# ON BOUNDARY VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS OF THE FORM $\mathcal{L}^+ A(u, \mathcal{L}u) = f$

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Abstract. Generalizedly posed boundary value problems for equations of the types $\mathcal{L}^+ A\mathcal{L}u = f$ and $\mathcal{L}^+ A(u, \mathcal{L}u) = f$, where $\mathcal{L}$ is some general differential operation with smooth matrix coefficients in a general bounded domain $\Omega$ and $A(\cdot,\cdot)$ is some continuous operator in the vector spaces $L_2(\Omega)$, are introdused and studied.

Let $\Omega$ be an arbitrary bounded domain in the space $\mathbf{R}^n$ with the boundary $\partial\Omega = \overline{\Omega}\backslash\Omega$,

$$\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha, D^\alpha = (-i\partial)^{|\alpha|}/\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}, \alpha \in \mathbf{Z}_+^n, |\alpha| = \sum_k \alpha_k$$

be some differential operation with smooth complex $j\times k-$matrix coefficients $a_\alpha(x)$, i.e. its elements belong to the space $C^\infty(\Omega)$, $\mathcal{L}^+\cdot = \sum_{|\alpha|\le m} D^\alpha(a_\alpha^*(x)\cdot)$, $a_\alpha^* = \overline{a_\alpha}^{\,t}$ be a formally adjoint differential operation and let $A : L_2^j(\Omega) \to L_2^j(\Omega)$ be some continuous linear or nonlinear operator. We shall consider at first the equation of the form

$$\mathcal{L}^+ A \mathcal{L} u = f. \tag{1}$$

and boundary value problems for them.

**1.** We call to mind general facts about extensions of a differential operator in a domain (see [2,4,6]). The closing of the operator, which is given on the space $(C_0^\infty(\Omega))^k$ by means of the operation $\mathcal{L}$, in the norm of the graph $\|u\|_L^2 = \|u\|^2_{L_2^k(\Omega)} + \|\mathcal{L}u\|^2_{L_2^j(\Omega)}$ is called a minimal operator $L_0$ in the space $L_2^k(\Omega)$. Below we shall often miss out vector indexes for an ease of the writing but one can easily restore them.

A contraction of the operator, wich is generated by the operation $\mathcal{L}$ in the space $\mathcal{D}'(\Omega)$, onto the domain of the definition $D(L) = \{u \in L_2(\Omega) | \, \mathcal{L}u \in L_2(\Omega)\}$, $L = \mathcal{L}|_{D(L)}$ is called a maximal operator $L$. The space $D(L)$ is some Hilbert space with scalar product of the norm $\|\cdot\|_L$ as well as its close subspace $D(L_0)$, which is the domain of the definition of the operator $L_0$. The kernel $\ker L$ is closed in the spaces $D(L)$ and $L_2(\Omega)$, the kernel $\ker L_0$ is closed in the spaces $D(L)$ and $\ker L$. Let consider another expansion of the operator $\mathcal{L}|_{C_0^\infty(\overline{\Omega})}$, which we define $\tilde{L}$. This is the operator with the definition domain $D(\tilde{L})$, which is the closing of the space $C^\infty(\overline{\Omega}) = \{u \in C^\infty(\Omega) | \, \exists U \in C^\infty(\mathbf{R}^n), U|_\Omega = u\}$ , in the norm of the graph $\|\cdot\|_L$.

We shall consider the following conditions:

$$\text{the operator } L_0 : D(L_0) \to L_2(\Omega) \text{ has a continuous left-inverse;} \tag{2}$$

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the operator $L_0^+ : D(L_0^+) \to L_2(\Omega)$ has a continuous left-inverse; (3)

$$\tilde{L} = (L_0^+)^*; \tag{4}$$

$$\tilde{L}^+ = (L_0)^*. \tag{5}$$

Note, that the first condition means the fulfilment of the estimate: $\exists C > 0, \forall \varphi \in C_0^\infty(\Omega)$, $\|\varphi\|_{L_2(\Omega)} \leq C\|\mathcal{L}\varphi\|_{L_2(\Omega)}$ . It is well known that $L = (L_0^+)^*$ and $L^+ = (L_0)^*$, so that the conditions (4),(5) mean the equalities $D(L) = D(\tilde{L})$, $D(L^+) = D(\tilde{L}^+)$. The conditions (2),(3),(4),(5) was introduced in connection with the study of the concept of the well-posed boundary value problem which we also remind here (see [4,6]). We define the Cauchy space $C(L)$ as $D(L)/D(L_0)$([4]). A homogeneous linear boundary value problem is by definition ([4]) the problem to find a solution $u \in D(L)$ of the relations

$$Lu = f,\ \Gamma u \in B, \tag{6}$$

where $\Gamma : D(L) \to C(L)$ is the mapping of the factorization, $B$ is some linear set in $C(L)$. The boundary condition $\Gamma u \in B$ generates the subspace $D(L_B) = \Gamma^{-1}(B)$ of the space $D(L)$ and an operator $L_B$, which is a contraction of the operator $L$ on the space $D(L_B)$ and which is some expansion of the operator $L_0$. This operator $L_B$ is closed if and only if the linear space $B$ is closed in $C(L)$ or the space $D(L_B)$ is closed in $D(L)$ [4]. The boundary value problem is called well-posed and the operator $L_B$ is called a solvable expansion of the operator $L_0$ if the operator $L_B : D(L_B) \to L_2(\Omega)$ has a continuous two-sided inverse.

STATEMENT 1. *There exists a solvable expansion of the operator $L_0$ and there exists a well-posed boundary value problem for the equation $Lu = f$ if and only if the conditions (2) and (3) are fulfilled.*
See proofs of this statement in the works of M.Vishik [6] and L.Hörmander [4].

**2.** The function $u \in D(L_B)$ satisfying the integral identity

$$< A \cdot L_B u,\ Lv >=< f,\ v > \tag{7}$$

for each function $v \in D(L_B)$, will be called a generalized solution of the problem $\Gamma u \in B, \Gamma^+ ALu \in B^+ (B^+ \in C(L^+)$ gives the conjugate to (6) problem), generated of the problem (6), in the domain $\Omega$ for the equation (1) with any function $f \in D'(L_B)$. The integral identity (7) means the equation

$$\widetilde{L}_{BA} u = L'_B \cdot A \cdot L_B u = f. \tag{8}$$

In particular, the probem (7) will be called a generalized Dirichlet problem if $B = 0$ (i.e. $L_B = L_0$) and a generalized Neumann problem if $B = C(L)$.

The generalized boundary value problem (7) will be called well-posed if the operator $\widetilde{L}_{BA} = L'_B \cdot A \cdot L_B : D(L_B) \to D'(L_B)$ has a continuous two-sided inverse $M : D'(L_B) \to D(L_B)$ and normally well-posed if for each function $f \in D'(L_B)$, which is orthogonal to the space $\ker L_B$, there exists an unique to within an additive component $h \in \ker L_B$ function $u \in D(L_B)$, which is a generalized solution of the equation (8) and which continuous depends on $f$.

These definitions imply the following statement.

STATEMENT 2. *The problem (7) with a continuous in $L_2(\Omega)$ operator $A$ is normally well-posed if and only if the operator $L_B$ is normally solvable and the operator $P \cdot A$ is a homeomorphism of the closed space $\mathrm{Im} L_B$ onto itself, where $P : L_2(\Omega) \to \mathrm{Im} L_B$ is the orthogonal projector. The problem (7) is well-posed if and only if this problem is normally solvable and* $\ker L_B = 0$.

In particular, the following statements are correct.

STATEMENT 3. *A generalized Dirichlet problem (7) with $A = id$ is well-posed if and only if the condition (2)is fulfilled.*

STATEMENT 4. *A generalized Neumann problem (7) with $A = id$ is normally well-posed if and only if the operator $L$ is normal solvable. In particular it holds if the condition (3) is fulfilled.*

From the statements of the work [1] it follows the correctness of the following facts.

STATEMENT 5. *Let $\Omega$ be a bounded domain $\Omega$ with the smooth boundary. The conditions (2), (3) are satisfied and generalized Dirichlet problem for the equation (1) is well-posed and the generalized Neumann problem for the same equation is normally correct if the operator $\mathcal{L}$ is one of indicated below:*

1) *$\mathcal{L}$ is a scalar operator with constant coefficients;*

2) *$\mathcal{L}$ is an operator of the real principal type of the form*

$$\mathcal{L} = P_0 + \sum_{j=1}^{N} c_j(x)P_j, \tag{9}$$

*where $P_j$ are operators of orders less then $m = \deg P_0$;*

3) *$\mathcal{L}$ is an operator of the constant strength of the form (9) with analitical in the domain $\Omega' \supset \overline{\Omega}$ coefficients, where $P_j$ are operators with constant coefficients of strengthes less then of the operator $P_0$,*

4) *$\mathcal{L}$ is a matrix operator with constant coefficients satisfying the condition of Panejach-Fuglede.*

5) *$\mathcal{L}$ is a matrix operator, properly elliptic by Douglis-Nirenberg.*

EXAMPLE 1. One can show that the normal solvability of the operator $L$ is equivalent to the fulfilment of the inequality $\exists C > 0, \forall u \in D(L)$, $\| u\|^2_{L_2(\Omega)} - \|P_{\ker} u\|^2_{L_2(\Omega)} \leq C\|Lu\|^2_{L_2(\Omega)}$,where $P_{\ker} : L_2(\Omega) \to \ker M$ is the orthogonal projector. For $L = \nabla$ we have $\ker L = \{const\}$, $P_{\ker} : u \to \frac{1}{\mathrm{meas}\Omega}\int_\Omega u(x)\,dx$ and the last inequality in this case has the form of the Poincare inequality: $\exists C > 0, \forall u \in C^\infty(\overline{\Omega})$, $\| u\|^2_{L_2(\Omega)} \leq \frac{1}{\mathrm{meas}\Omega}(\int_\Omega u\,dx)^2 + C\,\|\nabla u\|^2_{L_2(\Omega)}$. Thus, the statement 2.7 asserts that the generelized Neumann problem for Poisson equation is normally correct in a bounded domain $\Omega$ if and only if in this domain such the Poincare inequality is fulfilled.

The statement 2 implies the following statement.

STATEMENT 6. *Assume that the operator $A : L_2(\Omega) \to L_2(\Omega)$ is continuous and the expansion $L_B$ is solvable. Then*

1) *the problem (7) is well-posed if and only if the operator $A$ is a homeomorphism,*

2) *the problem (7) has a generalized solution if the operator $A$ is surjective.*

EXAMPLE 2. Let us consider as the operator $\mathcal{L}$ any operator with constant coefficients and as the operator $A$ an Urysohn operator $Au(x) = u(x) + \mu \int\limits_\Omega K(x,t,u(t))\,dt$, where $\forall x,t \in \Omega, \forall \xi_1,\xi_2 \in \mathbf{R}, |K(x,t,\xi_1)-K(x,t,\xi_2)| \leq K_1(x,t)\,|\xi_1-\xi_2|$ with a measurable Fredholm kernal $K_1 : \Lambda^2 = \int\limits_{\Omega\times\Omega} K_1^2(x,t)\,dx < \infty$ and assume that this operator $A$ is continuous acted in $L_2(\Omega)$ (there are different sets of conditions on $K$ for this, see for example [7]). Then, as is known, the equation $u = \mu Au + f$ has an unique solution $u \in L_2(\Omega)$ for any $f \in L_2(\Omega)$ if $|\mu| < \Lambda^{-1}$ with the estimate $\| u\| \leq C(\Lambda, \| f \|)$ and a continuous dependence $u$ on $f$. Therefore by the statement 6 the generalized Neumann problem, $B = C(L)$, for the equation $\mathcal{L}^+ A\,\mathcal{L}\,u = g$ has an unique to within an additive component $h \in \ker L$ solution $u \in D(L)$ for any function $g \in D'(L)$, which is orthogonal to the space $\ker L$. For instance, the Neumann problem $\Delta(u(x) + \mu \int\limits_\Omega K(x,t,\Delta\,u(t))\,dt) = g(x)$, $A\Delta u|_{\partial\Omega} = 0, (A\Delta\,u)'_\nu|_{\partial\Omega} = 0$, where $\Delta$ is the Laplace operator, admits the generalized setting and such problem is solvable for these $K, \mu$ and $f \in (H^2(\Omega))'$. One can bring a lot of examples with a convertible integral operator $A$ and a solvable expansion $L_B$, taken, for example, from the statement 5.

**3.** Note, that the previous definitions are unsuitable for a consideration of the operator depedence on lowest derivatives, for instance, we have no in the example 2 any possibility to consider the equation of the example 2 where $K = K(x, t, u(t), \nabla u(t), \Delta u(t))$. The following scheme is intended just for this case.

Let us now consider once more case of the operator $\widetilde{L}_{BA}$ acting as $\widetilde{L}_{BA}\ u = L'_B\ A(u, L_B u) = L'_B\ \widetilde{A}(Ku, L_B u)$, where $K : D(L_B) \to L_2(\Omega)$ is some compact operator, the expansion $L_B$ is normal solvable, $\widetilde{A} : L_2(\Omega) \times \mathrm{Im}L \to \mathrm{Im}L$ is an continuous operator such that $PA(u, w) = \widetilde{A}(Ku, w)$ with the same orthoprojector $P$, $A : D(L_B) \times \mathrm{Im}L \to L_2(\Omega)$.

We shall consider the following conditions:

$$\forall v \in L_2(\Omega), \text{the operator } \widetilde{A}(v, \cdot) : \mathrm{Im}L \to \mathrm{Im}L \text{ is a homeomorphism,} \tag{10}$$

$$\text{the homeomorphism } (\widetilde{A}\,(v, \cdot))^{-1} : \mathrm{Im}L \to \mathrm{Im}L \text{ is uniformly bounded,} \tag{11}$$

i.e. there exists a function $\beta : \mathbf{R}^+ \to \mathbf{R}^+$, $\beta(r) = R$ such that the image $(\widetilde{A}\ (v, \cdot))^{-1}(S(0, r))$ of the every ball $S(0, r) \subset \mathrm{Im}L$ of radius $r$ hit into the ball $S(0, R) \subset \mathrm{Im}L$ for each $v \in L_2(\Omega)$. Note that to verify the condition (11) it suffices to prove the condition:

$$\text{the mapping } (\widetilde{A}\,(v, \cdot))^{-1} : L_2(\Omega) \to L_2(\Omega) \text{ is uniformly bounded} \tag{12}$$

if the operator $\widetilde{A}(v, \cdot) : L_2(\Omega) \to L_2(\Omega)$ is given on all $L_2(\Omega)$ and is converted.

The function $u \in D(L_B)$ satisfying the integral identity

$$< A(u, L_B u),\ Lv >=< f, v > \tag{13}$$

for every function $v \in D(L_B)$, will be called **a generalized solution** of the problem $\Gamma u \in B$, $\Gamma^+ A(u, Lu) \in B^+$, generated of the problem (6), in the domain $\Omega$ for the equation

$$\mathcal{L}^+ A(u, \mathcal{L} u) = f \tag{14}$$

with an arbitrary function $f \in D'(L_B)$. The integral identity (13) means the equation

$$\widetilde{L}_{BA} u = L'_B \cdot \widetilde{A}(Ku, L_B u) = f. \tag{15}$$

The generalized boundary value problem (13) will be called **solvable** if $\forall f \in D'(L_B), \exists u \in D(L_B)$ such that the equality (15) is satisfied and **well-posed** if the operator $\widetilde{L}_{BA} : D(L_B) \to D'(L_B)$ has a continuous two-sided inverse $M : D'(L_B) \to D(L_B)$. This definition implies the following facts.

STATEMENT 7.*Assume that the expansion $L_B$ is normal solvable and* $\ker L_B = 0$.

1). *In order that a generalized problem* (13) *be solvable (well-posed) it is necessary and sufficient that the equation $PA(u, L_B u) = f$ be solvable for each function $f \in \mathrm{Im}L_B$ (the operator $PA$ has a continuous inverse for these $f$).*

2). *In order that a generalized problem* (13) *be solvable it is sufficient that the conditions* (10),(11) *be fulfilled.*

*Proof.* The point 1) follows (just as in the statement 2) from that the mapping $L_B : D(L_B) \to L_2(\Omega)$ is an isomorphism onto its image and $\ker L'_B \perp \mathrm{Im}L_B$.

2). Let $f \in D'(L_B)$ be an arbitrary function. We have by the statement 6 and the condition (10) that the mapping

$$T : D(L_B) \ni u \to L_B^{-1}(PA(u, \cdot))^{-1}((L'_B)^{-1} f) \in D(L_B)$$

is a completely continuous operator. For each ball $F \ni f$ we have also by the condition (11) that there exists a ball $U \subset \mathrm{Im}L_B$ containing the preimage $(\widetilde{A}\,(Ku, \cdot)^{-1}((L'_B)^{-1}F), \forall u$. Then the compact mapping $L_B T L_B^{-1}$ transfers the closure $\overline{U}$ of the ball $U$ into itself. We can now employ the wellknown Schauder principle and obtain that the mapping $L_B T L_B^{-1}$ has a fixed point, therefore the problem (13) is solvable.

*Remark.* We would like to have a possibility to see on the place $Ku$ a set of any differential expressions, but we should require that the operators of this expressions be compact. Here we come to the following definition. We shall call the differential operation $\mathcal{M}$ $B-$ subordinate to the operation $\mathcal{L}$ and write $\mathcal{M} \prec\prec_B \mathcal{L}$ *if* $D(M) \supset D(L_B)$ and the operator $I \circ M : D(L_B) \to L_2(\Omega)$ with embidding operator $I : \mathrm{Im} M|_{D(L_B)} \to L_2(\Omega)$ is compact. Here the inclusion is dense and means the presence of the a priori estimate

$$\| u\|_L \geq C\| u\|_M \text{or that is the same } \| Lu\|_{L_2(\Omega)} + \| u\|_{L_2(\Omega)} \geq C\| Mu\|_{L_2(\Omega)}$$

*for all* $u \in D(L_B)$. If the operator $L_B$ is normally solvable and $\ker L_B = 0$, then it has a left inverse and the last estimate implies that $\| Lu\|_{L_2(\Omega)} \geq C\| Mu\|_{L_2(\Omega)}$ for the same $u$. Remind that in the work [5] L.Hormander introdused comparisons $\mathcal{M} \prec \mathcal{L}$ and $\mathcal{M} \prec\prec \mathcal{L}$ for scalar differential operations with constant coeffitients, where $\mathcal{M} \prec \mathcal{L}$ means the inclusion $D(M_0) \supset D(L_0)$, i.e. the same a priori estimate but for all $u \in C_0^\infty(\Omega)$, and $\mathcal{M} \prec\prec \mathcal{L}$ means the compactness of the operator $I \circ M : D(L_0) \to L_2(\Omega)$ with the embidding operator $I : \mathrm{Im} M|_{D(L_0)} \to L_2(\Omega)$. In [4] there are conditions on the operator symbols for such comparisons. Of course, the obtaining of any conditions for such comparisons in the different operator classes is a big and hard problem.

EXAMPLE 3. Let us consider the equation

$$\Delta \left(u(x) + \mu \int\limits_\Omega K\left(x, t, u(t), \nabla u(t), \Delta\, u(t)\right) dt\right) = f(x)$$

where the function $K(x, t, \eta_0, \eta_1, ..., \eta_n, \xi)$ satisfies the same conditions just as in the example 3.1, with $K_1(x,t)$ independent of $\eta$. Then the conditions (10),(11) are fulfilled ($\mathrm{Im}\Delta = L_2(\Omega)$) and the generalized Neumann problem for the considered equation has a solution $u \in D(\Delta)$ for each $f \in D'(\Delta)$, $f \perp \ker \Delta$ if $|\mu| < \Lambda^{-1}$by the statement 7 and considerations of the example 3. One can consider equations of high order and also substitute any differential operator $L$ with constant coefficients (or in the same way other operator from the statement 5) for $\Delta$ inside and outside in the equation and obtain the same solvability statement about the generalized Neumann or other problem but then one should use the substitution of operators $L_j \prec\prec_B L$ for $\nabla, \nabla^2, ...$, where the last comparison was determined in the remark.

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