_1(1/h_j-k)-$$

$$(x/h_j-k)\Phi(x/h_j-k)+\frac{x-1}{\alpha+1}(k+\alpha/h_j)\Phi(-k)+\frac{x+\alpha}{\alpha+1}(1/h_j-k)\Phi(1/h_j-k)\Big]. \tag{14}$$

More often the functions $\varphi(x)$ and $\psi(x)$ are specified as the tables (see the known package MatLab). The functions $\Psi(x)$, $\Psi_1(x)$, $\Phi(x)$, and $\Phi_1(x)$ are derived as accordingly tabulated ones. The uniform writing (12), (13) is very convenient for calculations of such form of the function representation

Boundary value Problem with a right-hand side from W_i. We shall choose $j>0$ and corresponding expansion (4) of the space $L_2(\mathbf{R})$. We shall search an approximate solution of a problem (7) - (8) as

$$u_{ap}(x)=\sum_k v_{j,k}+\sum_{m>i\geq j}\sum_k \alpha_{i,k}w_{i,k}, \tag{15}$$

where the coefficients $\alpha_{i,k}$ and β_k will be found using the condition that the function $f(x)-Lu_{ap}(x)$ has projections only on subspaces W_i at $i>M$.

We start from searching of an approximate solution in the case, when the right-hand side of the equation (7) belongs to a subspace of the most general type (see (4)). If the right-hand side of the equation (7) belongs to a subspace W_j, we shall search an approximate solution of a problem as

$$u_j(x) = \sum_k \gamma_k w_{j,k}(x) . \tag{16}$$

In this section we shall omit for brevity the first index of the function w and note

$$w_k(x) = w_{j,k}(x).$$

If the support of the function $\psi_{j,k}(x)$ is inside the domain then the function $w_k(x)$, its first $w_{k_x}^{'}(x)$ and second $w_{k_{xx}}^{''}(x)$ derivations vanish at the boundary of support, i.e.

$$w_k((k-M)/2^j) = w_k((k+M)/2^j) = w_{k_x}^{'}((k-M)/2^j) =$$

$$= w_{k_x}^{'}((k+M)/2^j) = w_{k_{xx}}^{''}((k-1)/2^j) = w_{k_{xx}}^{''}((k+1)/2^j) = 0.$$

Besides, as it is easily seen at realization of the condition (5) the derivative $w_{k_x}^{'}(x)$ is the antiderivative function of $\psi_{j,k}(x)$ and accordingly has the form

$$w_{k_x}^{'}(x) = \int\limits_{(k-M)/2^j}^{x} \psi_{j,k}(\xi)d\xi \,,\ (k-M)/2^j \le x \le (k+M)/2^j .$$

The system of the functions $w_k(x)$ is almost orthogonal and similar to the system $\psi_{j,k}(x)$ in the sense that the scalar product $\langle w_m(x), \psi_{j,k}(x)\rangle$ is nonzero only when the remainer of the indexes k and m modulo 2^j is less than reduced support length $\psi_{j,k}(x)$ (i.e. $\mid k-m \mid < 2M \ll 2^j$). The expanding on functions $w_k(x)$ is similar to the representation of the form (6), when the expansion includes both scaling functions and wavelets themselves. Since the used system of wavelets according to the assumption (5), has two zero first moment, the system $\psi_{j,k}(x)$, as well as the main part of the operator L in (7), "distinguishes badly" the linear functions. Thus, using of almost linear functions $w_k(x)$ with $k < M$ and $k > 2^j M$, with the supports fitting to all the domain $0 < x < 1$ allows to improve essentially the finite-dimensional approximation of a solution (15), and, hence to choose the less 2^j.

Substituting of (16) in (7), multiplicating by $\psi_{j,m}(x)$, $-M+1 \le m \le 2^j + M - 1$ and integrating over our domain (from 0 to 1) lead us to the system of linear equations with respect to the unknowns coefficients γ_k. The matrix of this system has $2 \times 2M - 1$ nonzero diagonals. It's diagonal prevalence is easily seen. Really, at $M \le k \le 2^j - M$ the result of action of the operator L from (7) on the function $w_k(x)$

$$Lw_k(x) = \psi_{j,k}(x) + b(x) \int\limits_{(k-M)/2^j}^{x} \psi_{j,k}(\xi)d\xi + c(x) \int\limits_{(k-M)/2^j}^{x} (x-\xi)\psi_{j,k}(\xi)d\xi$$

is nonzero only on at the support of corresponding $\psi_{j,k}(x)$ and the scalar product with $\psi_{j,m}(x)$ vanishs due to nonintersecting supports. At $-M < k < M$ and $2^j - M < k < 2^j + M$ outside the support of the function $\psi_{j,m}(x)$, the function $w_k(x)$ and its derivative are the linear functions. Thus, it is easily seen, that this matrix has the first and the last nonzero $2M$ columns.

For the sake of simplicity we restrict ourselves to the case $M = 1$.

The support of function $\psi_{j,k}(x)$ will belong to the segment $[(k-1)h, (k+1)h]$. Thus, to calculate the coefficients γ_k we obtain the following system of the linear equations:

$$\mathbf{L}_w \vec{\gamma} = \vec{F},$$

where the matrix $\mathbf{L}_w$ has the form

$$\mathbf{L}_w = \begin{pmatrix} b_0 & a_0 & 0 & 0 & & 0 & 0 & s_0 \\ c_1 & b_1 & a_1 & 0 & \dots & 0 & 0 & s_1 \\ r_2 & c_2 & b_2 & a_2 & & 0 & 0 & s_2 \\ & \vdots & & & \ddots & & & \\ r_{N-2} & 0 & 0 & 0 & & b_{N-2} & a_{N-2} & s_{N-2} \\ r_{N-1} & 0 & 0 & 0 & \dots & c_{N-1} & b_{N-1} & a_{N-1} \\ r_N & 0 & 0 & 0 & & 0 & c_N & b_N \end{pmatrix}.$$

Denote

$$l_i(x) = b(x) w_{i_x}'(x) + c(x) w_i(x)$$

The elements of a matrix $\mathbf{L}_w$ has the following form :

$$b_0 = \int_0^h \left[|\ \psi_{j,0}(x)\ |^2 + l_0(x)\psi_{j,0}(x) \right] dx,$$

$$b_i = 1 + \int_{(i-1)h}^{(i+1)h} l_i(x)\psi_{j,i}(x)dx\,,\ i=1,\dots,N-1;$$

$$b_N = \int_{(N-1)h}^{1} \left[|\ \psi_{j,N}(x)\ |^2 + l_N(x)\psi_{j,N}(x) \right] dx\,,$$

$$a_0 = \int_0^h l_1(x)\psi_{j,0}(x)dx,\ a_i = \int_{(i-1)h}^{ih} l_i(x)\psi_{j,i}(x)dx,\ i=1,\dots,N-2;$$

$$a_{N-1} = \int_{(N-1)h}^{1} l_N(x)\psi_{j,N-1}(x)dx\,,$$

$$c_1=\int\limits_0^{2h} l_0(x)\psi_{j,1}(x)dx,\ c_i=\int\limits_{(i-1)h}^{ih} l_{i-1}(x)\psi_{j,i}(x)dx,\ i=2,\dots,N;$$

$$r_0=r_1=0,\ r_i=\int\limits_{(i-1)h}^{(i+1)h} l_0(x)\psi_{j,i}(x)dx,\ i=2,\dots,N-1;$$

$$r_N=\int\limits_{(N-1)h}^{1} l_0(x)\psi_{j,N}(x)dx\ ,$$

$$s_0=\int\limits_0^{h} l_N(x)\psi_{j,1}(x)dx,$$

$$s_i=\int\limits_{(i-1)h}^{(i+1)h} l_N(x)\psi_{j,i}(x)dx,\ i=1,\dots,N-2;\ s_{N-1}=s_N=0.$$

It is obvious, that all integrals in these expressions for the coefficients have the order of h , if the coefficients of the equation (7) are bounded.

The solution of obtained systems can be found by circular sweep - type method [15]. Let A_w is the 3 - diagonal matrix

$$A_w=\begin{pmatrix} b_0 & a_0 & 0 & 0 & & 0 & 0 & 0 \\ c_1 & b_1 & a_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & c_2 & b_2 & a_2 & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \\ 0 & 0 & 0 & 0 & & b_{N-2} & a_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & c_{N-1} & b_{N-1} & a_{N-1} \\ 0 & 0 & 0 & 0 & & 0 & c_N & b_N \end{pmatrix}.$$

As a result of multiplying by the inversed to $\mathbf{A}_w$ matrix we obtain the system of equations with nonzero only principal diagonal elements and the first and the last columns:

$$\mathbf{A}_w^{-1}\mathbf{L}_w\vec{\gamma}=\vec{G}=\mathbf{A}_w^{-1}\vec{F}\ .$$

Here

$$\mathbf{A}_w^{-1}\mathbf{L}_w=\begin{pmatrix} 1+\alpha_0 & 0 & \dots & \beta_0 \\ \alpha_1 & 1 & & \beta_1 \\ & & \ddots & \\ \alpha_N & 0 & & 1+\beta_N \end{pmatrix}\ ,\ \vec{\alpha}=\mathbf{A}_w^{-1}\vec{r}\,,\vec{\beta}=\mathbf{A}_w^{-1}\vec{s}\,.$$

The vectors $\vec{G}$, $\vec{\alpha}$ and $\vec{\beta}$ are easily found by usual sweep method [2].

It is easily seen, that from the obtained system one can extract the following subsystem:

$$(1+\alpha_0)\gamma_0+\beta_0\gamma_N=G_0\ ,$$

$$\alpha_N\gamma_0 + (1 + b_N)\gamma_N = G_N\,.$$

The determinant of this subsystem is equal to

$$\Delta = 1 + \alpha_0 + \beta_N + \left(\alpha_0\beta_N - \beta_0\alpha_N\right),$$

We obtain:

$$\gamma_0 = [(1+\beta_N)G_0 - \beta_0 G_N]/\Delta\,, \quad \gamma_N = [-\alpha_0 G_0 + (1+\alpha_N)G_N]/\Delta\,,$$

$$\gamma_i = G_i - \alpha_i\gamma_1 - \beta_i\gamma_N\,,\ i = 1,\ldots,N-1\,.$$

Boundary value problem with a right-hand side from V_j. If the right-hand side of function $f(x)$ belongs to a subspace V_j, then whule searching an approximate solution as a linear combination of functions $v_{j,k}$ (see (10)) the matrix L_v, unlike the matrix L_w, turns out to be solid. Inside of zero elements in the matrix L_w we have:

$$Lv_{k,l} = \begin{cases} h\left(b_l + \alpha c_l + c_l^+\right)(kh-1)/(\alpha+1) & \text{for } l + M < k < 2^j - M \\ h\left(b_l - c_l + c_l^+\right)(kh+\alpha)/(\alpha+1) & \text{for } \quad M < k < l - M \end{cases}$$

Here

$$b_i = \int_0^1 b(x)\varphi_i(x)dx\,,\ c_i = \int_0^1 c(x)\varphi_i(x)dx\,,\ c_i^+ = \int_0^1 xc(x)\varphi_i(x)dx$$

are the coefficients of expansion of functions $b(x), c(x)$ and $xc(x)$ in the basis of the scaling functions of the subspace V_j.

Due to linearity of the lines of the matrix L_v over k, it can be transformed to the form of L_w after a trivial procedure.

The solution $u_j(x)$ from (16) is constructed so that $Lu_j(x)$ resulting from application of the operator L from (7) to u_j will be orthogonal to the subspace W_j. It is easily seen that the projection of function $Lu_j(x)$ on a subspace V_j will be the order of h (but nonzero). Therefore the right-hand side $f_1(x) = f(x) - Lu_j(x)$ has almost the same projection on a subspace V_j, as the function $f(x)$. An approximate solution with a new right-hand side can easily be obtained with the help of inversions of matrix L_v. It is obvious that the new right-hand side $f_2(x)$, has a projection on the subspace W_j of the order not above h and easily can be reexpanded due to the general theory of the multiscale analysis of wavelets with the help of relations (1) in the basis of the subspace V_{j+1}. Thus, we have passed from a problem with a right-hand side having projection on the subspace V_j to a new problem having projections on subspaces only of more high level and differ from initial function by values of order h. The change-over from one level to a more high level of a right-hand side can be extended. Thus, the contribution of the first amendment is decreased proportionally $h^k/2^{k(k-1)/2}$, where k is the number of transitions. This estimate designates a velocity of decrease of coefficients in expansion (17) in comparison with a velocity of decrease of expansion coefficients in basis of wavelets of a right-hand side of the equation.

By choosing $j \geq 0$ and corresponding expansion (6), we can search for an approximate solution beginning with right-hand parts being a projection of function $f(x)$ either on V_j or on W_j.

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