ENTROPY SOLUTIONS OF DIRICHLET PROBLEM FOR A CLASS OF NONLINEAR ELLIPTIC HIGH-ORDER EQUATIONS WITH L^1-DATA

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1. Introduction

In this paper we study Dirichlet problem for nonlinear elliptic high-order equations with coefficients satisfying a strengthened ellipticity condition. A class of nonlinear high-order equations with such a condition on coefficients and data which are within usual theory of monotone operators [7] has been introduced in [9], where Hölder continuity of generalized solutions of equations of that class has been established. Unlike [9] equations under consideration in the present paper have L^1-right-hand sides. Their solvability does not follow directly from results of the theory of monotone operators and on the whole it is a rather difficult problem.

In this connection we note that a theory of existence and uniqueness of solutions of Dirichlet problem for nonlinear elliptic second-order equations with L^1-data has been constructed in [1]. First results on solvability of high-order equations with L^1-right-hand sides, namely on solvability of Dirichlet problem for fourth-order equations with coefficients satisfying a strengthened ellipticity condition, have been obtained in [3] and [4] with the use of the approach proposed in [1]. However, the realization of that approach finds in general a series of significant difficulties which are conditioned by some particularities of high-order equations as compared with the second-order ones. Overcoming of these difficulties has required to develop additional techniques (see details in [4]).

Following in general outline the approach of [1] in the present paper we extend some ideas of [3] and [4] for nonlinear elliptic equations of arbitrary even order greater than four. In so doing, new moments are connected with the use of some interpolation inequalities, such as Nirenberg-Gagliardo inequality [8], and in general with consideration of the intermediate order derivatives of the functions involved.

2. Initial assumptions

Let $m, n \in \mathbb{N}$ be numbers such that $m \geqslant 3$, $n > 2(m-1)$. These inequalities imply that $n(m-1) - 2 > 0$ and

$$\frac{2n(m-2)}{n(m-1)-2} < \frac{2(m-1)}{m} < \frac{n}{m}, \qquad 1 < \frac{2n(m-2)}{n(m-1)-2} < 2\,.$$

Let $p \in \mathbb{R}$ be a number such that

$$\frac{2n(m-2)}{n(m-1)-2} < p < \frac{n}{m}. \tag{2.1}$$

This inequality implies that $p(m-1) - 2(m-2) > 0$.

We set

$$\overline{p} = \frac{2p}{p(m-1) - 2(m-2)}.$$

By virtue of (2.1) we have $\max(\overline{p}, mp) < n$.

Let $q \in \mathbb{R}$ be a number such that

$$\max(\overline{p}, mp) < q < n. \tag{2.2}$$

Remark 2.1. If $p < 2(m-1)/m$, then $\overline{p} > mp$; if $p \geqslant 2(m-1)/m$, then $\overline{p} \leqslant mp$.

3. Functional spaces

Let Ω be a bounded open set of $\mathbb{R}^n$.

We set

$$q^* = \frac{nq}{n-q}.$$

It is well known (see [6]) that $\overset{\circ}{W}{}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ and there exists a positive constant c' depending only on n, q and such that for every $u \in \overset{\circ}{W}{}^{1,q}(\Omega)$,

$$\left(\int_\Omega |u|^{q^*} dx\right)^{1/q^*} \leqslant c' \sum_{|\alpha|=1} \left(\int_\Omega |D^\alpha u|^q \, dx\right)^{1/q}. \tag{3.1}$$

We denote by $W^{1,q}_{m,p}(\Omega)$ the set of all functions $u \in W^{1,q}(\Omega)$ having for every n-dimensional multiindex α, $|\alpha| = m$, the weak derivative $D^\alpha u \in L^p(\Omega)$. $W^{1,q}_{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \, dx\right)^{1/p}.$$

We denote by $\overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ the closure in $W^{1,q}_{m,p}(\Omega)$ of the set $C_0^\infty(\Omega)$.

We denote by Λ the set of all n-dimensional multiindices α such that $1 \leqslant |\alpha| \leqslant m$.

For every $\alpha \in \Lambda$ we set

$$q_\alpha = \left[\frac{|\alpha| - 1}{p(m-1)} + \frac{m - |\alpha|}{q(m-1)}\right]^{-1}.$$

LEMMA 3.1. *Let $h \in C^1(\mathbb{R})$, $0 \leqslant c < b$, and let the following conditions be satisfied: $0 \leqslant h \leqslant b$ in $\mathbb{R}$ and $|h'| \leqslant ch$ in $\mathbb{R}$. Let $u \in C_0^\infty(\Omega)$. Then for every n-dimensional multiindex β, $2 \leqslant |\beta| \leqslant m-1$,*

$$\Big(\int_\Omega |D^\beta u|^{q_\beta} h(u)\,dx\Big)^{\frac{1}{q_\beta}} \leqslant c''\Big(\sum_{|\alpha|=m}\int_\Omega |D^\alpha u|^p h(u)\,dx\Big)^{\frac{|\beta|-1}{p(m-1)}}$$

$$\times\Big(\sum_{|\alpha|=1}\int_\Omega |D^\alpha u|^q h(u)\,dx\Big)^{\frac{m-|\beta|}{q(m-1)}} + cc'''\Big(\sum_{|\alpha|=1}\int_\Omega |D^\alpha u|^q h(u)\,dx\Big)^{\frac{|\beta|}{q}},$$

where the positive constant c'' depends only on m, n, and the positive constant c''' depends only on m, n, b and $\operatorname{meas}\Omega$.

Proving this result we use the inequality $q > \max(\overline{p}, mp)$ and the integration by parts for the integrals over Ω of the functions $|D^\beta u|^{q_\beta} h(u)$, $2 \leqslant |\beta| \leqslant m-1$. We perform this integration by analogy with [9] where instead of (2.1) and (2.2) it was supposed only that $p \geqslant 2$ and $mp < q < n$.

From Lemma 3.1 we deduce the following result.

LEMMA 3.2. *Let $u \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$. Then for every n-dimensional multiindex β, $2 \leqslant |\beta| \leqslant m-1$, there exists the weak derivative $D^\beta u$, $D^\beta u \in L^{q_\beta}(\Omega)$, and the following inequality holds*

$$\Big(\int_\Omega |D^\beta u|^{q_\beta}\,dx\Big)^{\frac{1}{q_\beta}} \leqslant c''\Big(\sum_{|\alpha|=m}\int_\Omega |D^\alpha u|^p\,dx\Big)^{\frac{|\beta|-1}{p(m-1)}}\Big(\sum_{|\alpha|=1}\int_\Omega |D^\alpha u|^q\,dx\Big)^{\frac{m-|\beta|}{q(m-1)}}.$$

This inequality is well known Nirenberg-Gagliardo interpolation inequality [8].

With the use of Lemmas 3.1 and 3.2 we establish the following results.

LEMMA 3.3. *Let $\{u_i\} \subset \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$, $u \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$, and let $u_i \to u$ weakly in $\overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$. Let β be an n-dimensional multiindex such that $1 \leqslant |\beta| \leqslant m-1$, and let $\tilde{q}_\beta \in (1, q_\beta)$. Then $D^\beta u_i \to D^\beta u$ strongly in $L^{\tilde{q}_\beta}(\Omega)$.*

LEMMA 3.4. *Let $h \in C^1(\mathbb{R})$, $b > 0$, and let the following conditions be satisfied: $0 \leqslant h \leqslant b$ in $\mathbb{R}$ and $|h'| \leqslant bh$ in $\mathbb{R}$. Let for every n-dimensional multiindex β, $2 \leqslant |\beta| \leqslant m-1$, we have $\tilde{q}_\beta \in (1, q_\beta)$. Then for every $u \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ and $\varepsilon \in (0,1)$ we have*

$$\int_\Omega \Big\{\sum_{2\leqslant|\beta|\leqslant m-1} |D^\beta u|^{\tilde{q}_\beta}\Big\} h(u)\,dx \leqslant \varepsilon \int_\Omega \Big\{\sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=m} |D^\alpha u|^p\Big\} h(u)\,dx + (\tilde{c}/\varepsilon)^{\tilde{m}},$$

where the positive constant $\tilde{m}$ depends only on m, p, q, $\{\tilde{q}_\beta : 2 \leqslant |\beta| \leqslant m-1\}$ and the positive constant $\tilde{c}$ depends only on m, n, b and $\operatorname{meas}\Omega$.

LEMMA 3.5. *Let $h \in C^m(\mathbb{R})$, $h(0) = 0$, and let the function h and its derivatives $h^{(i)}$, $i = 1, \dots, m$, be bounded in $\mathbb{R}$. Let $u \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$. Then $h(u) \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ and the following assertions hold:*

(i) if $|\alpha| = 1$, then $D^\alpha h(u) = h^{(1)}(u) D^\alpha u$ a.e. in Ω;

(ii) if $2 \leqslant |\alpha| \leqslant m$, then

$$|D^\alpha h(u) - h^{(1)}(u) D^\alpha u| \leqslant c'_{n,m} \left(\sum_{i=2}^{|\alpha|} |h^{(i)}(u)| \right) \sum_{1 \leqslant |\beta| < |\alpha|} |D^\beta u|^{\frac{|\alpha|}{|\beta|}} \quad a.e. \text{ in } \Omega,$$

where the positive constant $c'_{n,m}$ depends only on m and n.

4. Functional set $\overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$

Let $\{h_k\}$ be a sequence of functions such that for every $k \in \mathbb{N}$, $h_k \in C^m(\mathbb{R})$,

$$h_k(s) = s \qquad \text{if} \quad |s| \leqslant k, \tag{4.1}$$

$$h_k(s) = b_1 k \operatorname{sign} s \qquad \text{if} \quad |s| \geqslant 2k, \tag{4.2}$$

$$0 \leqslant h_k^{(1)} \leqslant 1 \qquad \text{in} \quad \mathbb{R}, \tag{4.3}$$

$$|h_k^{(i)}| \leqslant b_2 k^{1-i} \qquad \text{in} \quad \mathbb{R}, \quad i = 2, \dots, m, \tag{4.4}$$

where b_1 and b_2 are some absolute positive constants.

We denote by $\overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ such that:

1) for every $k \in \mathbb{N}$, $\quad h_k(u) \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$;

2) $\displaystyle\sup_{k \in \mathbb{N}} \frac{1}{k} \int_\Omega \left\{ \sum_{|\alpha|=1} |D^\alpha h_k(u)|^q + \sum_{|\alpha|=m} |D^\alpha h_k(u)|^p \right\} dx < +\infty.$

Observe that

$$\overset{\circ}{W}{}^{1,q}_{m,p}(\Omega) \subset \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega).$$

However, the converse inclusion is not true.

We set

$$r = \frac{n(q-1)}{n-1}, \quad r^* = \frac{n(q-1)}{n-q}.$$

Note that by virtue of (2.2) $rp > q$.

Now we state a series of propositions which describe some properties of the set $\overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$.

PROPOSITION 4.1. *Let $u \in \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$. Then for every $\lambda \in (0, r^*)$, $\quad u \in L^\lambda(\Omega)$.*

PROPOSITION 4.2 *Let $u \in \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$ and $\alpha \in \Lambda$. Then there exists the weak derivative $D^\alpha u$ and the following assertions hold:*

1) *for every* $\lambda \in (1, rq_\alpha/q)$, $\quad D^\alpha u \in L^\lambda(\Omega)$;

2) *for every* $k \in \mathbb{N}$, $\quad D^\alpha u = D^\alpha h_k(u)$ *a.e. in* $\{|u| \leqslant k\}$.

PROPOSITION 4.3. *Let* $u \in \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$. *Then*

1) *for every* $\lambda \in (1, r)$, $\quad u \in \overset{\circ}{W}{}^{1,\lambda}(\Omega)$;

2) *for every* $\lambda \in (1, rp/q)$, $\quad u \in \overset{\circ}{W}{}^{m,\lambda}(\Omega)$.

In conclusion of this section we remark that the definition of the set $\overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$ does not depend on the choice of a sequence of functions of $C^m(\mathbb{R})$ with the properties (4.1)–(4.4).

5. Statement of the problem and definitions of its solutions

We shall use the following notation: $\mathbb{R}^{n,m}$ is the space of all functions $\xi : \Lambda \to \mathbb{R}$; if $u \in L^1_{\text{loc}}(\Omega)$ and the function u has the weak derivatives $D^\alpha u, \alpha \in \Lambda$, then $\nabla_m u : \Omega \to \mathbb{R}^{n,m}$ is the mapping such that for every $x \in \Omega$ and $\alpha \in \Lambda$, $(\nabla_m u\,(x))_\alpha = D^\alpha u\,(x)$.

Let c_1, c_2, c_3 be positive constants, g_1, g_2 be non-negative functions, $g_1, g_2 \in L^1(\Omega)$, for every n-dimensional multiindex α, $2 \leqslant |\alpha| \leqslant m-1$, $\tilde{q}_\alpha$ be a number such that $\tilde{q}_\alpha \in (1, q_\alpha)$. Let for every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R}^{n,m} \to \mathbb{R}$ be a Carathéodory function. We shall suppose that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{n,m}$,

$$\sum_{|\alpha|=1} |A_\alpha(x,\xi)|^{\frac{q}{q-1}} + \sum_{|\alpha|=m} |A_\alpha(x,\xi)|^{\frac{p}{p-1}} + \sum_{2\leqslant|\alpha|\leqslant m-1} |A_\alpha(x,\xi)|^{\frac{\tilde{q}_\alpha}{\tilde{q}_\alpha-1}}$$

$$\leqslant c_1\Big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p + \sum_{2\leqslant|\alpha|\leqslant m-1} |\xi_\alpha|^{\tilde{q}_\alpha}\Big\} + g_1(x)\,, \tag{5.1}$$

$$\sum_{|\alpha|=1,m} A_\alpha(x,\xi)\xi_\alpha \geqslant c_2\Big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p\Big\} - c_3 \sum_{2\leqslant|\alpha|\leqslant m-1} |\xi_\alpha|^{\tilde{q}_\alpha} - g_2(x)\,. \tag{5.2}$$

Let c_4 be a positive constant, and let for every n-dimensional multiindex α, $1 \leqslant |\alpha| \leqslant m-1$, r_α and $\tilde{r}_\alpha$ be numbers such that $1 < \tilde{r}_\alpha \leqslant r_\alpha < rq_\alpha/q$. We shall suppose that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{n,m}$, $\xi \neq \xi'$,

$$\sum_{\alpha\in\Lambda} [A_\alpha(x,\xi) - A_\alpha(x,\xi')](\xi_\alpha - \xi'_\alpha) > -c_4 \sum_{1\leqslant|\alpha|\leqslant m-1} (1+|\xi_\alpha|+|\xi'_\alpha|)^{r_\alpha-\tilde{r}_\alpha}|\xi_\alpha - \xi'_\alpha|^{\tilde{r}_\alpha}\,. \tag{5.3}$$

Let $f \in L^1(\Omega)$. We shall consider the following Dirichlet problem:

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_m u) = f \quad \text{in} \quad \Omega\,, \tag{5.4}$$

$$D^\alpha u = 0, \quad |\alpha| \leqslant m-1, \quad \text{on } \partial\Omega\,. \tag{5.5}$$

DEFINITION 5.1. An entropy solution of problem (5.4), (5.5) is a function $u \in \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$ satisfying the following condition: there exist $c > 0$, $\gamma > 0$ and $\lambda_\alpha \in (1, rq_\alpha/q)$, $1 \leqslant |\alpha| \leqslant m-1$, such that for every $\varphi \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u)(D^\alpha u - D^\alpha \varphi) \Big\} h_k^{(1)}(u-\varphi)\, dx$$

$$\leqslant \int_\Omega f h_k(u-\varphi)\, dx + c\Big(1 + \sum_{1\leqslant|\alpha|\leqslant m-1} \int_\Omega |D^\alpha \varphi|^{\lambda_\alpha} dx\Big) k^{-\gamma}.$$

DEFINITION 5.2. A W-solution of problem (5.4), (5.5) is a function $u \in \overset{\circ}{W}{}^{m,1}(\Omega)$ satisfying the following conditions:

1) for every $\alpha \in \Lambda$, $A_\alpha(x, \nabla_m u) \in L^1(\Omega)$;
2) for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u) D^\alpha \varphi \Big\} dx = \int_\Omega f\varphi\, dx\,.$$

A connection between entropy and weak solutions of the problem under consideration is described by the following result.

THEOREM 5.3. *Let*

$$q > \frac{mp(n-1) + n(m-1)^2}{n-1+(m-1)^2}\,,$$

and let $\tilde q_\alpha \in (1, rq_\alpha/q)$, $2 \leqslant |\alpha| \leqslant m-1$. *Let* u *be an entropy solution of problem* (5.4), (5.5). *Then* u *is a* W*-solution of problem* (5.4), (5.5).

In the next section we shall state results on existence of solutions of the problem under consideration. The question on uniqueness of an entropy solution of this problem is resolved positively (under additional conditions on the coefficients A_α). It can be studied in the same way as an analogous question was investigated in [4]. However, one cannot expect uniqueness of a W-solution of problem (5.4), (5.5).

6. Existence of solutions

THEOREM 6.1. *There exists a* W*-solution of problem* (5.4), (5.5).

THEOREM 6.2. *Let*

$$q > \frac{m(n-1) + n(m-1)^2}{n-1+p(m-1)^2}\, p\,.$$

Then there exists an entropy solution of problem (5.4), (5.5).

Let us consider briefly the scheme of the proof of these theorems.

Let $\{f_l\} \subset L^\infty(\Omega)$ be a sequence such that $\|f_l - f\|_{L^1(\Omega)} \to 0$ and for every $l \in \mathbb{N}$, $\|f_l\|_{L^1(\Omega)} \leqslant \|f\|_{L^1(\Omega)}$. Due to inequalities (5.1)–(5.3), Lemma 3.3 and results of the theory of monotone operators (see, for instance Theorem 1.2 of Chapter II in [2]) we have: if $l \in \mathbb{N}$, then there exists $u_l \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$ such that for every $v \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_m u_l) D^\alpha v \Big\}\, dx = \int_\Omega f_l v\, dx\,. \tag{6.1}$$

A significant part in the proof of the above-stated theorems consists in deriving of some uniform estimates for the functions u_l. We establish such estimates with the use of a sequence of functions of $C^\infty(\mathbb{R})$ which is introduced below. We only note that the role of these functions is analogous to that of the standard truncations T_k in the case of second-order equations with L^1-data (see [1]). However, in our case, where the functions T_k cannot be used, particularities of high-order equations dictate the necessity of presence of some important properties of smooth functions to be used instead of the truncations T_k. Just functions which we define below have necessary properties. In the case of fourth-order equations with L^1-data such a kind of functions has been introduced in [3] and [4].

Let $\psi_0 \in C^\infty(\mathbb{R})$ be a function such that $\psi_0 = 0$ in $(-\infty, 0]$, $\psi_0 = 1$ in $[1, +\infty)$ and ψ_0 is increasing in $[0, 1]$. We define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = 1 - (1 - e^{-s})\psi_0(s)\,, \quad s \in \mathbb{R}\,.$$

We fix $s_0 \in (0, 1)$ such that for every $s \in (0, s_0)$, $\psi(s) \geqslant 1/2$, and for every $i \in \{1, \dots, m\}$ and $s \in (0, s_0)$, $|\psi^{(i)}(s)| \leqslant 1/2$. Define

$$c_0 = 1 + \frac{m!}{\psi_0(s_0)} \Big\{ \sum_{j=1}^{m} \max_{[0,1]} |\psi_0^{(j)}| \Big\}\, e\,.$$

Then for every $i \in \{1, \dots, m\}$ and $s \in (0, +\infty)$ we have $|\psi^{(i)}(s)| \leqslant c_0 \psi(s)$.

Define $\chi : (0, +\infty) \to \mathbb{R}$ by

$$\chi(s) = \int_0^s \psi(t)\, dt\,, \quad s \in (0, +\infty)\,.$$

Now for every $k \in \mathbb{N}$ we define $\chi_k : \mathbb{R} \to \mathbb{R}$ by

$$\chi_k(s) = \begin{cases} s & \text{if } \ |s| \leqslant k\,, \\ \Big[1 + \chi\Big(\dfrac{|s| - k}{k}\Big)\Big]\, k \operatorname{sign} s & \text{if } \ |s| > k\,. \end{cases}$$

For every $k \in \mathbb{N}$ we have $\chi_k \in C^\infty(\mathbb{R})$,

$$|\chi_k| \leqslant 3k \quad \text{in } \mathbb{R}, \tag{6.2}$$

$$0 < \chi_k^{(1)} \leqslant 1 \quad \text{in } \mathbb{R}, \tag{6.3}$$

$$|\chi_k^{(i)}| \leqslant c_0 k^{1-i} \chi_k^{(1)} \quad \text{in } \mathbb{R}, \quad i = 2, \dots, m\,. \tag{6.4}$$

Due to Lemma 3.5 for every $k, l \in \mathbb{N}$ we have $\chi_k(u_l) \in \overset{\circ}{W}{}^{1,q}_{m,p}(\Omega)$. Taking arbitrary $k, l \in \mathbb{N}$ and putting $\chi_k(u_l)$ in (6.1) instead of v with the use of inequalities (5.1), (5.2), Lemmas 3.1, 3.2, 3.4, 3.5 and properties (6.2)–(6.4) we establish the estimate

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} |D^\alpha u_l|^{q_\alpha} \Big\} \chi_k^{(1)}(u_l)\, dx \leqslant c_5 k \tag{6.5}$$

with a positive constant c_5 which does not depend on k and l.

Estimate (6.5) and inequality (3.1) allow us to obtain for every $k, l \in \mathbb{N}$ the following estimates:

$$\operatorname{meas}\{|u_l| \geqslant k\} \leqslant c_6 k^{-r^*},$$
$$\operatorname{meas}\{|D^\alpha u_l| \geqslant k\} \leqslant c_7 k^{-rq_\alpha/q}, \quad \alpha \in \Lambda,$$

where the positive constants c_6 and c_7 do not depend on k and l. By virtue of these esimates we have: for every $\lambda \in (0, r^*)$, the sequence $\{u_l\}$ is bounded in $L^\lambda(\Omega)$; for every $\alpha \in \Lambda$ and $\lambda \in (0, rq_\alpha/q)$ the sequence $\{D^\alpha u_l\}$ is bounded in $L^\lambda(\Omega)$. Due to the last fact and (5.1) we obtain that for every $\alpha \in \Lambda$ and $\lambda \in \big(0, \frac{rq_\alpha}{q(q_\alpha-1)}\big)$ the sequence $\{A_\alpha(x, \nabla_m u_l)\}$ is bounded in $L^\lambda(\Omega)$.

Using the above-stated properties and some estimate for the measure of the sets

$$\Big\{ \sum_{\alpha\in\Lambda} |D^\alpha u_l - D^\alpha u_j| \geqslant t \Big\}, \quad t > 0,$$

we establish existence of an increasing sequence $\{l_i\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{\mathcal{H}}{}^{1,q}_{m,p}(\Omega)$ such that:

$$D^\alpha u_{l_i} \to D^\alpha u \quad \text{a.e. in } \Omega \quad \text{and strongly in } L^1(\Omega), \quad |\alpha| \leqslant m,$$
$$A_\alpha(x, \nabla_m u_{l_i}) \to A_\alpha(x, \nabla_m u) \quad \text{strongly in } L^1(\Omega), \quad \alpha \in \Lambda.$$

On the base of the results obtained we prove that the function u is a W-solution of problem (5.4), (5.5) and, under the condition on q given in the statement of Theorem 6.2, an entropy solution of the same problem.

7. On some generalizations

The above-stated results one can extend in regard to Dirichlet problem for the class of equations of the form (5.4) with coefficients satisfying for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{n,m}$ the inequalities

$$\sum_{1\leqslant|\alpha|\leqslant m} [\nu_\alpha(x)]^{-\frac{1}{p_\alpha-1}} |A_\alpha(x,\xi)|^{\frac{p_\alpha}{p_\alpha-1}} \leqslant c_1 \sum_{1\leqslant|\alpha|\leqslant m} \nu_\alpha(x)|\xi_\alpha|^{p_\alpha} + g_1(x),$$

$$\sum_{|\alpha|=1,m} A_\alpha(x,\xi)\xi_\alpha \geqslant c_2 \sum_{|\alpha|=1,m} \nu_\alpha(x)|\xi_\alpha|^{p_\alpha} - c_3 \sum_{2\leqslant|\alpha|\leqslant m-1} \nu_\alpha(x)|\xi_\alpha|^{p_\alpha} - g_2(x)$$

with given exponents p_α and weighted functions ν_α, $1 \leqslant |\alpha| \leqslant m$, positive constants c_1, c_2, c_3 and non-negative functions $g_1, g_2 \in L^1(\Omega)$.

This class of equations includes as a particular case equations introduced in [9]. Moreover, this class is a natural extension of the class of degenerate anisotropic fourth-order equations described in [5].

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