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## A FREE BOUNDARY PROBLEM ARISING IN BIOLOGY

This work was intended as an attempt to investigate a simple mathematical model cell. This model represents a problem with free boundary for system of two differential equations, one of which is parabolic and another one is elliptic. We apply a method which consists of the following. First, we construct a special system of difference-differential approximating elliptic problems, then we prove some uniform estimates and pass to the limit. We prove the existence of the classical solution.

### 1. Introduction.

In paper [1], the authors consider a physico-chemical model of a self-maintaining protocell which undergoes a process of growth and dissolution that mimics biological cells. The protocell can be visualized as having a porous structure maintained by building materials with concentration $C$; the structure is sustained only as long as $C$ exceeds a critical concentration $C^*$. Metabolism is maintained by nutrient material with concentration $u$ which is distributed in the entire space with $u = \tau$ at $\infty$ $(\tau > 0)$. $C$ and $u$ satisfy a coupled system of reaction diffusion equations:

$$c\frac{\partial C}{\partial t} - \Delta C = u, \quad \Delta u = 0 \text{ in the cell },$$

$$\Delta \sigma = 0 \text{ outside the cell},$$

on the boundary of the cell

$$C = C^*, \quad V_n = -\frac{\partial C}{\partial n} - \beta, \ \beta > 0,$$

$n$ is the exterior normal, $V_n$ is the velocity of the boundary, $\beta$ is modelled by disintegration, $C$ is concentration of the building material of the cell, $\sigma$ is concentration of the nutrient material. The authors established various estimates and proved the existence and uniqueness of the solution.

We think that it is more natural for this mathematical model to consider a two-phase medium, replacing one of a boundary conditions by the Stefan condition. Besides, we consider the problem in the three-dimensional space without assuming that the solution is radially symmetric.

*Statement of the problem.* Let $D = \{x \in \mathbb{R}^3 : |x| < R\}$, $D_T = D \times (0, T)$. The problem is to find the functions $c(x,t)$, $u(x,t)$ and domains $\Omega_T$, $G_T$, which satisfy:

$$\Delta c - a\frac{\partial c}{\partial t} = -u(x,t), \qquad \Delta u = \chi(\Omega_T)u \quad \text{in} \quad \Omega_T \cup G_T, \tag{1.1}$$

where

$$\Omega_T = \{(x,t) \in D_T : c(x,t) > 1\}, \quad G_T = D_T \setminus \overline{D}_T,$$

$a, T$ are positive constants, $\Omega_T$ is the domain of the cell, $\chi(\Omega_T)$ is the characteristic function of the domain $\Omega_T$.

On the known boundary $\partial D_T$

$$c(x,t) = 0, \qquad u(x,t) = \varphi(x,t) > 0. \tag{1.2}$$

On the unknown (free) boundary $\gamma_T = \partial\Omega_T \cap D_T = \partial G_T \cap D_T$

$$c^+(x,t) = c^-(x,t) = 1, \;\; V_n = \frac{\partial c^-}{\partial n} - \frac{\partial c^+}{\partial n}, \tag{1.3}$$

where $n$ is the exterior normal of the domain $\Omega_T$, $V_n$ is the velocity of the boundary points in the direction n, $c^+, c^-$ are corresponding limit values of the function.

The initial conditions are

$$c(x,0) = \psi(x) > 0, \;\; \Omega_0 = \{x \in D : \psi(x) > 1\}. \tag{1.4}$$

The aim of this paper is to study the existence of the global classical solution. The paper is organized as follows. In section 2 we construct a difference-differential approximation for our problem and study its properties. The properties of the fundamental solutions are considered in section 3. In section 4 we prove uniform estimates and pass to the limit.

Note that similar methods have been used in [ 2], [ 3].

**2. Construction of the approximating problem.**

We shall assume that the problem (1.1)-(1.4) has a classical solution. We multiply the equation (1) by a smooth function $\eta(x,t)$ which vanishes on $\partial D_T$ and integrate by parts:

$$\int_{D_T} [\nabla c \nabla \eta + c_t \eta - u\eta - \chi(c)\eta_t]\, dxdt = 0.$$

For any $\varepsilon > 0$ we introduce a function $\chi_\varepsilon(\tau) \in C^\infty(\mathbb{R}^1)$:

$$\chi_\varepsilon(\tau) = 0 \; \forall \tau \le 1 - \varepsilon, \;\; \chi_\varepsilon(\tau) = 1 \; \forall \tau \ge 1, \;\; \chi'_\varepsilon(\tau) \ge 0.$$

We define the functions $\{c^\varepsilon(x,t), u^\varepsilon(x,t)\}$ as the solutions of following problem:

$$\begin{gathered} \Delta c^\varepsilon(x,t) - a\tfrac{\partial c^\varepsilon(x,t)}{\partial t} = \tfrac{\partial \chi_\varepsilon(c^\varepsilon(x,t))}{\partial t} - u^\varepsilon(x,t), \\ c^\varepsilon(x,t) = 0 \;\text{ on } \partial D \times [0,T), \;\; c^\varepsilon(x,0) = \psi(x) \;\text{ in } \overline{D}. \end{gathered} \tag{2.1}$$

$$\Delta u^\varepsilon(x,t) - \chi_\varepsilon(c^\varepsilon(x,t))u^\varepsilon(x,t) = 0, \;\; u^\varepsilon(x,t) = \varphi(x,t) \text{ on } \partial D \times [0,T). \tag{2.2}$$

THEOREM 2.1. *Let*

$$\psi(x) \in C^{2+\alpha}(\overline{D}), \; \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T})$$

*and assume that the corresponding consistency conditions hold at $t = 0, x \in \partial D$. Then this problem is solvable and*

$$\left\| c^\varepsilon(x,t) \right\|_{H^{2+\alpha,1+\alpha/2}(\overline{D_T})} + \left\| c^\varepsilon(x,t) \right\|_{H^{2+\alpha,1+\alpha/2}(\overline{D_T})} \le \frac{M_1}{\varepsilon^\nu} \tag{2.3}$$

$$\left\| u^\varepsilon(x,t) \right\|_{C^{1+\alpha}(\overline{D})} \le M_2, \;\; \alpha \in (0,1), \;\; u^\varepsilon(x,t) > 0, \; c^\varepsilon(x,t) > 0, \tag{2.4}$$

*where positive constants $M_1, M_2, \nu$ do not depend on $\varepsilon$.*

*Proof.* As the coefficient of the function $u^\varepsilon(x,t)$ in the equation (2.2) satisfies the inequality

$$0 \le \chi_\varepsilon(c^\varepsilon(x,t)) \le 1,$$

then, as is well known, see [4], we have the following estimate in $\overline{D}_T$:

$$0 \le u^\varepsilon(x,t) \le \max_{\overline{D_T}} \varphi(x,t).$$

From this estimate it follows that the functions $\{u^\varepsilon(x,t)\}$ satisfy the Poisson equation with respect to the variable $x$ with a bounded right-hand side. Then, as is well-known, the estimate (2.6) holds. Let us transform the equation (2.1) to the form:

$$\frac{1}{a+\chi_\varepsilon'[c^\varepsilon(x,t)]}\Delta c^\varepsilon(x,t) - \frac{\partial c^\varepsilon(x,t)}{\partial t} = -u^\varepsilon(x,t),$$

$$0 < c_1\varepsilon \le \frac{1}{a+\chi_\varepsilon'[c^\varepsilon(x,t)]} \le c_2,$$

where $c_1, c_2$ are absolute constants. Solvability questions of the problem (2.1) in Hölder spaces have been investigated in [5].

We divide the cylinder $D_T$ by the planes $t = kh, k = 1,2,...N, Nh = T$, integrate equation (2.5) with respect to the variable $t$ from $(k-1)h$ to $kh$ and multiply the equation by $1/h$. After simple transformations we obtain:

$$\Delta c_k^\varepsilon(x) - a\frac{c_k^\varepsilon(x) - c_{k-1}^\varepsilon(x)}{h} = \frac{\chi_\varepsilon(c_k^\varepsilon(x)) - \chi_\varepsilon(c_{k-1}^\varepsilon(x))}{h} - u_k^\varepsilon(x) - f_k^\varepsilon, \tag{2.5}$$

$$c_k^\varepsilon(x) = 0 \quad \text{on } \partial D, \quad c_0^\varepsilon(x) = \psi(x) \quad \text{in} \quad \overline{D}, \tag{2.6}$$

$$\Delta u_k^\varepsilon(x) - \chi_\varepsilon(c_k^\varepsilon(x))u_k^\varepsilon(x) = 0, \tag{2.7}$$

$$u_k^\varepsilon(x) = \varphi_k(x) = \varphi(x,kh) \quad \text{on} \quad \partial D. \tag{2.8}$$

where

$$c_k^\varepsilon(x) = c^\varepsilon(x,kh),\ u_k^\varepsilon(x) = u^\varepsilon(x,kh),\ \left\|f_k^\varepsilon\right\|_{H^{\alpha,\alpha/2}(\overline{D_T})} \le \frac{M}{\varepsilon^\nu}h^{\alpha/2}, \tag{2.10}$$

where $M$ does not depend $k, \varepsilon$.

REMARK 2.1. *The equation (2.5) can be transformed to the form*

$$\Delta c_k^\varepsilon(x) - a_k^\varepsilon(x)\frac{c_k^\varepsilon(x) - c_{k-1}^\varepsilon(x)}{h} = -u_k^\varepsilon(x) - f_k^\varepsilon, \tag{2.11}$$

*where* $a_k^\varepsilon(x) = a + \int\limits_0^1 \chi_\varepsilon'[c_{k-1}^\varepsilon(x) + \tau(c_k^\varepsilon(x) - c_{k-1}^\varepsilon(x))]d\tau.$

**3. The fundamental solutions and its properties.**

To study properties of the solution of (2.5), we shall use an integral representation. In order to do that we shall construct the fundamental solutions. Consider the family $\{\Gamma_{m-k+1}(|x-x_0|)\}$, $k = 1,2,...m$ defined as follows

$$\Gamma_{m-k+1}(|x-x_0|) = \frac{ih}{2\pi a_m}\oint\limits_{\partial L} \frac{(\sinh\sqrt{z}R)^{-1}\sinh\sqrt{z}(R-|x-x_0|)dz}{4\pi|x-x_0|(1-\frac{zh}{a_m})(1-\frac{zh}{a_{m-1}})(1-\frac{zh}{a_k})}, \tag{3.1}$$

where

$$L = \{z = \xi + i\eta : Re z > -\frac{\pi^2}{R^2},\ |z| < \varrho\},\ \left(\frac{a_1}{h},0\right),\left(\frac{a_2}{h},0\right),...\left(\frac{a_m}{h},0\right) \in L,\ \ a_i > 0.$$

The numerator and denominator of the integrand have the same branch point $z = 0$. Therefore, in the domain $L$ it is possible to choose a univalent branch of the integrand by setting, for example, $\sqrt{1} = 1$.

Denote

$$\omega_{m-k+1}(z) = (z - \frac{a_m}{h})(z - \frac{a_{m-1}}{h})...(z - \frac{a_k}{h}).$$

The integral (3.1) can be calculated through the theory of residues. It gives

$$\Gamma_{m-k+1}(|x-x_0|) = \frac{ih}{2\pi a_m} \oint_{\partial L} \frac{\sinh[\sqrt{z}(R-|x-x_0|)]\omega_{m-k+1}(0)dz}{4\pi|x-x_0|\sinh[\sqrt{z}R]\omega_{m-k+1}(z)} =$$

$$= -\frac{h}{a_m} \sum_{l=k}^{l=m} \frac{\omega_{m-k+1}(0)}{\omega'_{m-k+1}(\frac{a_l}{h})} K(\frac{a_l}{h}, |x-x_0|),$$

where

$$K(z, |x-x_0|) = \frac{\sinh[\sqrt{z}(R-|x-x_0|)]}{4\pi|x-x_0|\sinh[\sqrt{z}R]}.$$

**Property 1.** *If $|x-x_0| \neq 0$, then the functions $\{\Gamma_{m-k+1}(|x-x_0|\}$ satisfy the equations*

$$\Delta\Gamma_{m-k+1} - a_k \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{h} = 0, \ \forall k = 1...(m-1), \tag{3.2}$$

*if $k = n$, then*

$$\Delta\Gamma_1 - a_m \frac{\Gamma_1}{h} = 0, \quad \Gamma_1(|x-x_0|) = \frac{\sinh\sqrt{\frac{a_m}{h}}(R-|x-x_0|)}{4\pi|x-x_0|\sinh\sqrt{\frac{a_m}{h}}R}.$$

This statement can be obtained by direct calculation.

**Property 2.** *Let $K_\delta(x_0)$ denote the ball with its center at the point $x_0$ and radius $\delta$. Then*

$$\lim_{\delta\to 0} \oint_{\partial K_\delta(x_0)} \frac{\partial\Gamma_{m-k+1}}{\partial n} ds = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if } k \neq m, \end{cases}$$

*where $n$ is the inner normal.*

*Proof.* Indeed, if $k < m$, then

$$\lim_{\delta\to 0} \frac{ih}{2\pi a_m} \oint_L \int_{\partial K_\delta(x_0)} \left( \frac{\sinh[\sqrt{z}(R-\delta)]}{4\pi\delta^2\sqrt{z}R} - \frac{\sqrt{z}\cosh[\sqrt{z}(R-\delta)]}{4\pi\delta\sinh(\sqrt{z}R)} \right) ds \frac{\omega_{m-k+1}(0)}{\omega_{m-k+1}(z)} dz =$$

$$= \lim_{\delta\to 0} \frac{ih}{2\pi a_m} \oint_L \int_{K_\delta(x_0)} \frac{\omega_{m-k+1}(0)}{\omega_{m-k+1}(z)} dz = 0,$$

as the sum of residues at all singular points, including the infinity, is equal to zero. If $k = m$ then we obtain

$$\lim_{\delta\to 0} \int_{K_\delta(x_0)} \frac{\partial\Gamma_1}{\partial n} ds = \frac{ih}{2\pi a_m} \oint_L \frac{dz}{1 - \frac{zh}{a_v}} = 1.$$

**Property 3.** *Let* $\{u_k(x)\} \in C^2(\overline{D}\}$ *and*

$$\Delta u_k - \frac{a_k u_k - a_{k-1}u_{k-1}}{h} = -\frac{f_k - f_{k-1}}{h},$$

*where* $f_k \in C(D)$ *is given function. Then we have the following integral representation*

$$u_m(x_0) = \int\limits_{K_R(x_0)} \frac{(a_0u_0 - f_0)\Gamma_m(|x-x_0|)}{h}dx - \sum_{k=1}^{m} \int\limits_{\partial K_R(x_0)} u_k \frac{\partial \Gamma_{m-k+1}}{\partial n} ds +$$

$$+\sum_{k=1}^{m} \int\limits_{\partial K_R(x_0)} f_k \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{h} dx. \qquad (3.3)$$

This integral representation follows from the previous properties of the fundamental solutions and Green's formula for the elliptic equations.

**Property 4.**

$$\int\limits_{K_R(x_0)} \frac{\Gamma_m(|x-x_0|)}{h}dx \le \int\limits_{K_R(x_0)} \frac{\Gamma_{m-1}(|x-x_0|)}{h}dx \le ... \le \int\limits_{K_R(x_0)} \frac{\Gamma_1(|x-x_0|)}{h}dx =$$

$$= \frac{1}{a_m}\left(1 - \frac{\sqrt{\frac{a_m}{h}}}{\sinh(\sqrt{\frac{a_m}{h}}R)}\right), \ \Gamma_m(|x-x_0|) \le \frac{1}{4\pi|x-x_0|}. \qquad (3.4)$$

*Proof.* If $x \ne x_0$, then

$$\Delta\Gamma_{m-k+1} - a_k \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{h} = 0, \ \forall k = 1...(m-1).$$

In addition to that, we have

$$\lim_{|x-x_0|\to 0} 4\pi|x-x_0|\Gamma_{m-k+1}(|x-x_0|) = \begin{cases} 1, \text{ if } k = m, \\ 0, \text{ if } k \ne m, \end{cases} \ \Gamma_{m-k+1}(R) = 0. \qquad (3.5)$$

The maximum principle implies

$$\Gamma_{m-k+1}(|x-x_0|) > 0 \text{ in } K_R(x_0), \ \frac{\partial\Gamma_{m-k+1}}{\partial n} < 0 \text{ on } \partial K_R(x_0).$$

Hence $\forall k < m$

$$\int\limits_{K_R(x_0)} a_k \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{h} = \int\limits_{K_R(x_0)} \Delta\Gamma_{m-k+1}(|x-x_0|)dx < 0.$$

The second estimate (3.4) also follows from the maximum principle.

**Property 5.** *The functions* $\{\Gamma_{m-k+1}(|x-x_0|) - \Gamma_{m-k}(|x-x_0|)\}$ *change the sign on the interval* $0 < |x-x_0| < R$ *no more than once.*

*Proof.* Denote $r = |x - x_0|$. The functions $\{r\Gamma_{m-k+1}(r)\}$ satisfy the equations

$$\frac{d^2}{dr^2}(r\Gamma_{m-k+1}(r)) - a_k \frac{r\Gamma_{m-k+1}(r) - r\Gamma_{m-k}(r)}{h} = 0, \quad k = 1, 2, ..., m-1.$$

If $k = m - 1$ in this equation then

$$\frac{d^2}{dr^2}(r\Gamma_2(r)) - a_{m-1} \frac{r\Gamma_2(r) - r\Gamma_1(r)}{h} = 0,$$

$$\lim_{r \to 0} r\Gamma_1(r) = 4\pi, \quad \Gamma_1(R) = 0.$$

This implies that $r\Gamma_1(r) > 0 \; \forall r \in (0, R)$. Near the point $x = x_0$ the function $r\Gamma_2(r) - r\Gamma_1(r) < 0$ and it satisfies the following conditions:

$$\frac{d^2}{dr^2}(r\Gamma_2(r) - r\Gamma_1(r)) - a_{m-1} \frac{r\Gamma_2(r) - r\Gamma_1(r)}{h} = -a_m \frac{r\Gamma_1(r)}{h} < 0. \tag{3.6}$$

Let us assume now that the function $r\Gamma_2(r) - r\Gamma_1(r)$ changes its sign at a point $r_{2,1} \in (0, R)$. The maximum principle implies that this function can not take negative values as at the ends of the interval $(r_{2,1}, R)$ it is equal to zero, and inside the interval it satisfies the equation (3.6). Let us prove, that the function $r\Gamma_3(r) - r\Gamma_3(r)$ changes its sign on the interval $0 < |x - x_0| < R$ no more than once. This function satisfies the equation

$$\frac{d^2}{dr^2}(r\Gamma_3(r) - r\Gamma_2(r)) - a_{m-2} \frac{r\Gamma_3(r) - r\Gamma_3(r)}{h} = -a_{m-1} \frac{r\Gamma_2(r) - r\Gamma_1(r)}{}$$

and at the end points of the interval $(0, R)$ it is equal to zero. Notice that the right-hand side of this equation is positive near the point $r = 0$. We will prove at first that function $r\Gamma_3(r) - r\Gamma_2(r)$ is negative near the point $r = 0$ . Suppose that on an interval $(0, r_{2,1})$ there is a point $r_{3,2}$, at which the considered function is equal to zero. Then on this interval, as follows from the equation, this function cannot have a positive maximum and is, consequently, negative. Let us suppose now that the point at which the function is equal to zero, belongs to the interval $(r_{2,1}, R)$. Then the function $r\Gamma_3(r) - r\Gamma_2(r)$ can not change its sign from plus to minus at this point because on the interval $(r_{3,2}, R)$ the function cannot have a negative minimum. Besides, from

$$\int_{K_R(x_0)} a_{m-2} \frac{\Gamma_3 - \Gamma_2}{h} dx = \int_{K_R(x_0)} \Delta\Gamma_3 dx = \int_{\partial K_R(x_0)} \frac{\partial \Gamma_3}{\partial n} ds < 0,$$

it follows that the function $r\Gamma_3(r) - r\Gamma_2(r)$ can not be everywhere positive. Thus, there is an interval $(0, r_{3,2})$, in which it is negative. Assume that at the point $r_{3,2}$ the considered function changes the sign from minus to plus. Then on the interval $(0, r_{2,1})$ the function $r\Gamma_3(r) - r\Gamma_2(r)$ cannot be equal to zero, as it can not reach on it a positive maximal value. Similarly, there can not be changes of sign on the interval $(r_{2,1}, R)$. Let $r_{3,2}$ belong to the interval $(r_{2,1}, R)$. Then on the interval $(r_{2,1}, R)$ the function is everywhere positive. Thus, the considered function can change the sign on the interval $(0, R)$ no more than once.

To complete the proof it suffices to apply the method of mathematical induction.

**Property 6.** *We will denote by* $r_{k,k-1}$ *the points, where the functions* $\Gamma_k(r)-\Gamma_{k-1}(r)$, $k=1,2...m$, *are equal to zero. Then we have the inequality* $r_{k,k-1}\le r_{k+1,k}$, *and*

$$r_{2,1}=\sqrt{h}\frac{\ln a_{m-1}-\ln a_m}{\sqrt{a_{m-1}}-\sqrt{a_m}}+o(h)\ \ h\to 0,\ \left|\frac{\partial\Gamma_{k-1}(r)}{\partial r}\right|_{r=R}\le\left|\frac{\partial\Gamma_k(r)}{\partial r}\right|_{r=R}. \tag{3.7}$$

*Proof.* The functions $\Gamma_k(r)-\Gamma_{k-1}(r)$ satisfy the equation

$$\Delta(\Gamma_{k+1}(|x-x_0|)-\Gamma_k(|x-x_0|))-a_{m-k}\frac{\Gamma_{k+1}(|x-x_0|)-\Gamma_k(|x-x_0|)}{h}=$$

$$=-a_{m-k+1}\frac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{h}. \tag{3.8}$$

We construct an integral representation at the center of the sphere $K_{r_{k,k+1}}(x_0)$

$$\Gamma_{k+1}(0)-\Gamma_k(0)=\int\limits_{K_{r_{k,k+1}}(x_0)}a_{m-k+1}\frac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{h}\Gamma_1(|x-x_0|)dx=$$

$$=-\int\limits_{\partial K_{r_{k,k+1}}(x_0)}(\Gamma_{k+1}(|x-x_0|)-\Gamma_k(|x-x_0|))\frac{\partial\Gamma_1}{\partial n}ds,$$

$$\Gamma_1(|x-x_0|)=\frac{\sinh\sqrt{\frac{a_{m-k}}{h}}(r_{k+1,k}-|x-x_0|)}{4\pi|x-x_0|\sinh\sqrt{\frac{a_{m-k}}{h}}r_{k+1,k}}.$$

Taking into account that the function $\Gamma_{k+1}(r)-\Gamma_k(r)$ is equal to zero at the points $r=0, r_{k+1,k}=0$, we obtain

$$0=\int\limits_{K_{r_{k,k+1}}(x_0)}a_{m-k+1}\frac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{h}\Gamma_1(|x-x_0|)dx.$$

If $r_{k,k-1}>r_{k+1,k}$, then this equality is impossible.

Formula (3.1) implies

$$4\pi|x-x_0|\Gamma_1(|x-x_0|)=e^{-|x-x_0|\sqrt{\frac{a_m}{h}}}+O\left(e^{-R\sqrt{\frac{a_m}{h}}}\right),$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|)=\frac{a_{m-1}}{a_{m-1}-a_m}\left(e^{-|x-x_0|\sqrt{\frac{a_m}{h}}}-e^{-|x-x_0|\sqrt{\frac{a_{m-1}}{h}}}\right)+$$

$$+O\left(e^{-R\sqrt{\frac{a_{min}}{h}}}\right),\ \text{if } a_{m-1}\ne a_m,$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|)=\frac{|x-x_0|}{2\sqrt{a_m h}}e^{-|x-x_0|\sqrt{\frac{a_m}{h}}}+O\left(e^{-R\sqrt{\frac{a_m}{h}}}\right)\ \text{if } .a_{m-1}=a_m.$$

From here we obtain

$$r_{2,1}=\sqrt{h}\frac{\ln a_{m-1}-\ln a_m}{\sqrt{a_{m-1}}-\sqrt{a_m}}+o(h)\ \ h\to 0.$$

In particular, if $a_{m-1} = a_m$, then

$$r_{2,1} = \sqrt{h}\frac{2}{\sqrt{a_m}}.$$

The function $\Gamma_2(r) - \Gamma_1(r)$ changes the sign once. Therefore, as follows from the equations (3.8), the functions $\Gamma_k(r) - \Gamma_{k-1}(r)$ change the sign once too. It means that the inequalities (3.7) hold.

**Property 7.** *We have the following estimate*

$$\left|\frac{\partial \Gamma_N}{\partial n}\right| \le M_1 \left\{\frac{1}{q^N R}\exp\{-M_2\frac{R}{\sqrt{h}}\} + \exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}\right\}, \tag{3.9}$$

*where $q \ge 2$, and positive constants $M_1, M_2$ do not depend on $N, h, R$.*

*Proof.* Let us estimate integral

$$\left.\frac{\partial \Gamma_N}{\partial n}\right|_{\partial K_R(x_0)} = \frac{-ih}{2\pi a_N}\oint\limits_L \frac{\sqrt{z}}{2\pi R\sinh(\sqrt{z}R)}\frac{dz}{(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})}, \tag{3.10}$$

where $\partial L$ is the boundary of the domain

$$L = \{z : \varrho = |z| < \frac{(1+q)\max\limits_{1\le k\le N} a_k}{h}, Rez = b_0 > -\frac{\pi^2}{R^2}, b_0 < 0, \varrho > \frac{\max\limits_{1\le k\le N} a_k}{h}\}.$$

Let us represent the integral (3.10) as a sum of two terms: $I_1$ and $I_2$, where $I_1$ denotes the integral along the part of the curve $\partial L$ which is an arch of a circle, and $I_2$ denotes the integral along the part of the contour which lies inside the straight line $Rez = b_0$. Let us estimate the integral $I_1$. The estimates

$$\left|(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})\right| \ge \left|\frac{|z|h}{a_{\max}} - 1\right|^N \ge q^N,$$

$$\left|\sinh(\sqrt{z}R)\right| \ge \sinh[\sqrt{|z|}R\cos(argz/2)] = \sinh[\sqrt{|z|}R\cos\varphi],$$

where $\varphi \to \frac{\pi}{4}$, if $h \to 0$, imply

$$|I_1| \le c_1\frac{1}{q^N R}\exp\{-c_2\frac{R}{\sqrt{h}}\},$$

where the constants $c_1$ and $c_2$ do not depend on $h$. Let us now estimate the integral $I_2$. As $Rez = b_0$, we obtain

$$\left|(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})\right| \ge \left(1 + h\frac{|b_0|}{a_{\max}}\right)^{\frac{T}{h}}.$$

Assume $b_0 = -\frac{\pi^2}{2R^2}$. Then $\left|\frac{\sqrt{|z|}R}{\sinh\sqrt{z}R}\right| \le c_4$. Thus we obtain

$$|I_2| \le c_5 \exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}.$$

The constants $c_3, c_4, c_5$ do not depend on $h$.

### 4. Uniform estimates. Passage to the limit.

THEOREM 4.1. *Let the following conditions hold:*

$$\psi(x) \in C^{2+\alpha}(\overline{D}),\ \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T})$$

*and we will assume that corresponding consistency conditions hold at $t=0, x \in \partial D$. Then $\forall h>0, \forall \varepsilon>0$, such that*

$$\varepsilon^{2\nu} \geq h^{(2+\alpha)\sigma-1},\ \ \frac{1}{2+\alpha} < \sigma < \frac{1}{2}, \tag{4.1}$$

*there exists a constant $c$, which does not depend on $h, \varepsilon, k$, such that the following estimate holds:*

$$\max_{x\in\overline{D},1\leq k\leq N}\left|\frac{c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)}{h}\right| + \max_{x\in\overline{D_\varepsilon},1\leq k\leq N}\left|\frac{\partial c_k^\varepsilon(x)}{\partial x_i}\right| \leq c, \tag{4.2}$$

*where*

$$D_\varepsilon = D\setminus\overline{\omega_0(\varepsilon)},\ \ \omega_0(\varepsilon)=\{x\in D: 1<\psi(x)<1+\varepsilon\}.$$

*If the functions $\{F_k^\varepsilon(x)\}$ are twice differentiable and*

$$\| F_k^\varepsilon(x) \|_{C^{2+\alpha}(\overline{D})} \leq \frac{ch^{\alpha/2}}{\varepsilon^{2\nu}},$$

*then the following estimate holds*

$$\|c_k^{\varepsilon\prime\prime}(x)\|_{C^\alpha(\overline{D_\varepsilon^k})} \leq c, \tag{4.3}$$

*where*

$$D_\varepsilon^k = D\setminus\overline{\omega_k(\varepsilon)},\ \ \omega_k(\varepsilon)=\{x\in D: 1<c_k^\varepsilon(x)<1+\varepsilon\}.$$

*Proof.* Let $x_0$ be an arbitrary point in $D$. Let us rewrite the equation (2.9) in the following form

$$\Delta[c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)] - a_k^\varepsilon(x_0)\frac{c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)}{h} + a_{k-1}^\varepsilon(x_0)\frac{c_{k-1}^\varepsilon(x)-c_{k-2}^\varepsilon(x)}{h} =$$

$$= -(a_k^\varepsilon(x_0)-a_k^\varepsilon(x))\frac{c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)}{h} + (a_{k-1}^\varepsilon(x_0)-a_{k-1}^\varepsilon(x))\frac{c_{k-1}^\varepsilon(x)-c_{k-2}^\varepsilon(x)}{h} - $$

$$-F_k^\varepsilon(x)+F_{k-1}^\varepsilon(x).$$

In order to obtain the estimate for the functions $\{c_k^\varepsilon - c_{k-1}^\varepsilon\}$ we will use the integral representation (3.3). Denote by $K_R(x_0)$ the sphere with the center at the point $x_0$ of radius $R=h^\sigma$. We obtain

$$c_m^\varepsilon(x_0)-c_{m-1}^\varepsilon(x_0) = \int_{K_R(x_0)} [\Delta c_0(x)+F_1^\varepsilon(x)]\Gamma_m(|x-x_0|)dx - \sum_{k=1}^m \int_{\partial K_R(x_0)} [c_k^\varepsilon - c_{k-1}^\varepsilon]\frac{\partial\Gamma_{m-k+1}}{\partial n}ds +$$

$$+\sum_{k=1}^{m}\int\limits_{K_R(x_0)}(a_k^\varepsilon(x_0)-a_k^\varepsilon(x))\frac{c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)}{h}[\Gamma_{m-k+1}(|x-x_0|)-\Gamma_{m-k}(|x-x_0|)]dx+$$

$$+\sum_{k=1}^{m}\int\limits_{K_R(x_0)}F_k^\varepsilon(x)[\Gamma_{m-k+1}(|x-x_0|)-\Gamma_{m-k}(|x-x_0|)dx=I_1+I_2+I_3+I_4.$$

Let us estimate every term. The relations (2.3) and (3.4) imply

$$|I_1|\le\frac{h}{a_m^\varepsilon(x_0)}\max_{x\in\overline{D}}(|\Delta c_0(x)|+|F_1^\varepsilon(x)|)\le\frac{h}{a}\max_{x\in\overline{D}}|\Delta c_0(x)|+c_1\frac{h^{1+\alpha/2}}{\varepsilon^\nu}.$$

From (2.3), (3.7) and (3.9) it follows that

$$|I_2|\le\frac{T}{h}\max_{x\in\overline{D},1\le k\le N}|c_k^\varepsilon-c_{k-1}|M_1\left\{\frac{1}{q^NR}\exp\{-M_2\frac{R}{\sqrt{h}}\}+\exp\left\{-\frac{T\pi^2}{R^2a_{\max}}\right\}\right\}\le$$

$$\le\frac{c_2}{\varepsilon^\nu}\exp\left\{-\frac{c_3\varepsilon}{h^{2\sigma}}\right\}\le\frac{c_2}{h^{\alpha/4}}\exp\left\{-\frac{c_3}{h^{2\sigma-\alpha/2}}\right\}=o(h),$$

where the constants $c_2, c_3$ do not depend on $h,\varepsilon$.

From (2.3), (2.4) and from properties (4), (6) of the fundamental solutions we obtain

$$|I_3|\le\sum_{k=1}^{m}\int\limits_{K_R(x_0)}\frac{c|x-x_0|^\alpha}{\varepsilon^\nu}\max_{x\in\overline{D},1\le k\le N}|c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)|\frac{\Gamma_{m-k+1}-\Gamma_{m-k}}{h}dx\le$$

$$\le\int\limits_{K_{r_{2,1}}(x_0)}\frac{c|x-x_0|^\alpha}{\varepsilon^\nu}\max_{x\in\overline{D},1\le k\le N}|c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)|\frac{\Gamma_1(|x-x_0|)}{h}dx+$$

$$+\int\limits_{K_R\setminus K_{r_{2,1}}(x_0)}\frac{c|x-x_0|^\alpha}{\varepsilon^\nu}\max_{x\in\overline{D},1\le k\le N}|c_k^\varepsilon(x)-c_{k-1}^\varepsilon(x)|\frac{1}{h|x-x_0|}dx\le$$

$$\le ch\left(\frac{h^\alpha}{\varepsilon^{2\nu}}+\frac{h^{(2+\alpha)\sigma-1}}{\varepsilon^{2\nu}}\right).$$

The similar estimate takes place for $I_4$. Near the boundary of $D$ the equations (2.5) and (2.7) turn into linear equations with constant coefficients. Therefore, the appropriate estimates can be easily obtained.

We differentiate the equation (2.5) with respect to one of the variables $x_i$ and transform it to the following form:

$$\Delta c_k^{\varepsilon\prime}-\frac{b_k^\varepsilon(x_0)c_k^{\varepsilon\prime}-b_{k-1}^\varepsilon(x_0)c_{k-1}^{\varepsilon\prime}}{h}=-\frac{(b_k^\varepsilon(x_0)-b_k^\varepsilon(x))c_k^{\varepsilon\prime}-(b_{k-1}^\varepsilon(x_0)-b_{k-1}^\varepsilon(x))c_{k-1}^{\varepsilon\prime}}{h}-F_k^{\varepsilon\prime},$$

where $b_k^\varepsilon(x)=a+\chi_\varepsilon'(c_k^\varepsilon)$.

We use the property 3. It gives

$$c_m^{\varepsilon\prime}(x_0)=\int\limits_{K_R(x_0)}b_0^\varepsilon c_0^{\varepsilon\prime}(x)\frac{\Gamma_{m-k+1}(|x-x_0|)}{h}dx-\sum_{k=1}^{m}\int\limits_{\partial K_R(x_0)}c_k^{\varepsilon\prime}(x)\frac{\partial\Gamma_{m-k+1}}{\partial n}ds+$$

$$+\sum_{k=1}^{m}\int\limits_{K_R(x_0)}(b_k^\varepsilon(x_0)-b_k^\varepsilon(x))c_k^{\varepsilon\prime}(x)\frac{\Gamma_{m-k+1}(|x-x_0|)-\Gamma_{m-k}(|x-x_0|)}{h}dx+$$

$$\sum_{k=1}^{m}\int\limits_{K_R(x_0)}F_k^{\varepsilon\prime}(x)\Gamma_{m-k+1}(|x-x_0|)dx. \tag{4.4}$$

From this integral representation, applying the same reasoning as above, we obtain the second part of the estimate (4.2). The proof of (4.3) is quite similar.

THEOREM 4.2. *Let the following conditions be satisfied:*

$$\psi(x)\in C^{2+\alpha}(\overline{D}),\ \varphi(x,t)\in H^{2+\alpha,1+\alpha/2}(\overline{D_T}),\ \min_{x\in\overline{D}}|\nabla\psi(x)|>0$$

*and we will assume that corresponding consistency conditions hold at* $t=0, x\in\partial D$. *Then* $\forall h>0, \forall\varepsilon>0$, *such that*

$$\varepsilon^{2\nu}\geq h^{(2+\alpha)\sigma-1},\quad \frac{1}{2+\alpha}<\sigma<\frac{1}{2},$$

*there exists a constant* $c$, *which does not depend on* $h,\varepsilon,k$, *such that the following estimate holds:*

$$|\nabla c_k^\varepsilon(x)|\geq c>0\ \ \forall x\in D,\ \ k=1,2,...N. \tag{4.5}$$

*Proof.* Notice that the first term in the right-hand side of (4.4) can be estimated as follows

$$\left|\int\limits_{K_R(x_0)}b_0^\varepsilon c_0^{\varepsilon\prime}(x)\frac{\Gamma_{m-k+1}(|x-x_0|)}{h}dx\right|\geq c\min_{x\in\overline{D}}|\psi'(x)|+o(h^{\alpha/2}).$$

Similarly to the previous theorem it is possible to prove that all other terms in the right-hand side of (4.4) have limits equal to zero when $h\to 0$.

Let the function $\eta(x,t)\in C^{2,1}(\overline{D})$ be equal to zero on $(\partial D\times(0,T))\cup(D\times(t=T))$, $\eta_k(x)=\eta(x,kh)$. We multiply (2.5) and (2.7) by $h\eta_k(x)$, integrate it over $D$, and take the sum over $k$ from 1 to $N$. After simple transformations we obtain

$$h\sum_{k=1}^{N}\int\limits_{D}\{\nabla c_k^\varepsilon\nabla\eta_k+\frac{a}{h}(c_k^\varepsilon-c_{k-1}^\varepsilon)\eta_k-\chi_\varepsilon(c_k^\varepsilon)\frac{\eta_k-\eta_{k-1}}{h}+(u_k^\varepsilon+f_k^\varepsilon)\eta_k\}dx=0,$$

$$h\sum_{k=1}^{N}\int\limits_{D}\{\nabla u_k^\varepsilon\nabla\eta_k+\chi_\varepsilon(c_k^\varepsilon)u_k^\varepsilon\eta_k\}dx=0.$$

Let us denote by $\overline{c}(x,t,h,\varepsilon)$, $\overline{u}(x,t,h,\varepsilon)$ the piecewise linear interpolations of the functions $\{c_k^\varepsilon(x)\},\{u_k^\varepsilon(x)\}$ with respect to the variable $t$,

$$c(x,t)=\lim_{\varepsilon,h\to 0}\overline{c}(x,t,h,\varepsilon),\ u(x,t)=\lim_{\varepsilon,h\to 0}\overline{u}(x,t,h,\varepsilon),$$

where $h,\varepsilon$ satisfy the conditions (4.1). The possibility of passage to the limit follows from the statements proved above. As a result we obtain

THEOREM 4.3. *Let the following conditions be satisfied:*

$$\psi(x) \in C^{2+\alpha}(\overline{D}),\ \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T}),\ \min_{x\in\overline{D}} |\nabla\psi(x)| > 0$$

*and we will assume that corresponding consistency conditions hold at* $t = 0, x \in \partial D$. *Then* $\forall T > 0$ *there exists a solution of the problem (1.1)-(1.4) and*

$$c(x,t) \in C(\overline{D_T}) \cap \left(H^{2+\alpha,1+\alpha/2}(\overline{\Omega_T}) \times H^{2+\alpha,1+\alpha/2}(\overline{G_T})\right),\ u(x,t) \in C^{1+\alpha,1+\alpha/2}(\overline{D_T});$$

*the free boundary is given by the graph of a function from* $H^{2+\alpha,1+\alpha/2}$ *class.*


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