THE INVERSE SCATTERING PROBLEM FOR ACOUSTIC WAVES IN UNBOUNDED INHOMOGENEOUS MEDIUM

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Abstract. The inverse scattering problem of determining the complex absorption coefficient of an unbounded, penetrable and inhomogeneous medium from a knowledge of the far-field patterns of the scattered fields corresponding to many incident time-harmonic plane acoustic waves is solved by using the orthogonal projection method of Colton–Monk.

1. Introduction. This paper considers the inverse scattering problem of determining the absorption of sound in an inhomogeneous, penetrable and unbounded medium from a knowledge of the far–field patterns of the scattered fields corresponding to many incident time-harmonic plane waves. The inverse scattering problem for acoustic waves in an inhomogeneous, penetrable medium is difficult to solve since it is both nonlinear and improperly posed. Therefore, to simplify the problem, most existing inverse algorithms reconstruct the sound velocity assuming that the absorption coefficient is negligible (cf. [3], [4] and [2]). We will not make this assumption, but instead of attempting to reconstruct the sound velocity, we shall reconstruct the absorption coefficient by assuming that the inhomogeneity of the medium is only caused by its absorption coefficient.

Our interest in reconstructing the absorption coefficient is motivated by its medical applications. In particular, for biological tissue, a knowledge of the absorption coefficient may provide important diagnostic information since absorption for some biological media is very sensitive to structural changes. For instance, the absorption of a normal tissue may differ considerably from the absorption of a pathological tissue, while other acoustical parameters of the medium, such as density and speed of sound, may differ only slightly. Generally, density and sound velocity are only weakly dependent on structural changes (cf. [5], [6]). Also a knowledge of the absorption coefficient may provide significant information for diffraction tomography since the absorption coefficient for different types of biological media varies notably, while the speed of sound varies much less [9].

There are two main methods for solving the inverse medium problem, (cf. [2], [3]). The first approach, *Linear Methods*, is to simplify the problem by linearization, while the second approach, *Nonlinear Methods*, considers the full nonlinear problem. Our approach is patterned after the *Nonlinear Method* discovered by Colton and Monk (cf. [2], [3], [4]). The main idea of this method is to stabilize the inverse problem by formulating it as a nonlinear optimization problem.

2000 *Mathematics Subject Classification.* 35P25, 76Q05.

First, we derive the mathematical formulation and sketch the solution of the direct scattering problem for acoustic waves in a penetrable, unbounded inhomogeneous medium. The inhomogeneity of the medium is caused by its complex absorption coefficient. Then we present the solution of the inverse scattering problem by using the Colton–Monk method. Since in our case the absorption coefficient is complex valued, our approach is valid for all wave numbers.

The case in which the scattering object is a penetrable, inhomogeneous medium, with inhomogeneity caused by an absorption coefficient having compact support, has been solved in [8]. This note is an extension of the main results of [8] to the case when the absorption coefficient no longer has compact support. See [2] and [8] for application of the Colton–Monk method to the case of impenetrable media.

2. The direct scattering problem. Consider the propagation of acoustic plane waves through an inhomogeneous medium in three-dimensional Euclidean space. The behavior of an acoustic wave is determined by the frequency of the propagating wave and by the following three properties of the medium: its density, its sound velocity and its absorption. Assume that the wave frequency is a known constant. Let $c(x)$ denote the local speed of sound and $\gamma(x) \geq 0$ the absorption (attenuation, damping) coefficient, where $x \in \mathbf{R}^3$.

In the linearized theory of acoustics the wave motion can be determined from a velocity potential $U(x,t)$, (cf. [1]). Assuming that the density is slowly varying, it can be shown that U satisfies the following damped wave equation

$$\frac{\partial^2 U}{\partial t^2} + \gamma \frac{\partial U}{\partial t} - c^2 \Delta U = 0 .$$

Suppose the acoustic propagation is a time–harmonic, plane wave with frequency $\omega > 0$, so that $U(x,t) = u(x)e^{-i\omega t}$. Let $c(x) = c_0$.

Under these assumptions u, the space-dependent part of the velocity potential, satisfies the reduced wave equation

$$\Delta u + k^2 \mu(x) u = 0$$

where the wave number $k = \omega / c_0$ and

$$\mu = 1 + i\frac{\gamma(x)}{k} .$$

The function μ is called the index of refraction.

Assume that an incoming time harmonic acoustic wave $u^i(x) = e^{ik(\hat{\alpha},x)}$ is scattered by the inhomogeneous medium having an unknown index of refraction. Here, $(\hat{\alpha},x)$ denotes the scalar or dot product between $\hat{\alpha}$ and x.

The above considerations motivate the following mathematical problem:

Determine a bounded, twice continuously differentiable function u such that

$$\Delta u + k^2\big[1 + im(x)\big]u = 0 \qquad \text{in } \mathbf{R}^3 \tag{2.1}$$

$$u = e^{ik(\hat{\alpha},x)} + u^s \qquad \text{in } \mathbf{R}^3 \tag{2.2}$$

$$\lim_{r\to\infty} r\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0 , \tag{2.3}$$

where m is a positive, continuously differentiable function in $\mathbf{R}^3$ satisfying the growth condition

$$m(x) \le \frac{C}{|x|^{3+s}} \qquad \text{for } |x| > R\,, \tag{2.4}$$

for some positive constants C, s and R.

One can show that a bounded, continuous solution u of the integral equation

$$u(x) = e^{ik(\hat{\alpha},x)} + ik^2 \iint_{\mathbf{R}^3} \Phi(x,y)m(y)u(y)\,dy \quad x \in \mathbf{R}^3 \tag{2.5}$$

where

$$\Phi(x,y) = \frac{\exp[ik|x-y|]}{4\pi|x-y|}\,,$$

satisfies the conditions (2.1)-(2.3).

Let us denote by u^s the function

$$u^s(x) = ik^2 \iint_{\mathbf{R}^3} \Phi(x,y)m(y)u(y)\,dy\,.$$

The function u^s satisfies the Sommerfeld condition (2.3). Let us define the far–field pattern $F(\hat{x};k,\hat{\alpha})$ by

$$F(\hat{x};k,\hat{\alpha}) = \lim_{r\to\infty} re^{-ikr}u^s(x)$$

where $x = r\hat{x}$ and $|\hat{x}| = 1$, assuming this limit exists. We note that if u is continuous and bounded in $\mathbf{R}^3$ then the far–field pattern exists and is equal to

$$F(\hat{x};k,\hat{\alpha}) = \frac{ik^2}{4\pi} \iint_{\mathbf{R}^3} \exp[-ik(\hat{x},y)]m(y)u(y)\,dy\,. \tag{2.6}$$

The proof of (2.6) follows from the Lebesgue dominated convergence theorem and the inequality (2.4).

Direct Scattering Problem: Assuming that m and u^i are known, compute $F(\hat{x};k,\hat{\alpha})$.

One can show that the incident wave and the function m uniquely determine the far–field pattern.

For this purpose define the Hilbert space $L^2_m(\mathbf{R}^3)$ by

$$L^2_m(\mathbf{R}^3) = \left\{ v\colon\ v \text{ is measurable}, \iint_{\mathbf{R}^3} m(x)|v(x)|^2\,dx < \infty \right\}$$

with the inner product and the norm given by $(f,g)_{L^2_m} = \iint_{\mathbf{R}^3} m(x)f(x)\overline{g(x)}\,dx$ and $\|v\|^2_{L^2_m} = \iint_{\mathbf{R}^3} m(x)|v(x)|^2\,dx$. Then we can rewrite the integral equation (2.5) as

$$f_\alpha = u - ik^2 Tu \tag{2.7}$$

where $f_\alpha(x) = e^{ik(\hat{\alpha},x)}$. One can show that $T : L^2_m(\mathbf{R}^3) \to L^2_m(\mathbf{R}^3)$ is a compact linear operator in $L^2_m(\mathbf{R}^3)$.

As in the case when m has compact support, the integral equation (2.7) is equivalent to the scattering problem (2.1)-(2.3), provided the solution of (2.7) is continuous and bounded.

THEOREM 2.1. *There exists a unique, continuous and bounded solution* $u \in L^2_m(\mathbf{R}^3)$ *of equation* (2.7).

The proof of this theorem follows from the Fredholm alternative, the Sommerfeld condition (2.3) and the regularity properties of the Newtonian potential.

Now we turn our attention to the inverse scattering problem.

3. The inverse scattering problem.

INVERSE SCATTERING PROBLEM: From a knowledge of the far–field patterns $F(\hat{x}; k, \hat{\alpha}_j)$ corresponding to the incident plane waves u^i_j with direction $\hat{\alpha}_j, j = 1, ..., n$, determine the function m.

The inverse scattering problem is difficult to solve since it is both nonlinear and improperly posed. In particular, there is no a linear relationship between F, u^i and the function m. Small perturbations in F may result in either large changes in m or an unsolvable problem.

To solve the inverse scattering problem let us define H to be the vector subspace

$$H = \operatorname{span}\Big\{ j_l(k|x|)Y_l^m(\hat{x}) \,:\, l = 0, 1, 2, \dots, \, -l \le m \le l \Big\}$$

of $L^2_m(\mathbf{R}^3)$ where $j_l(k|x|)$ is a spherical Bessel function and $Y_l^m(\hat{x})$ is a spherical harmonic. $\overline{H}$ is the closure of H in $L^2_m(\mathbf{R}^3)$.

Observe that the Herglotz wave function defined by

$$\omega(y) = \int_{\partial\Omega} g(\hat{x}) e^{ik(\hat{x}, y)} \, ds(\hat{x}) \tag{3.1}$$

with kernel from $L^2(\partial\Omega)$ belongs to $\overline{H}$.

We define the following radiation problem.

DEFINITION 3.1. The radiation problem is to find the pair of functions $\{v, \omega\}$, $v \in L^2_m(\mathbf{R}^3)$ and $\omega \in \overline{H}$, such that

$$\text{(a)} \qquad v(x) = \omega(x) + ik^2 \iint_{\mathbf{R}^3} \overline{\Phi}(x, y) m(y) v(y) \, dy \,,$$

$$\text{(b)} \qquad \text{For } \hat{x} \in \partial\Omega, \qquad \frac{k^2}{4\pi} \iint_{\mathbf{R}^3} e^{ik(\hat{x}, y)} m(y) v(y) \, dy = i \,.$$

One can prove the following uniqueness and the existence theorems:

THEOREM 3.1. *There exists at most one solution for the radiation problem.*

THEOREM 3.2. *There exists a solution for the radiation problem.*

The next result forms a relationship between the solution of the radiation problem and the solution of the problem defined by (2.1) - (2.3). This also gives the motivation for examination of the radiation problem.

THEOREM 3.3. *Suppose there exists a solution $\{v, \omega\}$ for the radiation problem such that ω is a Herglotz wave function with kernel g. Then if $F(\hat{x}, k; \hat{\alpha})$ is the far–field pattern corresponding to (2.1) - (2.4), we have*

$$\int_{\partial\Omega} F(\hat{x}, k; \hat{\alpha})\overline{g(\hat{x})}\, ds(\hat{x}) = 1 \,. \tag{3.2}$$

Our proofs of the Theorems 3.1-3 are long and complicated and hence cannot be presented here.

The main idea of the numerical solution of the inverse scattering problem by the Colton–Monk method is the following: First from a knowledge of the far–field patterns $F(\hat{x}; k, \hat{\alpha}_j)$ by using (3.2) one calculates an approximation of the function g. Next, by the equation (3.1) we define the Herglotz wave function ω. Finally, we reformulate the overdetermined radiation problem as a nonlinear optimization problem and solve it for the unknown function m. Since the inverse scattering problem is improperly posed, one has to combine the above steps and calculate m by the Tikhonov regularization method. For the numerical implementation of the Colton–Monk method see [4] and [2].

The orthogonal projection method of Colton–Monk has several advantages. Most previous solutions of the inverse problem require a direct scattering problem to be solved at each step of the iteration scheme, while the Colton-Monk method avoids this. Also, the number of unknown functions in the optimization scheme does not depend on the number of incident fields, but only depends on the number of different frequencies of the incoming waves. Moreover, if we have more data, we can determine the solution of the inverse problem with better accuracy without increasing the computational effort considerably.

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