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ON A FOURIER PROBLEM FOR EVOLUTION DIFFERENTIAL–FUNCTIONAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Introduction.

The boundary value problems with nonlocal boundary conditions have been treated in many works. Individually, Day [1] while studying quasistatic thermoelasticity posed a model involving parabolic equation with nonlocal boundary condition. This model has been expanded into more general problems [1,2]. The results of these papers concern the boundary value problems with an initial condition.

In the current paper, we consider Fourier Problem (the problem without initial conditions) for evolution differential-functional equations with nonlocal boundary conditions. Note that Fourier Problem for strongly non-linear parabolic equations of the second order with nonlocal boundary conditions of the type of periodicity was investigated in [5].

Let us introduce several concepts and symbols we need later. Let D be a domain in the space $\mathbb{R}^{n+1}_{x,t}$. Denote by $C^{\alpha,\alpha/2}(\overline{D})$, where α is a number from interval $[0;1]$, Banach space of real-valued functions, which are continuous in $\overline{D}$, if $\alpha = 0$, and Hölder continuous functions in $\overline{D}$ with exponent α, if $\alpha > 0$ (see definitions in [8], p.16). Denote by $C^{2+\alpha,1+\alpha/2}(\overline{D})$ a subspace of space $C^{\alpha,\alpha/2}(\overline{D})$ which consists of functions w such that $\{w_{x_ix_j}\ (\{i,j\} \subset \{1,...,n\}), w_t\} \subset C^{\alpha,\alpha/2}(\overline{D})$. The norms in these spaces are denoted by $||\cdot||^D_{\alpha,\alpha/2}$ and $||\cdot||^D_{2+\alpha,1+\alpha/2}$, respectively. If D is unbounded domain then denote by $C^{\alpha,\alpha/2}_{\rm loc}(\overline{D})$, $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\overline{D})$ the spaces of functions defined in $\overline{D}$ which restrictions on the closure of any bounded subdomain D' of domain D belong to $C^{\alpha,\alpha/2}(\overline{D'})$ and $C^{2+\alpha,1+\alpha/2}(\overline{D'})$, respectively $(\alpha \in [0;1])$. Set $C(\overline{D}) \stackrel{def}{=} C^{0,0}(\overline{D})$, $C_{\rm loc}(\overline{D}) \stackrel{def}{=} C^{0,0}_{\rm loc}(\overline{D})$. In the case when Q is conjugation of domain D and the part of its boundary, we denote by $C^{\alpha,\alpha/2}_{\rm loc}(Q)$, $C^{2+\alpha,1+\alpha/2}_{\rm loc}(Q)$ the spaces of functions which restrictions on closure of arbitrary bounded subdomain D' of domain D such that $\overline{D'} \subset Q$, belong to spaces $C^{\alpha,\alpha/2}(\overline{D'})$ and $C^{2+\alpha,1+\alpha/2}(\overline{D'})$, respectively $(\alpha \in [0;1])$.

1. Statement of the problem and formulation of main results.

Let $Q = \Omega \times (-\infty, T]$, $0 < T < +\infty$, Ω be a bounded domain in $\mathbb{R}^n_x$ with smooth boundary $\partial\Omega$, $\Sigma = \partial\Omega \times (-\infty, T]$.

We consider a problem

$$Pu(x,t) \equiv \frac{\partial u(x,t)}{\partial t} - Lu(x,t) + a(x,t)u(x,t)-$$

$$-f(x,t,u(x,t);u(\cdot,t)) = \hat{f}(x,t), \quad (x,t) \in Q, \tag{1}$$

$$Bu(x,t) \equiv u(x,t) - g(x,t,u(x,t);u(\cdot,t)) = h(x,t), \quad (x,t) \in \Sigma, \tag{2}$$

where

$$Lu(x,t) \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial u(x,t)}{\partial x_i};$$

$f(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q\times\mathbb{R}$, and $g(x,t,\eta;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\eta) \in \Sigma\times\mathbb{R}$, are families of functionals.

Henceforth this problem is called Problem (1),(2).

We impose the following main conditions on the data-in:

(A1) functions a_{ij}, a_i, a are continuous in Q, $\{i,j\} \subset \{1,...,n\}$;

(A2) $a_{ij} = a_{ji}$, $\{i,j\} \subset \{1,...,n\}$, and for arbitrary point $(x,t) \in Q$ and for all $\xi = (\xi_1,...,\xi_n) \in \mathbb{R}^n$ the following inequality holds

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqslant \mu(t)\sum_{s=1}^{n}\xi_s^2,$$

where $\mu(t) \geqslant 0, \quad t \in (-\infty, T]$;

(A3) for all $v \in C(\overline{\Omega})$ functions $f(x,t,\xi;v(\cdot))$, $(x,t,\xi) \in Q\times\mathbb{R}$, $g(x,t,\xi;v(\cdot))$, $(x,t,\xi) \in \Sigma\times\mathbb{R}$, are continuous, nondecreasing in ξ, i.e. for arbitrary $\{\xi^1,\xi^2\} \subset \mathbb{R}$, such that $\xi^1 \geqslant \xi^2$, the following inequalities hold:

$$f(x,t,\xi^1;v(\cdot)) - f(x,t,\xi^2;v(\cdot)) \geqslant 0, \qquad (x,t) \in Q,$$

$$g(x,t,\xi^1;v(\cdot)) - g(x,t,\xi^2;v(\cdot)) \geqslant 0, \qquad (x,t) \in \Sigma;$$

moreover, these functions are Lipschitz in ξ, more precisely, there exist functions $L^f(x,t)$, $(x,t) \in Q$, and $L^g(x,t)$, $(x,t) \in \Sigma$, such that for arbitrary $\{\xi^1,\xi^2\} \subset \mathbb{R}$ and $v \in C(\overline{\Omega})$

$$|f(x,t,\xi^1;v(\cdot)) - f(x,t,\xi^2;v(\cdot))| \leqslant L^f(x,t)|\xi^1-\xi^2|, \quad (x,t) \in Q,$$

$$|g(x,t,\xi^1;v(\cdot)) - g(x,t,\xi^2;v(\cdot))| \leqslant L^g(x,t)|\xi^1-\xi^2|, \quad (x,t) \in \Sigma;$$

(A4) functionals $f(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q\times\mathbb{R}$, and $g(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in \Sigma\times\mathbb{R}$, are G â teaux differentiable, more precisely, for arbitrary $v \in C(\overline{\Omega})$ there exist linear and continuous functionals $f'_c(x,t,\xi;v(\cdot),\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q\times\mathbb{R}$, and $g'_c(x,t,\xi;v(\cdot),\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in \Sigma\times\mathbb{R}$, such that

$$\frac{d}{ds}f(x,t,\xi;v(\cdot)+sh(\cdot))\big|_{s=0} = f'_c(x,t,\xi;v(\cdot),h(\cdot)), \quad (x,t,\xi) \in Q\times\mathbb{R},$$

$$\frac{d}{ds}g(x,t,\xi;v(\cdot)+sh(\cdot))\big|_{s=0}= g'_c(x,t,\xi;v(\cdot),h(\cdot)), \quad (x,t,\xi)\in\Sigma\times\mathbb{R},$$

for all $h\in C(\overline{\Omega})$; the following inequalities hold:

$$f'_c(x,t,\xi;v(\cdot),h(\cdot))\geqslant 0, \quad g'_c(x,t,\xi;v(\cdot),h(\cdot))\geqslant 0, \quad \text{if} \quad h\geqslant 0;$$

moreover, assume that

$$||f'_c(x,t,\xi;v(\cdot),h(\cdot))||\leqslant K^f(x,t)||h||_{C(\overline{\Omega})}, \quad (x,t,\xi)\in Q\times\mathbb{R}, \quad v\in C(\overline{\Omega}),$$

$$||g'_c(x,t,\xi;v(\cdot),h(\cdot))||\leqslant K^g(x,t)||h||_{C(\overline{\Omega})}, \quad (x,t,\xi)\in \Sigma\times\mathbb{R}, \quad v\in C(\overline{\Omega}),$$

where $K^f(x,t),\ (x,t)\in Q$, $K^g(x,t),\ (x,t)\in\Sigma$, are some functions;

(A5)

$$\inf_{x\in\Omega}\big(a(x,t)-F(x,t)\big)\geqslant a_0(t), \qquad t\in(-\infty,T],$$

where $F(x,t)\overset{\text{def}}{=}K^f(x,t)+L^f(x,t),\quad (x,t)\in Q$, and a_0 is continuous in $(-\infty,T]$ function;

(A6)

$$G(x,t)<1, \quad (x,t)\in\Sigma,$$

where $G(x,t)\overset{\text{def}}{=}K^g(x,t)+L^g(x,t), \quad (x,t)\in\Sigma;$

(A7) $\hat{f}\in C_{\text{loc}}(Q),\ h\in C_{\text{loc}}(\Sigma)$.

For the convenience of formulating and proving the results, without loss of generality let us make additional assumption

(A0) $f(x,t,0,0)=0,\ (x,t)\in Q, \quad g(x,t,0,0)=0,\ (x,t)\in\Sigma.$

In the sequel we assume that conditions (A0)-(A7) hold.

DEFFINITION 1. *A function $u\in C^{2,1}_{\text{loc}}(Q)\cap C_{\text{loc}}(\overline{Q})$ is called a solution of Problem (1),(2) if it satisfies equation (1) and boundary condition (2).*

Before formulating of main results of the work we introduce some notations and concepts.

Let b be an arbitrary continuous in $(-\infty,T]$ function. We denote by $V(b)$ a set of continuous differentiable in $(-\infty,T]$ functions ν which satisfy

$$\nu'(t)<b(t), \quad t\in(-\infty,T], \qquad \int_{-\infty}^{T}(b(t)-\nu'(t))\,dt=+\infty.$$

Let us note that when $b(t)=b_0,\ t\in(-\infty,T]$, where $b_0=\text{const}$, then the set of functions $\{ct,\ t\in(-\infty,T]: c\in\mathbb{R},\ c<b_0\}$ is subset of $V(b)$.

Let H be one of the sets Q, $\overline{Q}$ or Σ, and $\nu\in V(a_0)$. Denote by $E_\nu(H)$ a set of continuous functions $q(x,t),\ (x,t)\in H$, which satisfy an inequality

$$|q(x,t)|\leqslant Ke^{-\nu(t)}, \quad (x,t)\in H,$$

where $K \geqslant 0$ is a constant which may depend on q.

We denote by $\Phi^{\alpha,\alpha/2}_{x,t;\mathrm{loc}}$, where $\alpha \in (0;1]$, a space of families of functionals $f(x,t,\xi;\cdot): C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q \times \mathbb{R}$, such that for arbitrary numbers $t_0 < T$, $l_1 > 0$, $l_2 > 0$ there exists a constant $K \geqslant 0$ which satisfies an inequality

$$|f(x^1,t^1,\xi;v(\cdot)) - f(x^2,t^2,\xi;v(\cdot))| \leqslant K\big[|x^1 - x^2|^\alpha + |t^1 - t^2|^{\alpha/2}\big]$$

for arbitrary $\{(x^1,t^1),\ (x^2,t^2)\} \subset \Omega \times (t_0,T]$, and any $\xi \in [-l_1,l_1]$ and $v \in C(\overline{\Omega})$, $||v||_{C(\overline{\Omega})} \leqslant l_2$.

THEOREM 1 (AN APRIORI ESTIMATE OF THE SOLUTION). *Let $\hat{f}/(a_0 - \nu') \in E_\nu(Q)$ and $h/(1-G) \in E_\nu(\Sigma)$ for some ν from $V(a_0)$. Then the solution of Problem (1),(2) u from $E_\nu(\overline{Q})$ satisfies the following estimate:*

$$|u(x,t)| \leqslant \max\Big\{ \sup_{(y,\tau)\in\Sigma} \frac{|h(y,\tau)|e^{\nu(\tau)}}{1 - G(y,\tau)},\ \sup_{(y,\tau)\in Q} \frac{|\hat{f}(y,\tau)|e^{\nu(\tau)}}{a_0(\tau) - \nu'(\tau)}\Big\} \cdot e^{-\nu(t)} \equiv M_0 e^{-\nu(t)} \qquad (3)$$

for all $(x,t) \in \overline{Q}$.

THEOREM 2 (UNIQUENESS OF THE SOLUTION). *The solution of Problem (1),(2) from class $E_\nu(\overline{Q})$, where $\nu \in V(a_0)$, is unique.*

For all $k \in \mathbb{N}$ let us denote by $Q^k = Q \cap \{(x,t) : t > -k\}$, $\Sigma^k = \Sigma \cap \{(x,t) : t > -k\}$ and define a function $u_k \in C^{2,1}_{\mathrm{loc}}(Q^k) \cap C(\overline{Q^k})$, as the solution of Problem:

$$P_k u_k(x,t) = \hat{f}_k(x,t), \qquad (x,t) \in Q^k, \qquad (1_k)$$

$$B_k u_k(x,t) = h_k(x,t), \qquad (x,t) \in \Sigma^k, \qquad (2_k)$$

$$u_k(x,-k) = 0, \qquad x \in \overline{\Omega}. \qquad (3_k)$$

Here

$$P_k w(x,t) \equiv \frac{\partial w(x,t)}{\partial t} - Lw(x,t) + a(x,t)w(x,t) - f_k(x,t,w(x,t);w(\cdot,t))$$

and $B_k w(x,t) \equiv w(x,t) - g_k(x,t,w(x,t);w(\cdot,t))$ $\forall w \in C^{2,1}(\overline{Q})$, $f_k(x,t,\xi;\cdot) = \zeta(t+k)f(x,t,\xi;\cdot)$, $(x,t,\xi) \in Q \times \mathbb{R}$, $g_k(x,t,\xi;\cdot) = \zeta(t+k)g(x,t,\xi;\cdot)$, $(x,t,\xi) \in \Sigma \times \mathbb{R}$, $h_k(x,t) = \zeta(t+k)h(x,t)$, $(x,t) \in \Sigma$, $\hat{f}_k(x,t) = \hat{f}(x,t)\zeta(t+k)$, $(x,t) \in \overline{Q}$, where ζ is smooth and monotonic in $\mathbb{R}$ function such that $\zeta(t) = 0$ if $t \leqslant 1/2$, $\zeta(t) = 1$ if $t \geqslant 1$, $k \in \mathbb{N}$.

THEOREM 3 (EXISTENCE OF THE SOLUTION). *Let the following conditions hold for some $\alpha \in (0;1]$ and $\nu \in V(a_0)$:*

(B1) $\{a_{ij}, a_i, a\} \subset C^{\alpha,\alpha/2}_{\mathrm{loc}}(\overline{Q})$, $\partial a_{ij}/\partial x_s \in C_{\mathrm{loc}}(\overline{Q})$, $\{i,j,s\} \subset \{1,...,n\}$; $\mu(t) > 0$, $t \in (-\infty,T]$;

(B2) $f \in \Phi^{\alpha,\alpha/2}_{x,t;\mathrm{loc}}$;

(B3) $\hat{f} \in C^{\alpha,\alpha/2}_{\mathrm{loc}}(\overline{Q})$, $\hat{f}/(a_0 - \nu') \in E_\nu(\overline{Q})$; $h \in C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\overline{Q})$, $h/(1-G) \in E_\nu(\Sigma)$;

(B4) for all $k \in \mathbb{N}$ Problem $(1_k)-(3_k)$ has a solution $u_k \in C^{2+\alpha,1+\alpha/2}(\overline{Q^k})$.

Then Problem (1),(2) has a solution u in $E_\nu(\overline{Q}) \cap C^{\alpha,\alpha/2}_{\mathrm{loc}}(\overline{Q}) \cap C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(Q)$.

COROLLARY. *Let conditions (B1)-(B3) of theorem 3 hold, moreover, the following conditions are true: (C1) there exist a family of functionals* $g^*(x,t,\xi;\cdot): C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi;\cdot) \in \overline{Q}\times\mathbb{R}$, *and a function* $\eta \in C_0^\infty(\Omega)$, $0 \leqslant \eta \leqslant 1$, *such that* $g(x,t,\xi;\cdot) = g^*(x,t,\xi;\cdot)$, $(x,t,\xi) \in \Sigma\times\mathbb{R}$, *and for arbitrary* $(x,t) \in \overline{Q}$ *and* $v \in C(\overline{\Omega})$ $g^*(x,t,0;(\eta v)(\cdot)) = g^*(x,t,v(x);v(\cdot))$, *moreover,* $\tilde{g}(x,t) \stackrel{\text{def}}{=} g^*(x,t,0;(\eta w)(\cdot,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ *for any* $w \in C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\overline{Q})$;

(C2) $\partial\Omega \in C^{2+\alpha}$.

Then the statement of Theorem 3 is true.

REMARK 1. *The functionals* $g(x,t,\xi;\cdot): C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in \Sigma\times\mathbb{R}$, *from Bitsadze-Samarskii boundary condition*

$$g(x,t,\xi;v(\cdot)) = \sum_{k=1}^{K} g_k(x,t)v(\xi_k(x)), \quad (x,t,\xi) \in \Sigma\times\mathbb{R}, \ v \in C(\overline{\Omega}),$$

satisfy the condition (C1) of corollary if $K \in \mathbb{N}$, $g_k \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$, $g_k(x,t) \geqslant 0$, $(x,t) \in \Sigma$, $k \in \{1,...,K\}$, *and functions* ξ_k, $k \in \{1,...,K\}$, *are defined in* $\overline{\Omega}$ *with values in* $\overline{\Omega_\delta} = \{x \in \Omega : dist(x,\partial\Omega) \geqslant \delta\}$, $\delta > 0$, $\xi_k \in C^{2+\alpha}(\Omega)$.

Let Π_ν be a space of ordered pairs of functons $(\hat{f},h)$ such that $\hat{f} \in C^{\alpha,\alpha/2}_{\text{loc}}(\overline{Q})$, $\hat{f}/(a_0-\nu') \in E_\nu(\overline{Q})$, $h \in C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\overline{Q})$, $h/(1-G) \in E_\nu(\Sigma)$ for $\nu \in V(a_0)$. Let us assume that conditions (B1),(B2),(B4) are fulfilled. Then there exists unique solution of Problem (1),(2) in $E_\nu(\overline{Q})$ for arbitrary $(\hat{f},h) \in \Pi_\nu$, where $\nu \in V(a_0)$. In short, we write this as $u = NZ_\nu(\hat{f},h)$.

THEOREM 4 (CONTINUOUS DEPENDENCE ON DATA-IN). *Let conditions (B1),(B2),(B4) of Theorem 3 hold. Then for arbitrary* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $(\hat{f}_1,h_1)$, $(\hat{f}_2,h_2) \in \Pi_\nu$, *satisfying conditions:*

$$\sup_{(x,t)\in Q} \frac{|\hat{f}_1(x,t)-\hat{f}_2(x,t)|e^{\nu(t)}}{a_0(t)-\nu'(t)} < \delta \quad \text{and} \quad \sup_{(x,t)\in\Sigma} \frac{|h_1(x,t)-h_2(x,t)|e^{\nu(t)}}{1-G(x,t)} < \delta$$

the following inequality holds:

$$\sup_{(x,t)\in Q} |u_1(x,t)-u_2(x,t)|e^{\nu(t)} < \varepsilon,$$

where $u_i = NZ_\nu(\hat{f}_i,h_i)$, $i \in \{1,2\}$.

2. AUXILIARY STATEMENTS

Let t_0 be an arbitrary fixed number in $(-\infty,T)$. Set $Q_0 = \Omega\times(t_0,T]$ and $\Sigma_0 = \partial\Omega\times(t_0,T]$ where $0 < T < +\infty$.

LEMMA 1. *Let functions* $\tilde{u},\hat{u} \in C^{2,1}(Q_0)\cap C(\overline{Q_0})$ *fulfil the inequalities*

$$P\tilde{u}(x,t) < P\hat{u}(x,t), \quad (x,t) \in Q_0,$$

$$B\tilde{u}(x,t) < B\hat{u}(x,t), \quad (x,t) \in \Sigma_0, \qquad \tilde{u}(x,t_0) < \hat{u}(x,t_0), \quad x \in \overline{\Omega}. \tag{4}$$

Then $\tilde{u}(x,t) < \hat{u}(x,t)$ *for all* $(x,t) \in \overline{Q_0}$.

Proof. Let us assume a contradiction. Let $t^* \in (t_0, T]$ be a maximum value of variable t such that $\tilde{u}(x,t) < \hat{u}(x,t)$ for all $(x,t) \in \overline{Q_0} \cap \{(x,t) : t_0 \leqslant t < t^*\}$. Then there exists a point $x^* \in \overline{\Omega}$ such that $\tilde{u}(x^*,t^*) = \hat{u}(x^*,t^*)$.

It follows from the third inequality of (4) that $t^* > t_0$. Using (5) we next show that $(x^*,t^*) \notin \Sigma_0$. Assuming that $(x^*,t^*) \in \Sigma_0$ then for the function $w \stackrel{\text{def}}{=} \tilde{u} - \hat{u}$ by the second inequality of (4) and condition (A4) we have

$$0 = w(x^*,t^*) = g(x^*,t^*,\tilde{u}(x^*,t^*);\tilde{u}(\cdot,t^*)) - g(x^*,t^*,\hat{u}(x^*,t^*);\hat{u}(\cdot,t^*)) +$$

$$+ B\tilde{u}(x^*,t^*) - B\hat{u}(x^*,t^*) < 0,$$

but this contadicts the statement of lemma.

It leads to the fact that the function w in $\overline{Q_0} \cap \{(x,t) : t_0 \leqslant t \leqslant t^*\}$ takes a maximum value at the point $(x^*,t^*) \in Q_0$ and it equals zero. Thus, using condition (A2) we have

$$\partial w(x^*,t^*)/\partial t \geqslant 0, \ \partial w(x^*,t^*)/\partial x_s = 0, \ \sum_{i,j=1}^{n} a_{ij}(x^*,t^*)\partial^2 w(x^*,t^*)/\partial x_i \partial x_j \leqslant 0.$$

Hence and by condition (A4) it follows

$$P\tilde{u}(x^*,t^*) - P\hat{u}(x^*,t^*) \geqslant 0,$$

but this contradicts the third inequality of (4). □

LEMMA 2. *Assume that all conditions of Lemma 1 are fulfilled and the inequalities (4) are nonstrict. Then* $\tilde{u}(x,t) \leqslant \hat{u}(x,t)$ *for all* $(x,t) \in \overline{Q_0}$.

Proof. Let us consider an auxiliary function $\hat{u}_\lambda(x,t) = \hat{u}(x,t) + \lambda e^{m^* t}, (x,t) \in \overline{Q_0}$, where $\lambda > 0$, $m^* + a_0(t) > 0$, $t \in [0,T]$. Using conditions (A3)-(A5) we obtain

$$P\hat{u}_\lambda(x,t) = P\hat{u}(x,t) + \lambda e^{m^* t}(m^* + a(x,t)) - [f(x,t,\hat{u}(x,t) + \lambda e^{m^* t}; \hat{u}(\cdot,t) + \lambda e^{m^* t}) -$$

$$- f(x,t,\hat{u}(x,t);\hat{u}(\cdot,t))] > P\hat{u}(x,t), \ (x,t) \in Q_0.$$

Since $P\tilde{u}(x,t) \leqslant P\hat{u}(x,t)$, and $P\hat{u}(x,t) < P\hat{u}_\lambda(x,t)$, then $P\tilde{u}(x,t) < P\hat{u}_\lambda(x,t)$, $(x,t) \in Q_0$. It follows from conditions (A3), (A4) and (A6) that

$$B\hat{u}_\lambda(x,t) \geqslant B\hat{u}(x,t) + \lambda e^{m^* t}(1 - G(x,t)) > B\hat{u}(x,t), \quad (x,t) \in \Sigma_0.$$

Hence, using the fact that $\tilde{u}(x,t_0) < \hat{u}_\lambda(x,t_0)$, $x \in \overline{\Omega}$, in view of Lemma 1 we have $\tilde{u}(x,t) < \hat{u}_\lambda(x,t)$, if $(x,t) \in \overline{Q_0}$, $\lambda > 0$. Since $\lim\limits_{\lambda \to 0+} \hat{u}_\lambda(x,t) = \hat{u}(x,t)$, then $\tilde{u}(x,t) \leqslant \hat{u}(x,t)$, $(x,t) \in \overline{Q_0}$. □

LEMMA 3. *Let* $a_0(t) > 0$, $t \in [t_0,T]$. *Then an arbitrary function* $u \in C(\overline{Q_0}) \cap C^{2,1}(Q_0)$ *such that* Pu *is the bounded in* Q_0 *function, fulfil an estimate*

$$|u(x,t)| \leqslant \max\{\max_{y \in \overline{\Omega}} |u(y,t_0)|, \sup_{(y,\tau) \in \Sigma_0} \frac{|Bu(y,\tau)|}{1 - G(y,\tau)}, \sup_{(y,\tau) \in Q_0} \frac{|Pu(y,\tau)|}{a_0(\tau)}\}, (x,t) \in \overline{Q_0}. \tag{5}$$

Proof. Let $C \stackrel{\text{def}}{=} \max\{\max_{y\in\overline{\Omega}} |u(y,t_0)|, \sup_{(y,\tau)\in\Sigma_0} \frac{|Bu(y,\tau)|}{1-G(y,\tau)}, \sup_{(y,\tau)\in Q_0} \frac{|Pu(y,\tau)|}{a_0(\tau)}\}$. Consider a function $\hat{u} \equiv C$. It follows from conditions (A3)-(A5) and the choice of C that

$$P\hat{u}(x,t) \geqslant C(a(x,t) - F(x,t)) \geqslant C \cdot a_0(t) \geqslant Pu(x,t), \quad (x,t) \in Q_0. \tag{6}$$

Using condition (A6), in view of definition of function $\hat{u}$ we obtain

$$B\hat{u}(x,t) \geqslant C(1 - G(x,t)) \geqslant Bu(x,t), \quad (x,t) \in \Sigma_0. \tag{7}$$

It is obvious that $\hat{u}(x,t_0) \geqslant u(x,t_0)$, $x \in \overline{\Omega}$. Hence and by (6),(7) and Lemma 2 we obtain $\hat{u}(x,t) \geqslant u(x,t)$, $(x,t) \in \overline{Q_0}$. It can be shown analogously that $u(x,t) \geqslant -\hat{u}(x,t)$, $(x,t) \in \overline{Q_0}$. □

LEMMA 4. *Let $a_0(t) > 0$, $t \in [t_0,T]$. Then arbitrary functions $\{u_1,u_2\} \subset C(\overline{Q_0}) \cap C^{2,1}(Q_0)$ such that Pu_1, Pu_2 are bounded in Q_0 functions, fulfil an inequality*

$$|u_1(x,t) - u_2(x,t)| \leqslant \max\ \{\ ,\max_{y\in\Omega} |u_1(y,t_0) - u_2(y,t_0)|,$$

$$\max_{(y,\tau)\in\Sigma_0} \frac{|Bu_1(y,\tau) - Bu_2(y,\tau)|}{1 - G(y,\tau)},\ \max_{(y,\tau)\in Q_0} \frac{|Pu_1(y,\tau) - Pu_2(y,\tau)|}{a_0(\tau)}\},\ (x,t) \in \overline{Q_0}.$$

Proof. Let u_1, u_2 be the same as in the formulation of lemma. Then

$$f(x,t,u_1(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_2(\cdot,t)) = f(x,t,u_1(x,t);u_1(dot,t)) -$$

$$-f(x,t,u_2(x,t);u_1(\cdot,t)) + f(x,t,u_2(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_2(\cdot,t)).$$

We have

$$f(x,t,u_1(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_1(\cdot,t)) = f_*(x,t)u_{1,2}(x,t),$$

where $f_*(x,t) = \frac{f(x,t,u_1(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_1(\cdot,t))}{u_1(x,t)-u_2(x,t)}$, if $u_1(x,t) \neq u_2(x,t)$, otherwise $f_*(x,t) = 0$; $u_{1,2}(x,t) \stackrel{\text{def}}{=} u_1(x,t) - u_2(x,t), (x,t) \in \overline{Q_0}$.

By condition (A4) we obtain

$$f(x,t,u_2(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_2(\cdot,t)) = f_{**}(x,t;u_{1,2}(\cdot,t)),$$

where $f_{**}(x,t;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t) \in Q$, is a family of linear continuous and nondecreasing functionals.

The preceding facts lead to

$$f(x,t,u_1(x,t);u_1(\cdot,t)) - f(x,t,u_2(x,t);u_2(\cdot,t)) = \tilde{f}(x,t,u_{1,2}(x,t);u_{1,2}(\cdot,t)),$$

where $\tilde{f}(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q \times \mathbb{R}$, is a family of functionals with the same properties as the family $f(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in Q \times \mathbb{R}$.

Analogously

$$g(x,t,u_1(x,t);u_1(\cdot,t)) - g(x,t,u_2(x,t);u_2(\cdot,t)) = \tilde{g}(x,t,u_{1,2}(x,t);u_{1,2}(\cdot,t)),$$

where $\tilde{g}(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in \Sigma \times \mathbb{R}$, is a family of functionals with the same properties as $g(x,t,\xi;\cdot) : C(\overline{\Omega}) \to \mathbb{R}$, $(x,t,\xi) \in \Sigma \times \mathbb{R}$.

Denote $\hat{f}_k(x,t) \overset{def}{=} Pu_k(x,t)$, $(x,t) \in Q_0$, $k \in \{1,2\}$; $h_k(x,t) = Bu_k(x,t)$, $(x,t) \in \Sigma_0$, $k \in \{1,2\}$. Subtracting from equation (1) for u_1 the same equation for u_2 we obtain

$$\tilde{P}u_{1,2}(x,t) \equiv \frac{\partial u_{1,2}(x,t)}{\partial t} - Lu_{1,2}(x,t) + a(x,t)u_{1,2}(x,t)-$$

$$-\tilde{f}(x,t,u_{1,2}(x,t);u_{1,2}(\cdot,t)) = \cdot,t))] = \hat{f}_{1,2}(x,t), \quad (x,t) \in Q_0, \tag{8}$$

where $\hat{f}_{1,2}(x,t) \overset{\text{def}}{=} \hat{f}_1(x,t) - \hat{f}_2(x,t)$, $(x,t) \in Q_0$.

Analogously

$$\tilde{B}u_{1,2}(x,t) \equiv u_{1,2}(x,t) - \tilde{g}(x,t,u_{1,2}(x,t);u_{1,2}(\cdot,t)) = h_{1,2}(x,t), \ (x,t) \in \Sigma_0, \tag{9}$$

$$u_{1,2}(x,t_0) = u^1(x,t_0) - u^2(x,t_0), \quad x \in \Omega, \tag{10}$$

where $h_{1,2}(x,t) \overset{\text{def}}{=} h_1(x,t) - h_2(x,t)$, $(x,t) \in \Sigma_0$.

Using Lemma 3 and (8)-(10), we complete the proof . □

LEMMA 5. *Let $\nu \in V(a_0)$ and functions $\{u_1, u_2\} \subset C^{2,1}_{\text{loc}}(Q) \cap C_{\text{loc}}(\overline{Q})$ are such that $u_1 - u_2 \in E_\nu(\overline{Q})$, $(Pu_1 - Pu_2)/(a_0 - \nu') \in E_\nu(Q)$, $(Bu_1 - Bu_2)/(1-G) \in E_\nu(\Sigma)$. Then*

$$|u_1(x,t) - u_2(x,t)| \leqslant \max\{ \sup_{(y,\tau)\in\Sigma} \frac{|Bu_1(y,\tau) - Bu_2(y,\tau)|e^{\nu(\tau)}}{1 - G(y,\tau)},$$

$$\sup_{(y,\tau)\in Q} \frac{|Pu_1(y,\tau) - Pu_2(y,\tau)|e^{\nu(\tau)}}{a_0(\tau) - \nu'(\tau)} \}e^{-\nu(t)}, \quad (x,t) \in \overline{Q}. \tag{11}$$

Proof. First we consider the case $\nu = 0 \in V(a_0)$ and use the ideas of paper [7]. Denote $\hat{f}_k(x,t) \overset{def}{=} Pu_k(x,t)$, $(x,t) \in Q$, $k \in \{1,2\}$; $h_k(x,t) = Bu_k(x,t)$, $(x,t) \in \Sigma$, $k \in \{1,2\}$. Let $\lambda(t) = \int\limits_T^t a_0(\tau)\,d\tau$, $\tau \in (-\infty, T]$, and $\gamma \in (0;1)$. Let us multiply equation (1) and condition (2) for u_1 and u_2 by $e^{\gamma\lambda(t)}$. After simple transformations we obtain

$$P^\gamma \tilde{u}_{\gamma,k}(x,t) \equiv \frac{\partial \tilde{u}_{\gamma,k}(x,t)}{\partial t} - L\tilde{u}_{\gamma,k}(x,t) + \big(a(x,t) - \gamma a_0(t)\big)\tilde{u}_{\gamma,k}(x,t)-$$

$$- f(x,t,\tilde{u}_{\gamma,k}(x,t)e^{-\gamma\lambda(t)};\tilde{u}_{\gamma,k}(\cdot,t)e^{-\gamma\lambda(t)})e^{\gamma\lambda(t)} = \hat{f}_k(x,t)e^{\gamma\lambda(t)}, \ (x,t) \in Q, \tag{12}$$

$$B^\gamma \tilde{u}_{\gamma,k}(x,t) \equiv \tilde{u}_{\gamma,k}(x,t) - g(x,t,\tilde{u}_{\gamma,k}(x,t)e^{-\gamma\lambda(t)};\tilde{u}_{\gamma,k}(\cdot,t)e^{-\gamma\lambda(t)})e^{\gamma\lambda(t)} =$$

$$= h_{1,2}(x,t)e^{\gamma\lambda(t)}, \quad (x,t) \in \Sigma, \quad k \in \{1,2\}, \tag{13}$$

where $\tilde{u}_{\gamma,k}(x,t) = u_k(x,t)e^{\gamma\lambda(t)}$, $(x,t) \in Q$, $k \in \{1,2\}$.

Let t_* be an arbitrary negative number, $Q_* = \Omega \times (t_*, T]$, $\Sigma_* = \partial\Omega \times (t_*, T]$. It is easy to see that coefficients of the differential operators P^γ and B^γ fulfil the conditions similar to conditions (A0)-(A7) for the coefficients of the operators P and B with $a_0(1-\gamma)$ in place of a_0. Thus, in view of Lemma 4, by (12) and (13) we obtain

$$|\tilde{u}_{\gamma,1,2}(x,t)| \leqslant \max\{e^{\gamma\lambda(t_*)}\cdot \max_{y\in\overline{\Omega}} |u_{1,2}(y,t_*)|, e^{\gamma\lambda(T)}\cdot \sup_{(y,\tau)\in\ Sigma_*} \frac{|h_{1,2}(y,\tau)|}{1-G(y,\tau)},$$

$$\frac{e^{\gamma\lambda(T)}}{1-\gamma}\cdot \sup_{(y,\tau)\in Q_*} \frac{|\hat{f}_{1,2}(y,\tau)|}{a_0(\tau)}\}, \quad (x,t)\in\overline{Q_*}, \quad (14)$$

where $\tilde{u}_{\gamma,1,2} \stackrel{\text{def}}{=} \tilde{u}_{\gamma,1}(x,t)-\tilde{u}_{\gamma,2}(x,t)$, $(x,t)\in\overline{Q}$; $\hat{f}_{1,2}(x,t) \stackrel{\text{def}}{=} \hat{f}_1(x,t)-\hat{f}_2(x,t), (x,t)\in Q$; $\quad h_{1,2}(x,t) \stackrel{\text{def}}{=} h_1(x,t)-h_2(x,t)$, $(x,t)\in\Sigma$.

Since $u_1-u_2\in E_0(\overline{Q})$ then $|u_{1,2}(x,t)|\leqslant C_1$ for all $(x,t)\in\overline{Q}$, where $C_1\geqslant 0$ is a constant. Because $e^{\gamma\lambda(t_*)}\to 0$ as $t_*\to-\infty$ implies $e^{\gamma\lambda(t_*)}\cdot\max_{y\in\overline{\Omega}}|u_{1,2}(y,t_*)|\to 0$ as $t_*\to-\infty$. Hence, letting first $t_*\to-\infty$ in (14) and after letting $\gamma\to 0+$ we obtain (11) for $\nu=0$.

Let $\nu\in V(a_0), \nu\neq 0$. Let us multiply equation (1) and condition (2) for u_1 and u_2 by $e^{\nu(t)}$. After simple transformations we obtain (see (18),(19))

$$P_\nu\hat{u}_k(x,t)=\hat{f}_k(x,t)e^{\nu(t)}, \quad (x,t)\in Q, \quad k\in\{1,2\},$$

$$B_\nu\hat{u}_k(x,t)=h_k(x,t)e^{\nu(t)}, \quad (x,t)\in\Sigma, \quad k\in\{1,2\},$$

where $\hat{u}_k(x,t) = u_k(x,t)e^{\nu(t)}$, $(x,t)\in\overline{Q}$, $k\in\{1,2\}$; $P_\nu v(x,t)\equiv \partial v(x,t)/\partial t - Lv(x,t)+\big(a(x,t)-\nu'(t)\big)v(x,t)-f(x,t,v(x,t)e^{-\nu(t)};\ v(\cdot,t)e^{-\nu(t)})e^{\ nu(t)}$, $(x,t)\in Q$, $B_\nu v(x,t)\equiv v(x,t)-g(x,t,v(x,t)e^{-\nu(t)};\ v(\cdot,t)e^{-\nu(t)})e^{\nu(t)}$, $(x,t)\in\Sigma$, $v\in C^{2,1}_{\text{loc}}(Q)$. It is easy to see that coefficients of the differential operators P_ν and B_ν fulfill the conditions analogous to conditions (A1)-(A6) for the coefficients of the operators P and B with $a_0-\nu'$ in place of a_0. It is obvious that $\hat{u}_{1,2}\in E_0(\overline{Q})$. Hence, based on the proof of lemma for $\nu=0$ we complete the proof. □

3. Proof of basic results

Proof of Theorem 1. To obtain an apriori estimate of the solution of Problem (1),(2) it suffices to set $u_1=u$, $u_2=0$ and use Lemma 5.

Proof of Theorem 2 As a consequence of Lemma 5, we have the uniqueness of the solution.

Proof of Theorem 3. Consider first the case $\nu=0\in V(a_0)$. Let for all $k\in\mathbb{N}$ u_k be a solution of Problem $(1_k)-(3_k)$.

Based on Lemma 4 we have $u_k(x,t)=0$ for all $(x,t)\in\overline{\Omega}\times[-k,-k+1/2]$. Let us extend the function u_k by zero on $\overline{Q}\setminus\overline{Q^k}$ and denote these extensions again by u_k ($k\in\mathbb{N}$). It is obvious that $u_k\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ and u_k is a solution of Problem (1),(2) with $f_k, \hat{f}_k, g_k$, h_k in place of $f,\hat{f},g,h$, respectively, we denote this by $u_k = NZ^k_0(\hat{f}_k,h_k)$, $k\in\{1,2,...\}$. In view of Theorem 1 we have

$$|u_k(x,t)|\leqslant M_0, \quad (x,t)\in\overline{Q}, \quad k\in\mathbb{N}. \quad (15)$$

We show that the restrictions of terms of sequence $\{u_k\}$ on the closure $\overline{Q'}$ of an arbitrary bounded subdomain Q' of domain Q is the fundamental sequence in $C(\overline{Q'})$.

Let for arbitrary $\{k,l\} \subset \mathbb{N}$ $u_k = NZ_0^k(\hat{f}_k, h_k)$, $u_l = NZ_0^k(\hat{f}_l, h_l)$. Set $\lambda(t) \stackrel{\text{def}}{=} \int\limits_T^t a_0(\tau)\,d\tau$, $t \in (-\infty, T]$, and multiply the equalities (1_m) and (2_m), $m \in \{k,l\}$, by $e^{\frac{1}{2}\lambda(t)}$. After simple transformations we obtain

$$\frac{\partial \tilde{u}_m(x,t)}{\partial t} - L\tilde{u}_m(x,t) + (a(x,t) - \frac{1}{2}a_0(t))\tilde{u}_m(x,t) -$$

$$-f_m(x,t,\tilde{u}_m(x,t)e^{-\frac{1}{2}\lambda(t)}; \tilde{u}_m(\cdot,t)e^{-\frac{1}{2}\lambda(t)})e^{\frac{1}{2}\lambda(t)} = \hat{f}_m(x,t)e^{\frac{1}{2}\lambda(t)},\ (x,t) \in Q, \qquad (16)$$

$$\tilde{u}_m(x,t) - g_m(x,t,\tilde{u}_m(x,t)e^{-\frac{1}{2}\lambda(t)}; \tilde{u}_m(\cdot,t)e^{-\frac{1}{2}\lambda(t)})e^{\frac{1}{2}\lambda(t)} =$$

$$= h_m(x,t)e^{\frac{1}{2}\lambda(t)}, \quad (x,t) \in \Sigma, \qquad (17)$$

where $\tilde{u}_m(x,t) = u_m(x,t)e^{\frac{1}{2}\lambda(t)}$, $(x,t) \in \overline{Q}$, $m \in \{k,l\}$.

Let us show that for an arbitrary fixed natural number m the restrictions of terms of the sequence $\{u_k\}$ on $\overline{Q^m}$ compose fundamental sequence in $C(\overline{Q^m})$. We take an arbitrary value $\varepsilon > 0$ and fix it. Let $k_0 \in \mathbb{N}$ is such that $k_0 > m$ and $2M_0 \cdot e^{\frac{1}{2}\lambda(-k_0)} < \varepsilon$. Let k and l be arbitrary natural numbers greater than k_0 (constant M_0 is from (3)). It leads to $\hat{f}_k(x,t) - \hat{f}_l(x,t) = 0$, $(x,t) \in Q^{k_0}$, and $h_k(x,t) - h_l(x,t) = 0$, $(x,t) \in \Sigma^{k_0}$.

Consider the restrictions of equalities (16) and (17) on Q^{k_0}. Based on Lemma 4 with $\frac{1}{2}a_0(t)$ in place of $a_0(t)$ we obtain

$$|\tilde{u}_k(x,t) - \tilde{u}_l(x,t)| \leqslant \max_{y \in \overline{\Omega}} |\tilde{u}_k(y,-k_0) - \tilde{u}_l(y,-k_0)|, \quad (x,t) \in \overline{Q^{k_0}}. \qquad (18)$$

By (15) we have $\max\limits_{y \in \overline{\Omega}} |\tilde{u}_k(y,-k_0) - \tilde{u}_l(y,-k_0)| \leqslant 2M_0 \cdot e^{\frac{1}{2}\lambda(-k_0)} < \varepsilon$. This and (18) lead to $\max\limits_{(x,t)\in Q_{k_0}} |\tilde{u}_k(x,t) - \tilde{u}_l(x,t)| < \varepsilon$ for any $k,l \geqslant k_0$. Hence $\max\limits_{(x,t)\in\overline{Q^m}} |\tilde{u}_{k,l}(x,t)| < \varepsilon$.

Therefore we have that the restrictions of the sequence $\{\tilde{u}_k\}$ where $\tilde{u}_k(x,t) = u_k(x,t)e^{\frac{1}{2}\lambda(t)}$, $(x,t) \in \overline{Q}$, on the set $\overline{Q^m}$, where m is an arbitrary fixed natural number, is fundamental sequence in $C(\overline{Q^m})$. Thus, there exists function $u \in C_{\text{loc}}(\overline{Q})$ such that $u_k \to u$ uniformly as $k \to \infty$ on an arbitrary compact from Q. In view of (15) we have $|u(x,t)| \leqslant M_0$, $(x,t) \in \overline{Q}$.

Let us show that u is a solution of Problem (1),(2). Since for any $k \in \mathbb{N}$ $u_k = NZ_0^k(\hat{f}_k, h_k)$, we have

$$\frac{\partial u_k(x,t)}{\partial t} - Lu_k(x,t) + a(x,t)u_k(x,t) =$$

$$= f_k(x,t,u_k(x,t); u_k(\cdot,t)) + \hat{f}_k(x,t), \quad (x,t) \in Q. \qquad (19_k)$$

Let $\{\Omega_m\}_{m=1}^{\infty}$ be a sequence of domains $\Omega_m \subset \Omega$, $m \in \mathbb{N}$ such that $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_m$ *subset...*, $\bigcup\limits_{m\in\mathbb{N}} \Omega_m = \Omega$, $\text{dist}(\Omega_m, \partial\Omega_{m+1} > 0)$, $m \in \mathbb{N}$, ($\text{dist}(A,B)$ is a distance between the sets A and B). Set $Q_{(m)} = \Omega_m \times (-m, T]$, $m \in \mathbb{N}$.

Let m be an arbitrary fixed natural number. By (19_k) and Theorem 10.1 of monograph [8; pp.238-239] we obtain $||u_k||_{\alpha,\alpha/2}^{Q(m+1)} \leqslant C_2$, $k \in \mathbb{N}$, where $C_2 > 0$ is a constant which does not depend on k.

Using this, (19_k), conditions of Theorem 3, in view of Theorem 10.1 of monograph [8; p.400] we obtain $||u_k||_{2+\alpha,1+\alpha/2}^{Q_{(m)}} \leqslant C_3$, where $C_3 \geqslant 0$ is a constant which does not depend on k.

Hence, and by well-known properties of space $C^{2+\alpha,1+\alpha/2}(\overline{Q_{(m)}})$ it follows that for an arbitrary fixed number $\gamma \in (0,\alpha)$ there exists a subsequence of sequence $\{u_k\}_{k=1}^{\infty}$ which tends to u in $C^{2+\gamma,1+\gamma/2}(\overline{Q_{(m)}})$, the restrictions of u on $\overline{Q_{(m)}}$ belong to the space $C^{2+\alpha,1+\alpha/2}(\overline{Q_{(m)}})$. Hence, using diagonal process, we conclude that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of sequence $\{u_k\}_{k=1}^{\infty}$ such that for an arbitrary bounded domain $Q' \subset Q$ located on the positive distance from Σ the restrictions u_{k_j} on $\overline{Q'}$, $j \in \mathbb{N}$, compose a sequence which converges to the restriction of u on $\overline{Q'}$ in $C^{2+\gamma,1+\gamma/2}(\overline{Q'})$, and $u \in C^{2+\alpha,1+\alpha/2}_{\text{loc}}(Q)$. Hence and by (1_k) we obtain that u is a solution of equation (1). The fulfilment of condition (2) follows from (2_k) and uniform convergence of $\{u_k\}$ in $\overline{Q^m}$ for an arbitrary fixed natural number m. Thus the proof of Theorem 3 is complete in the case of $\nu = 0 \in V(a_0)$.

Let $\nu \in V(a_0)$, $\nu \neq 0$. Probem (1),(2) can be rewrited

$$\frac{\partial \hat{u}(x,t)}{\partial t} - L\hat{u}(x,t) + \big(a(x,t) - \nu'(t)\big)\hat{u}(x,t) -$$
$$-f(x,t,\hat{u}(x,t)e^{-\nu(t)};\hat{u}(\cdot,t)e^{-\nu(t)})e^{\nu(t)} = \hat{f}(x,t)e^{\nu(t)}, \quad (x,t) \in Q, \tag{20}$$

$$\hat{u}(x,t) - g(x,t,\hat{u}(x,t)e^{-\nu(t)};\hat{u}(\cdot,t)e^{-\nu(t)})e^{\nu(t)} = h(x,t)e^{\nu(t)}, \quad (x,t) \in \Sigma, \tag{21}$$

where $\hat{u}(x,t) = u(x,t)e^{\nu(t)}$, $(x,t) \in \overline{Q}$.

Problem (20),(21) is similar to Problem (1),(2) with $a_0 - \nu'$ in place of a_0 in the case of $\nu = 0 \in V(a_0)$. This completes the proof of Theorem 3. □

Proof of Corollary. For the sake of simplicity without loss of generality we consider only the case of $k = 1$, more precisely, we consider a problem

$$P_1 u_1(x,t) = \hat{f}_1(x,t), \qquad (x,t) \in Q^1, \tag{22}$$

$$B_1 u_1(x,t) = h_1(x,t), \qquad (x,t) \in \Sigma^1, \tag{23}$$

$$u_1(x,-1) = 0, \qquad x \in \Omega. \tag{24}$$

Further we use the arguments similar to that in [4].

Without loss of generality we take $a_0(t) > 0$, $t \in [-1,T]$. Let

$$v_0(x,t) = C, \qquad (x,t) \in \overline{Q^1},$$

where

$$C \stackrel{\text{def}}{=} \max\{ \sup_{(y,\tau)\in\Sigma^1} \frac{|h_1(y,\tau)|}{1 - G(y,\tau)}, \sup_{(y,\tau)\in Q^1} \frac{|\hat{f}_1(y,\tau)|}{a_0(\tau)} \}.$$

Define the sequence of functions $\{v_p\}_{p=0}^{\infty}$ by the rule: if the function $v_{p-1} \in C^{2+\alpha,1+\alpha/2}(\overline{Q^1})$ is known then a function $v_p \in C^{2+\alpha,1+\alpha/2}(\overline{Q^1})$ is a solution of problem

$$\hat{P}v_p(x,t) \equiv \frac{\partial v_p(x,t)}{\partial t} - Lv_p(x,t) + a(x,t)v_p(x,t) =$$

$$= f_1(x,t,v_{p-1}(x,t);v_{p-1}(\cdot,t)) + \hat{f}_1(x,t), \quad (x,t) \in Q^1, \tag{25}$$

$$v_p(x,t) = g_1^*(x,t,0;(\eta v_{p-1})(\cdot,t)) + h_1(x,t), \quad (x,t) \in \Sigma^1, \tag{26}$$

$$v_p(x,-1) = 0, \quad x \in \Omega, \tag{27}$$

where function $\eta \in C_0^{\infty}(\Omega)$ is from condition (C1) of corollary, $g_1^*(x,t,\xi;\cdot) = \xi(t+1)g^*(x,t,\xi;\cdot)$, $(x,t,\xi) \in \overline{Q} \times \mathbb{R}$.

It follows from Theorem 6.1 of monograph [8; p.513], that a sequence $\{v_p\}$ is correctly defined, that is, for arbitrary $p \in \mathbb{N}$ function $v_p \in C^{2+\alpha,1+\alpha/2}(\overline{Q^1})$ is uniquely found.

Further, using our assumptions and the results of monograph [8] it can be shown that sequence $\{v^p\}$ converges to the solution of Problem (22)-(24).

□

Proof of Theorem 4. Let $\varepsilon > 0$ be an arbitrary number and $(\hat{f}_1, h_1)$, $(\hat{f}_2, h_2) \in \Pi_\nu$ such that $\sup\limits_{(y,\tau)\in Q} \frac{|\hat{f}_1(y,\tau)-\hat{f}_2(y,\tau)|e^{\nu\tau}}{a_0(\tau)-\nu'(\tau)} < \varepsilon/3$, $\sup\limits_{(y,\tau)\in\Sigma} \frac{|h_1(y,\tau)-h_2(y,\tau)|e^{\nu\tau}}{1-\hat{K}(y,\tau)} < \varepsilon/3$, and $u_i = NZ_\nu(\hat{f}_i, h_i)$, $i \in \{1,2\}$. Hence, in view of Lemma 5 we complete the proof of Theorem 4. □

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