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ON A QUASILINEAR ANALOG OF GIDAS-SPRUCK THEOREM

Theorems on nonexistence of global solutions are proved for elliptic inequalities and systems containing the second powers of the unknown functions under the assumption of the continuity of the coefficients at the principal nonlinear part. In the case of constant coefficients, the critical value of the parameter, such that the nonexistence does not take place for its larger values, is found.

1. Introduction.

A famous Gidas-Spruck theorem [1] establishes the nonexistence of positive solutions of equation $-\Delta u = u^q$ in $\mathbb{R}^n$ under assumption $1 < q < \dfrac{n+2}{n-2}$. Afterwards different authors have obtained analogs of that theorem for a number of *semilinear* equations (see references in [2]). However investigation of *quasilinear* case is still far from completion. The most general result in that field is obtained in [2], Ch. 1: theorem on nonexistence of global solution is proved for *inequality*

$$-\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} a_{i,j}(x,u) \geq b(x)|u|^q, \tag{1}$$

where $a_{i,j}$ are Carathéodory functions of $n+1$ variables, satisfying condition

$$|a_{i,j}(x,s)| \leq a(x)|s|^p, \;\; x \in \mathbb{R}^n, \; s \in (-\infty, +\infty), \; i,j = \overline{1,n} \tag{2}$$

with a positive p and non-negative $a(x)$.

Besides that, in [3] a similar theorem is proved for inequalities with p-Laplacian i. e. for quasiliniear inequalities which cannot be reduced to (1). All the rest known up to now quasiliniear analogs of Gidas-Spruck theorem refer to radial solutions only (see [2] and references therein).

In this paper theorem on nonexistence of global solution (and, under certain assumptions, unimprovability of that result) is proved for inequality of kind (1) in which, however, condition (2) is not satisfied, and nonlinearity in the right-hand side is not a power function; in particular, it is allowed to be an exponential function. Let us note, that besides purely theoretical aspect (illustration of the fact that elaborated in [2] methods are applicable to a much more broad area of problems), the considered nonlinearities arise in different applications (see e. g. [4] and [5]).

2. Nonexistence of global solution.

In $\mathbb{R}^n$ we consider inequality

$$\Delta u + \alpha |\nabla u|^2 + \beta(x) e^{\gamma u} \leq 0, \tag{3}$$

where α, γ are real parameters, $\beta(x)$ is a measurable a. e. positive function and there exists a positive R_0 such that $\beta^{-\frac{\alpha}{\gamma}} \in L_{1,loc}\big(\{|x| > R_0\}\big)$.
The following assertion is valid:

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Theorem 1. *Let* $\frac{\gamma}{\alpha} > 0$, $\lim\limits_{R\to\infty} R^{n-2-\frac{2\alpha}{\gamma}} \int\limits_{1\le|\xi|\le\sqrt{2}} \beta^{-\frac{\alpha}{\gamma}}(R\xi)d\xi < \infty$. *Then no global classical solution of inequality (3) exist.*

Proof. Suppose to the contrary that there exists function $u(x)$ satisfying (in the classical sense) inequality (3) in $\mathbb{R}^n$. Following e. g. [6], Ch. V, § 1 (see also [7] and [8]) we define in $\mathbb{R}^n$ the following function:

$$v(x) \overset{\text{def}}{=} \frac{1}{\alpha} e^{\alpha u(x)-1}. \tag{4}$$

Then $\frac{\partial v}{\partial x_j} = e^{\alpha u(x)-1}\frac{\partial u}{\partial x_j}$, $\frac{\partial^2 v}{\partial x_j^2} = e^{\alpha u(x)-1}\left[\frac{\partial^2 u}{\partial x_j^2} + \alpha\left(\frac{\partial u}{\partial x_j}\right)^2\right]$; $j = \overline{1,n}$.

Thus $\Delta v = e^{\alpha u(x)-1}\left(\Delta u + \alpha|\nabla u|^2\right)$ i. e. $\Delta u + \alpha|\nabla u|^2 = e^{1-\alpha u(x)}\Delta v$.

On the other hand, it follows from (4) that $v(x)$ has a constant sign and this is the same as the sign of constant α. This yields:

$$e^{\alpha u(x)-1} = \alpha v(x) = |\alpha v(x)| \Rightarrow \alpha u(x) - 1 = \ln|\alpha v(x)| \Rightarrow u(x) = \frac{\ln|\alpha v(x)| + 1}{\alpha}.$$

Thus

$$\Delta u + \alpha|\nabla u|^2 = e^{-\ln|\alpha v(x)|}\Delta v = \frac{1}{|\alpha v(x)|}\Delta v,$$

$$e^{\gamma u(x)} = e^{\frac{\gamma}{\alpha}(\ln|\alpha v(x)|+1)} = e^{\frac{\gamma}{\alpha}} e^{\ln|\alpha v(x)|^{\frac{\gamma}{\alpha}}} = e^{\frac{\gamma}{\alpha}}|\alpha v(x)|^{\frac{\gamma}{\alpha}}.$$

Therefore the following inequality is valid:

$$\frac{\Delta v}{|\alpha v(x)|} + \beta(x)e^{\frac{\gamma}{\alpha}}|\alpha|^{\frac{\gamma}{\alpha}}|v(x)|^{\frac{\gamma}{\alpha}} \le 0,$$

i. e.

$$-\Delta v \ge |\alpha|^{\frac{\gamma}{\alpha}+1} e^{\frac{\gamma}{\alpha}}\beta(x)|v|^{\frac{\gamma}{\alpha}+1}. \tag{5}$$

Inequality (5) is a particular case of inequality (1) with $b(x) = |\alpha|^{\frac{\gamma}{\alpha}+1}e^{\frac{\gamma}{\alpha}}\beta(x)$, $q = \frac{\gamma}{\alpha} + 1$, $a_{i,j}(x) = \delta_i^j$. Thus condition (2) is satisfied with $a(x) \equiv 1, p = 1$. Then by the virtue of the assumption of Th. 1

$$\lim_{R\to\infty} R^{n-\frac{2q}{q-p}} \int\limits_{1\le|\xi|\le\sqrt{2}} \frac{a^{\frac{p}{q-p}}(R\xi)}{b^{\frac{p}{q-p}}(R\xi)}d\xi = \lim_{R\to\infty} R^{n-2-\frac{2\alpha}{\gamma}} \int\limits_{1\le|\xi|\le\sqrt{2}} b^{-\frac{\alpha}{\gamma}}(R\xi)d\xi < \infty.$$

Finally, under the assumptions of Th. 1 $p > 0$, $q > p$, b is measurable and a. e. positive. Then, by the virtue of Th. 3.1 of book [2], inequality (5) has no global classical (and even weak) solutions. We obtain a contradiction. □

Remark 1. *Inequality (3) can be written as*

$$-\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\left(\delta_i^j \frac{1}{\alpha} e^{\alpha u}\right) \ge \beta(x)e^{(\alpha+\gamma)u}.$$

The left-hand side of the latter inequality is a particular case of the left-hand side of inequality (1) however assumption (2) is satisfied with no p. This confirms that neither velocity of coefficients growth nor their power form are, in general, circumstances, restricting the applicability of the theory elaborated in [2].

REMARK 2. *In case of constant coefficient β Th. 1 obviously implies the following assertion (cf. Th. 2.1 of book [2]):*

COROLLARY 1. *Let $\beta > 0$, $0 < \dfrac{\gamma}{\alpha} \le \dfrac{2}{n-2}$. Then inequality*

$$\Delta u + \alpha|\nabla u|^2 + \beta e^{\gamma u} \le 0 \tag{6}$$

has no classical solutions in $\mathbb{R}^n$.

3. Unimprovability of the result.

For positive α the critical value $\dfrac{2}{n-2}$ is precise i. e. Corollary 1 is unimprovable in that sense (like Th. 2.1 of book [2]). To prove that introduce (for positive α) function

$$u_0(x) \overset{\text{def}}{=} -\frac{1}{\gamma}\ln(|x|^2+1) + \frac{\ln \alpha\varepsilon e}{\alpha}$$

(ε is a positive parameter), and substitute it to inequality (6).

$$e^{\gamma u_0} = e^{-\ln(|x|^2+1)+\frac{\gamma}{\alpha}\ln\alpha\varepsilon e} = \frac{1}{|x|^2+1}(\alpha\varepsilon e)^{\frac{\gamma}{\alpha}};$$

$$\frac{\partial u_0}{\partial x_j} = -\frac{1}{\gamma}\frac{2x_j}{|x|^2+1}, \ \frac{\partial^2 u_0}{\partial x_j^2} = \frac{1}{\gamma}\frac{4x_j^2 - 2\big(|x|^2+1\big)}{\big(|x|^2+1\big)^2};\ j=\overline{1,n} \Longrightarrow$$

$$\Delta u_0 = \frac{1}{\gamma}\frac{4|x|^2 - 2n|x|^2 - 2n}{\big(|x|^2+1\big)^2},\ |\nabla u_0|^2 = \frac{1}{\gamma^2}\frac{4|x|^2}{\big(|x|^2+1\big)^2}.$$

It yields:

$$\Delta u_0 + \alpha|\nabla u_0|^2 + \beta e^{\gamma u_0} =$$

$$= \frac{1}{\gamma}\frac{(4-2n)|x|^2 - 2n}{\big(|x|^2+1\big)^2} + \frac{4\alpha}{\gamma^2}\frac{|x|^2}{\big(|x|^2+1\big)^2} + \beta(\alpha\varepsilon e)^{\frac{\gamma}{\alpha}}\frac{1}{|x|^2+1} =$$

$$= \frac{2}{\gamma\big(|x|^2+1\big)^2}\left[\left(2 - n + \frac{2\alpha}{\gamma}\right)|x|^2 - n\right] + \beta(\alpha\varepsilon e)^{\frac{\gamma}{\alpha}}\frac{1}{|x|^2+1}.$$

Now suppose that $\dfrac{\gamma}{\alpha} > \dfrac{2}{n-2}$. Then $\dfrac{\alpha}{\gamma} < \dfrac{n-2}{2}$, i. e. $2 - n + \dfrac{2\alpha}{\gamma} < 0$. Taking into account that positivity of α implies positivity of γ, we obtain that for small enough ε function $u_0(x)$ satisfies inequality (6) in $\mathbb{R}^n$.

Unimprovability of Corollary 1 is proved.

4. Case of variable coefficient at high-order nonlinear terms.

The approach, described above, can be generalized for the case when coefficient α in inequality (3) is variable. More exactly, the following assertion is true:

THEOREM 2. *Let $q > 1, \beta(x)$ be measurable, $\beta > 0$ a. e. in $\mathbb{R}^n$, there exists $R_0 > 0$ such that $\beta^{\frac{1}{1-q}} \in L_{1,loc}(\{|x| > R_0\})$ and*

$$\lim_{R\to\infty} R^{n-\frac{2q}{q-1}} \int\limits_{1\le|\xi|\le\sqrt{2}} \beta^{\frac{1}{1-q}}(R\xi)d\xi < \infty.$$

Let $g \in C(-\infty, +\infty)$ and $\omega(s) \ge \left|\int\limits_0^s e^{\int_0^\tau g(t)dt} d\tau\right|^q e^{-\int_0^s g(\tau)d\tau}$ on $(-\infty, +\infty)$. Then inequality

$$\Delta u + g(u)|\nabla u|^2 + \beta(x)\omega(u) \le 0 \tag{7}$$

has no classical solutions in $\mathbb{R}^n$.

Proof. Suppose to the contrary that there exists function $u(x)$ satisfying (in the classical sense) inequality (7) in $\mathbb{R}^n$. Define in $\mathbb{R}^n$ function $v(x)$ as $f[u(x)]$, where

$$f(s) = \int\limits_0^s e^{\int_0^\tau g(t)dt} d\tau. \tag{8}$$

Then $\frac{\partial v}{\partial x_j} = f'(u)\frac{\partial u}{\partial x_j}, \frac{\partial^2 v}{\partial x_j^2} = f''(u)\Big(\frac{\partial u}{\partial x_j}\Big)^2 + f'(u)\frac{\partial^2 u}{\partial x_j^2}; j = \overline{1,n}.$

Hence $\Delta v = f'(u)\Delta u + f''(u)|\nabla u|^2$.

Further $f'(s) = e^{\int_0^s g(\tau)d\tau} > 0$ on $(-\infty, +\infty)$, $f''(s) = g(s)e^{\int_0^s g(\tau)d\tau}$, and therefore

$$\Delta v = f'(u)\big[\Delta u + g(u)|\nabla u|^2\big].$$

It implies, since f' is strictly positive on $(-\infty, +\infty)$, that the following inequality is valid in $\mathbb{R}^n$:

$$\frac{\Delta v}{f'(u)} + \beta(x)\omega(u) \le 0.$$

Then inequality $\frac{\Delta v}{f'(u)} + \beta(x)\left|\int\limits_0^u e^{\int_0^\tau g(t)dt} d\tau\right|^q e^{-\int_0^u g(\tau)d\tau} \le 0$ is also valid in $\mathbb{R}^n$ therefore inequality

$$\frac{\Delta v}{f'(u)} + \beta(x)\frac{|f(u)|^q}{f'(u)} \le 0$$

is also true in $\mathbb{R}^n$. It means, by the virtue of the strict positivity of f' in $(-\infty, +\infty)$, that in $\mathbb{R}^n$

$$\Delta v + \beta(x)|v|^q \le 0,$$

but it contradicts Th. 3.1 of book [2]. □

REMARK 3. $\Delta f(u) = \sum_{j=1}^n \frac{\partial}{\partial x_j}\Big[f'(u)\frac{\partial u}{\partial x_j}\Big] = \Big[\Delta u + g(u)|\nabla u|^2\Big]f'(u)$ *hence inequality (7) can be written as* $\Delta f(u) + \beta(x)e^{\int_0^u g(s)ds}\omega(u) \le 0$ *i. e. as*

$$-\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}[\delta_i^j f(u)] \ge \beta(x)\omega_q(u), \tag{9}$$

where ω_q is any function satisfying condition $\omega_q(s) \geq |f(s)|^q$ on $(-\infty, +\infty)$ and f is defined by (8).

Thus the left-hand side of inequality (9) is a particular case of the left-hand side of inequality (1), however, since continuous function g is arbitrary, *then condition (2), in general, can be broken (unlike [2]); as it is shown in § 2, that takes place even in the simplest case $g = const$. The right-hand side of inequality (9) is also, in general, different from the right-hand side of inequality (1).*

Remark 4. *In general, the approach, described above, can be also generalized for the case of singular coefficient $g(u)$ in inequality (7). However in this problem it does not lead to* new *results. Really, applying anzatses $f(s) = \int\limits_0^s e^{\frac{\alpha}{1-\mu}\tau^{1-\mu}} d\tau$ and $f(s) = s^{\alpha+1}$ instead of (8), we can find sufficient conditions of nonexistence of global positive solutions of inequality*

$$\Delta u + \frac{\alpha}{u^\mu}|\nabla u|^2 + \beta(x)\omega(u) \leq 0$$

for the cases $\alpha < 0, 0 < \mu < 1$ and $-1 < \alpha < 0, \mu = 1$ correspondingly (cf. [8], § 3). It is however easy to check that those conditions would be exactly the same as assumptions of Th 3.1 of book [2] if we take correspondingly $\delta_i^j \int\limits_0^s e^{\frac{\alpha}{1-\mu}\tau^{1-\mu}} d\tau$ and $\delta_i^j s^{\alpha+1}$ instead of functions $a_{i,j}(x,s)$; those $a_{i,j}(x,s)$ satisfy condition (2) with $0 < p \leq 1$.

5. Case of system of inequalities.

Consider the following system:

$$\Delta u_1 + g_1(u_1)|\nabla u_1|^2 + d_1\omega_1(u_1, u_2) \leq 0; \tag{10}$$

$$\Delta u_2 + g_2(u_2)|\nabla u_2|^2 + d_2\omega_2(u_1, u_2) \leq 0; \tag{11}$$

where $d_1 > 0 < d_2$, $g_1, g_2 \in C(-\infty, +\infty)$.

The following assertion is true:

Theorem 3. *Let $q_1 > 1 < q_2$, $\max\left\{\frac{2(q_1+1)}{q_1 q_2}, \frac{2(q_2+1)}{q_1 q_2}\right\} \geq n-2$. Let*

$$\omega_1(s_1, s_2) \geq \frac{\left|\int\limits_0^{s_2} e^{\int\limits_0^\tau g_2(t)dt} d\tau\right|^{q_1}}{e^{\int\limits_0^{s_1} g_1(\tau)d\tau}}, \quad \omega_2(s_1, s_2) \geq \frac{\left|\int\limits_0^{s_1} e^{\int\limits_0^\tau g_1(t)dt} d\tau\right|^{q_2}}{e^{\int\limits_0^{s_2} g_2(\tau)d\tau}}$$

on $(-\infty, +\infty)$. Them system (10),(11) has no classical solutions in $\mathbb{R}^n$.

Proof. Suppose to the contrary that there exists a vector-function $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ satisfying (in the classical sense) system (10),(11) in $\mathbb{R}^n$. Then on $\mathbb{R}^n$ we define vector-function

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} f_1[u_1(x)] \\ f_2[u_2(x)] \end{pmatrix},$$

where $f_j(s) = \int\limits_0^s e^{\int\limits_0^\tau g_j(t)dt} d\tau$; $j = 1, 2$. We obtain (see proof of Th. 2) that

$$\Delta v_j = f_j'(u)\left[\Delta u_j + g_j(u_j)|\nabla u_j|^2\right],$$

and f_1', f_2' are strictly positive on $\mathbb{R}^1$. It means that inequalities $\dfrac{\Delta v_j}{f_j'(u_j)} + d_j\omega_j(u_1, u_2) \le 0$ are valid in $\mathbb{R}^n$ for $j = 1, 2$. By the virtue of the condition of Th. 3 it implies (similarly to the proof of Th. 2) that inequalities

$$\frac{\Delta v_1}{f_1'(u_1)} + d_1\frac{|f_2(u_2)|^{q_1}}{f_1'(u_1)} \le 0, \quad \frac{\Delta v_2}{f_2'(u_2)} + d_2\frac{|f_1(u_1)|^{q_2}}{f_2'(u_2)} \le 0$$

are also true in $\mathbb{R}^n$. From that we deduce (since f_1', f_2' are strictly positive) that vector-function $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a classical solution of system

$$\Delta v_1 + d_1|v_2|^{q_1} \le 0, \ x \in \mathbb{R}^n; \tag{12}$$

$$\Delta v_2 + d_2|v_1|^{q_2} \le 0, \ x \in \mathbb{R}^n. \tag{13}$$

However, by the virtue of Th. 17.1 of book [2] system (12),(13) does not have even week solutions in case $\max\left\{\dfrac{2(q_1+1)}{q_1q_2}, \dfrac{2(q_2+1)}{q_1q_2}\right\} \ge n-2$, $0 \in (d_1, d_2)$, $1 \in (q_1, q_2)$.

We obtain a contradiction. □

REMARK 5. *Instead of (10),(11) one can consider the following system of more general kind:*

$$\Delta u_i + \sum_{j=1}^n a_{ij}(x, u_1, u_2)\left(\frac{\partial u_i}{\partial x_j}\right)^2 + d_i\omega_i(x, u_1, u_2) \le 0,$$

whose coefficients satisfy for $i = 1, 2$, $j = \overline{1, n}$ the following conditions:
there exist $g_1, g_2 \in C(-\infty, +\infty)$ such that

$$a_{ij}(x, s_1, s_2) \ge g_i(s_i),$$

$$\omega_1(x, s_1, s_2) \ge \frac{\left|\int\limits_0^{s_2} e^{\int\limits_0^\tau g_2(t)dt} d\tau\right|^{q_1}}{e^{\int\limits_0^{s_1} g_1(\tau)d\tau}}, \quad \omega_2(x, s_1, s_2) \ge \frac{\left|\int\limits_0^{s_1} e^{\int\limits_0^\tau g_1(t)dt} d\tau\right|^{q_2}}{e^{\int\limits_0^{s_2} g_2(\tau)d\tau}}$$

on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^1$.

In that case Th. 3 remains true under its conditions regarding relations between parameters n, q_1, q_2.

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