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## ASYMPTOTIC ANALYSIS OF THE SPECTRAL NEUMANN PROBLEM IN THICK MULTI-STRUCTURE OF TYPE 3:1:1

The spectral Neumann problem is considered in a thick multi-structure, which is the union of some three-dimensional domain (the junction's body) and a large number of  $\varepsilon$ -periodically situated thin cylinders along some curve (the joint zone) on the boundary of junction's body. The asymptotic behaviour (as  $\varepsilon \rightarrow 0$ ) of the eigenvalues and eigenfunctions is investigated. Three spectral problems form asymptotics for the eigenvalues and eigenfunctions of this problem, namely, the spectral Neumann problem in junction's body; some spectral problem in a plane domain, which is filled up by the thin cylinders in the limit passage (each eigenvalue of this problem has infinite multiplicity); and the spectral problem for some singular integral operator given on the joint zone. The Hausdorff convergence of the spectrum is proved, the leading terms of asymptotics are constructed (as  $\varepsilon \rightarrow 0$ ) and asymptotic estimates are justified for the eigenvalues and the eigenfunctions.

### 1. Introduction and statement of the problem.

A thick multi-structure (or thick junctions)  $\Omega_\varepsilon$  of type  $k : m : d$  is the union of some domain  $\Omega_0 \in \mathbb{R}^n$  (junction's body) and a large number of  $\varepsilon$ -periodically situated thin domains along some manifold (the joint zone) on the boundary of junction's body. Here  $\varepsilon$  is a small parameter and the type  $k : m : d$  of the junction refers to the limiting dimensions of the body, the joint zone, and each attached thin domains. This classification was given by T.A. Mel'nyk and S.A. Nazarov in [4]-[8].

Such multi-structures are prototypes of widely used engineering constructions such as long bridges on supports, frameworks of houses, industrial installations, spaceship grids as well as many other physical and biological systems with very distinct characteristic scales.

Thick junctions have special character of the connectedness, namely, there are points in a thick junction, which are at a short distance of order  $\mathcal{O}(\varepsilon)$ , but the length of all curves, which connect these points in the junction, is order  $\mathcal{O}(1)$ . As a result, many new difficulties appear in the asymptotic investigation of boundary value problems in thick multi-structures, for example, the loss of the coercitivity of differential operators in the limit passage as  $\varepsilon \rightarrow 0$ , the absence of extension operators that would be bounded uniformly in  $\varepsilon$  in the Sobolev space  $W_2^1$ , the power behaviour of junction-layer solutions at infinity.

The goal of the study of boundary-value problems in thick multi-structures is to describe the asymptotic behaviour of the solutions as  $\varepsilon \rightarrow 0$ , i.e., when the number of attached thin domains infinitely increases and their thickness vanishes. The reader can find extensive reviews on this theme in [4]-[8].

It follows from these papers that the asymptotic behaviour of the solution to a boundary-value problem in a thick junction essentially depends on the type of this junction. In addition, the type determines the scheme of investigation. Thick multi-structures of types  $k : m : d$ , where  $k - m \neq 1$ , are more strongly degenerated junctions and there were no full asymptotic investigations of spectral problems and boundary-value problems in such thick junctions for the present. If we consider some boundary value problem in a thick junction of type  $k : m : d$  ( $k - m \neq 1$ ), then any transmission conditions are not reasonable on the joint zone for the corresponding limiting problems in the Sobolev space  $W_2^1$ . This is the principle difficulty in contrast to boundary-value problems in thick junctions of types  $k : m : d$ ,  $k - m = 1$ .

**1.1. Statement of the problem.** A model thick junction  $\Omega_\varepsilon$  of type 3 : 1 : 1 consists

of the body

$$\Omega_0 = \{x \in \mathbb{R}^3 : |x_1| < h, |x_3| < b, -\varphi(x') < x_2 < 0\},$$

where  $x' = (x_1, x_3)$ , and a large number of  $\varepsilon$ -periodically situated thin cylinders

$$G_j(\varepsilon) = \{x : 0 \leq x_2 < l, \varepsilon^{-1}(x_1 - \varepsilon j, x_3) \in \omega\}, \quad j = 0, \pm 1, \dots, \pm N,$$

which are attached to the body along the segment  $I_h = \{x : |x_1| < h, x_2 = x_3 = 0\}$ , i.e.,  $\Omega_\varepsilon = \Omega_0 \cup G(\varepsilon)$ , where  $G(\varepsilon) = \cup_{j=-N}^N G_j(\varepsilon)$ . Here  $\varphi \in C^\infty([-h, h] \times [-b, b])$ ,  $\varphi \geq 1$ ;  $b > 1$ ;  $\omega$  is a plane domain with the smooth boundary and it is symmetric relative to the axis  $\{x_1 = 0\}$ . In addition,  $(0, 0) \in \omega \subset \{x' : x_1^2 + x_3^2 < \rho_0 < 1/4\}$ . The number  $N$  is a large positive integer, therefore the value  $\varepsilon = 2h/(2N + 1)$  is a small parameter, which characterizes the distance between the neighboring thin cylinders and their thickness.

The following spectral Neumann problem

$$-\Delta_x u(x, \varepsilon) = \lambda(\varepsilon) u(x, \varepsilon), \quad x \in \Omega_\varepsilon; \quad \partial_\nu u(x, \varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon, \quad (1)$$

is considered in this paper. Here  $\partial_\nu = \partial/\partial\nu$  is the derivative in the direction of outward normal to the surface  $\partial\Omega_\varepsilon$ .

A spectral stiff problem in  $\Omega_\varepsilon$  with concentrated masses on the thin cylinders was studied by the author and S.A. Nazarov in [8]. In this problem, passage to the limit as  $\varepsilon \rightarrow 0$  was accompanied by mass concentration within each thin cylinder and by the infinite increase of the stiffness of the cylinders. On the other hand, presence of a large parameter in the transmission condition at the joint zone leads to some simplification, namely, the corresponding limit spectral problem is reduced to the spectral problem for some singular integral operator  $J$  given on the segment  $I_h$ . This operator acts in the Hörmander spaces  $H_{\log,1}(I_h) \mapsto H_{\log,-1}(I_h)$ . Only the low frequency convergence of the eigenvalues and some series of the high frequency convergence of the spectrum were studied in [8].

As distinct from [8], three spectral problems form asymptotics of the eigenvalues and eigenfunctions of problem (1), namely, the spectral Neumann problem in junction's body  $\Omega_0$ ; the spectral problem in the rectangle  $D = I_h \times (0, l)$ , which is filled up by the thin cylinders in the limit passage (each eigenvalue of this problem has infinite multiplicity); and the spectral problem for the integral operator  $J$ .

**1.2. The corresponding equivalent operator problem.** As usual,  $\lambda(\varepsilon)$  is an eigenvalue of the problem (1) if there is a nonzero function (eigenfunction)  $u \in \mathcal{H}_\varepsilon := H^1(\Omega_\varepsilon)$  satisfying the integral identity

$$\langle u(\cdot, \varepsilon), v \rangle_\varepsilon = (\lambda(\varepsilon) + 1) (u(\cdot, \varepsilon), v)_{\Omega_\varepsilon}, \quad (2)$$

where  $\langle \cdot, \cdot \rangle_\varepsilon$  and  $(\cdot, \cdot)_{\Omega_\varepsilon}$  are the standard scalar products in  $\mathcal{H}_\varepsilon$  and in  $L_2(\Omega_\varepsilon)$  respectively. Define the operator  $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$  by the formula

$$\langle A_\varepsilon u, v \rangle_\varepsilon = \int_{\Omega_\varepsilon} uv \, dx \quad \text{for all } u, v \in \mathcal{H}_\varepsilon. \quad (3)$$

It is easy to verify that this operator is self-adjoint, positive, and compact. Now we can re-write problem (1) as the spectral problem for the operator  $A_\varepsilon$ :

$$A_\varepsilon u(\cdot, \varepsilon) = (\lambda(\varepsilon) + 1)^{-1} u(\cdot, \varepsilon). \quad (4)$$

Thus, for each fixed  $\varepsilon$  all eigenvalues of problem (1) can be put in order

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \dots \leq \lambda_n(\varepsilon) \leq \dots \rightarrow +\infty \quad \text{as } n \rightarrow +\infty, \quad (5)$$

with the classical convention of repeated values. The corresponding eigenfunctions can be orthonormalized by the following way

$$\int_{\Omega_0} u_n(x, \varepsilon) u_m(x, \varepsilon) dx + \int_{G_\varepsilon} u_n(x, \varepsilon) u_m(x, \varepsilon) dx = \delta_{n,m}, \quad \{n, m\} \in \mathbb{N}_0, \quad (6)$$

where  $\delta_{n,m}$  is the Kronecker symbol.

The aim is to study the asymptotic behaviour of the eigenvalues  $\{\lambda_n(\varepsilon) : n \in \mathbb{N}_0\}$  and eigenfunctions  $\{u_n(\cdot, \varepsilon) : n \in \mathbb{N}_0\}$  as  $\varepsilon \rightarrow 0$ .

## 2. Formal asymptotic representations.

By virtue of the minimax principle for eigenvalues it is easy to prove the following estimate

$$\lambda_n(\varepsilon) = \min_{E \in E_n} \max_{0 \neq v \in E} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 dx}{\int_{\Omega_\varepsilon} v^2 dx} \leq \frac{(\pi n)^2}{l^2}, \quad (7)$$

where  $E_n$  is the set of all subspaces of  $\mathcal{H}_\varepsilon$  with the dimension  $n$ .

Using inequality (7), condition (6) and the integral identity (2), we deduce the estimates for the eigenfunctions

$$\int_{\Omega_\varepsilon} |\nabla u_n(x, \varepsilon)|^2 dx = \lambda_n(\varepsilon) \leq \pi^2 n^2 l^{-2}. \quad (8)$$

The asymptotic behaviour of an eigenfunction of problem (1) depend essentially on the energy concentration of the corresponding proper-oscillation. This energy is proportional to the value  $\int_{\Omega_\varepsilon} |\nabla u_n(x, \varepsilon)|^2 dx$ . Due to estimate (8) and condition (6), the energy of some proper-oscillation is uniformly bounded with respect to  $\varepsilon$  and can be concentrated either in the junction's body or in the thin cylinders or uniformly in these sets for  $\varepsilon$  small enough.

We fix the index  $n$  for  $\lambda_n(\varepsilon)$  and  $u_n$ . In what follows we don't write this index. Having observed that  $L_2$ -norm of some bounded function on the thin cylinders is order  $\mathcal{O}(\varepsilon^{\frac{1}{2}})$ , we propose the following asymptotic representations for the eigenfunction  $u$  :

$$u(x, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^{\frac{k-1}{2}} v_{\frac{k-1}{2}}^+(x_1, x_2, \frac{x_1}{\varepsilon} - j, \frac{x_3}{\varepsilon}) \quad (9)$$

in the thin cylinder  $G_j(\varepsilon)$  ( $j = 0, \pm 1, \dots, \pm N$ );

$$u(x, \varepsilon) \sim v_0^-(x) + \sum_{k=2}^{\infty} \varepsilon^{\frac{k-1}{2}} v_{\frac{k-1}{2}}^-(x) \quad (10)$$

in the junction's body  $\Omega_0$ ;

$$u(x, \varepsilon) \sim w_0(x_1, \frac{x}{\varepsilon}) + \sum_{k=2}^{\infty} \varepsilon^{\frac{k-1}{2}} w_{\frac{k-1}{2}}(x_1, \frac{x}{\varepsilon}) \quad (11)$$

near the joint zone  $I_h$ ; and for the eigenvalue

$$\lambda(\varepsilon) \sim \sum_{k=1}^{\infty} \varepsilon^{\frac{k-1}{2}} \tau_{\frac{k-1}{2}}. \quad (12)$$

The expansions (9) and (10) are usually called *outer expansions*, the expansion (11) is called *inner expansions*.

REMARK 1. As we will show further, some coefficients of (9)-(12) depend on  $\ln \varepsilon$ . But, since the power scale  $\{\varepsilon^{\frac{k}{2}} : k \in \mathbb{N}\}$  gives the main terms of the asymptotic behaviour, we will not indicate the logarithmic dependence of these coefficients.

**2.1. The limiting spectral problem in the junction's body.** Let the energy of the corresponding proper-oscillation is concentrated in the junction's body  $\Omega_0$ , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_0} |\nabla u(x, \varepsilon)|^2 dx > 0$$

(we recall the index  $n$  is omitted). Since the corresponding values  $\lambda(\varepsilon)$  and  $\|u(\cdot, \varepsilon)\|_{\mathcal{H}_\varepsilon}$  are bounded with respect to  $\varepsilon$ , we can choose a subsequence of  $\{\varepsilon\}$  (still denoted by  $\{\varepsilon\}$ ) such that

$$u(\cdot, \varepsilon) \rightarrow v^- \neq 0 \text{ weakly in } H^1(\Omega_0) \text{ and } \lambda(\varepsilon) \rightarrow \mu \text{ as } \varepsilon \rightarrow 0.$$

Passing to the limit in the integral identity (2) with a test function, which vanishes on the thin cylinders and in a neighbourhood of the joint zone, we obtain that  $\mu$  is an eigenvalue and  $v^-$  is the corresponding eigenfunction of the following spectral Neumann problem

$$-\Delta v^-(x) = \mu v^-(x), \quad x \in \Omega_0; \quad \partial_\nu v^-(x) = 0, \quad x \in \partial\Omega_0. \quad (13)$$

Its spectrum  $\sigma_{\Omega_0}$  consists of non-decreasing sequence of finite-to-one eigenvalues

$$0 = \mu_0 < \mu_1 \leq \dots \leq \mu_n \leq \dots \rightarrow +\infty. \quad (14)$$

**2.2. Limiting relations in the rectangle  $D = (-h, h) \times (0, l)$ .** Assuming for the moment that the functions  $v_{\frac{k-1}{2}}^+$  in (9) are smooth, we write their Taylor series with respect to the first argument at the point  $x_1 = j\varepsilon$  and pass the "fast" variables  $\xi_1 = \varepsilon^{-1}x_1$ ,  $\xi_3 = \varepsilon^{-1}x_3$ . Then (9) takes the form

$$u(\varepsilon, x) \sim \sum_{k=0}^{\infty} \varepsilon^{k-\frac{1}{2}} V_{k-\frac{1}{2}}^j(x_2, \xi_1 - j, \xi_3) + \sum_{k=0}^{\infty} \varepsilon^k V_k^j(x_2, \xi_1 - j, \xi_3), \quad (15)$$

where

$$V_{-\frac{1}{2}}^j = v_{-\frac{1}{2}}^+(\varepsilon j, x_2, \xi_1 - j, \xi_3), \quad V_0^j = v_0^+(\varepsilon j, x_2, \xi_1 - j, \xi_3), \quad (16)$$

$$V_{k-\frac{1}{2}}^j = v_{k-\frac{1}{2}}^+(\varepsilon j, x_2, \xi_1 - j, \xi_3) + \sum_{m=1}^k \frac{(\xi_1 - j)^m}{m!} \frac{\partial^m v_{k-m-\frac{1}{2}}^+}{\partial x_1^m}(\varepsilon j, x_2, \xi_1 - j, \xi_3), \quad (17)$$

$$V_k^j = v_k^+(\varepsilon j, x_2, \xi_1 - j, \xi_3) + \sum_{m=1}^k \frac{(\xi_1 - j)^m}{m!} \frac{\partial^m v_{k-m}^+}{\partial x_1^m}(\varepsilon j, x_2, \xi_1 - j, \xi_3). \quad (18)$$

Here and further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion.

We substitute the series (15) and (12) in the equation of problem (1) and in the boundary conditions on the lateral surface  $S_j(\varepsilon)$  of the cylinder  $G_j(\varepsilon)$ . Since the Laplace operator takes the form  $\Delta_x = \varepsilon^{-2}\Delta_{\xi'} + \partial^2/\partial x_2^2$  in the variables  $\xi^h = (\xi_1, \xi_3)$  and  $x_2$ , the collection of coefficients of the same powers of  $\varepsilon$  brings to the following problems

$$\Delta_{\xi'} V_{\frac{k-1}{2}}^j = 0 \quad \text{in } \omega(j) = \{\xi : (\xi_1 - j, \xi_3) \in \omega\}, \quad \partial_{\nu(\xi')} V_{\frac{k-1}{2}}^j = 0 \quad \text{on } \partial\omega(j), \quad (19)$$

for  $k = 0, 1, 2, 3$ . Here  $\partial_{\nu(\xi')}$  is the derivative along the normal  $\nu(\xi')$ ,  $\xi' \in \partial\omega(j)$ . It follows from (19) that  $V_{\frac{k-1}{2}}^j$ ,  $k = 0, 1, 2, 3$ , are independent of  $\xi_1, \xi_3$ . Next we obtain the following problems

$$-\Delta_{\xi'} V_{\frac{3}{2}}^j = \partial_{x_2 x_2}^2 V_{-\frac{1}{2}}^j + \tau_0 V_{-\frac{1}{2}}^j \quad \text{in } \omega(j), \quad \partial_{\nu(\xi')} V_{\frac{3}{2}}^j = 0 \quad \text{on } \partial\omega(j); \quad (20)$$

$$-\Delta_{\xi'} V_2^j = \partial_{x_2 x_2}^2 V_0^j + \tau_0 V_0^j + \tau_{\frac{1}{2}} V_{-\frac{1}{2}}^j(x_2) \quad \text{in } \omega(j), \quad \partial_{\nu(\xi')} V_2^j = 0 \quad \text{on } \partial\omega(j), \quad (21)$$

where  $\partial_{x_i} = \partial/\partial x_i$ ,  $\partial_{x_i x_q}^2 = \partial^2/\partial x_i \partial x_q$ . The solvability conditions for the problems (20) and (21) respectively read as follows

$$-|\omega| \partial_{x_2 x_2}^2 V_{-\frac{1}{2}}^j = \tau_0 |\omega| V_{-\frac{1}{2}}^j, \quad -|\omega| \partial_{x_2 x_2}^2 V_0^j = |\omega| (\tau_0 V_0^j + \tau_{\frac{1}{2}} V_{-\frac{1}{2}}^j), \quad x_2 \in (0, l) \quad (22)$$

where  $|\omega|$  is the Lebesgue measure of the plane domain  $\omega$ .

Taking into account (16) and the fact that the points  $\varepsilon j$ ,  $j = 0, \pm 1, \dots, \pm N$ , form an  $\varepsilon$ -net for the interval  $I_h = (-h, h)$ , we can extend equations (22), defined initially on  $2N + 1$  segments, to the entire rectangle  $D = (-h, h) \times (0, l)$ :

$$-\partial_{x_2 x_2}^2 v_{-\frac{1}{2}}^+(x^0) = \tau_0 v_{-\frac{1}{2}}^+(x^0), \quad -\partial_{x_2 x_2}^2 v_0^+(x^0) = \tau_0 v_0^+(x^0) + \tau_{\frac{1}{2}} v_{-\frac{1}{2}}^+(x^0), \quad x^0 \in D. \quad (23)$$

Here  $x^0 = (x_1, x_2)$ . In accordance with the Neumann conditions on the external bases of the thin cylinders  $G_0(\varepsilon), G_{\pm 1}(\varepsilon), \dots, G_{\pm N}(\varepsilon)$ , we supplement (23) by the conditions

$$\partial_2 v_{-\frac{1}{2}}^+(x_1, l) = 0, \quad \partial_2 v_0^+(x_1, l) = 0, \quad x_1 \in I_h. \quad (24)$$

For other terms of asymptotic expansions (15) and (12) we can similarly obtain the following recurrence relations

$$-\partial_{x_2 x_2}^2 V_{k+\frac{1}{2}}^j(x_2) - \tau_0 V_{k+\frac{1}{2}}^j(x_2) = \sum_{m=0}^k (\tau_{k-m+1} V_{m-\frac{1}{2}}^j + \tau_{k-m+\frac{1}{2}} V_m^j), \quad x_2 \in (0, l), \quad (25)$$

$$-\partial_{x_2 x_2}^2 V_k^j(x_2) - \tau_0 V_k^j(x_2) = \tau_{\frac{1}{2}} V_{k-\frac{1}{2}}^j + \sum_{m=0}^{k-1} (\tau_{k-m} V_m^j + \tau_{k-m+\frac{1}{2}} V_{m-\frac{1}{2}}^j), \quad x_2 \in (0, l), \quad (26)$$

and the following conditions  $\partial_2 V_{k+\frac{1}{2}}^j(l) = 0$ ,  $\partial_2 V_k^j(l) = 0$ ;  $k \in \mathbb{N}_0$ . As before we extend these equations, defined initially on  $2N + 1$  segments, to the interior of  $D$ .

**2.3. The limiting spectral problem in the rectangle  $D$ .** Comparing the first term of the outer expansion (9) on the thin cylinders with the first term of the outer expansion

(10) in junction's body, we conclude that  $v_{-\frac{1}{2}}^+(x_1, 0) = 0$ ,  $x_1 \in I_h$ . In addition,  $v_{-\frac{1}{2}}^+$  satisfies the first relations (23) and (24). Thus,  $v_{-\frac{1}{2}}^+$  must be either trivial or an eigenfunction of the following problem

$$\begin{aligned} -\partial_{x_2 x_2}^2 v(x^0) &= \tau_0 v(x^0), \quad x^0 = (x_1, x_2) \in D; \\ v(x_1, 0) &= 0, \quad \partial_{x_2} v(x_1, l) = 0, \quad x_1 \in I_h. \end{aligned} \quad (27)$$

It should be noted that eigenvalues of problem (27) form the sequence

$$\sigma_D = \left\{ \left( \frac{\pi(2m-1)}{2l} \right)^2 : m \in \mathbb{N} \right\} \quad (28)$$

and each eigenvalue has infinite multiplicity. Indeed, for any function  $\psi \in C^\infty(I_h)$  the function  $v(x^0) = \psi(x_1)\varphi_m(x_2)$  satisfies the problem (27) if the function

$$\varphi_m(x_2) = \sin(\pi l^{-1}(m-2^{-1})x_2), \quad x_2 \in (0, l). \quad (29)$$

**2.4. The limiting spectral problem for problem (1).** Problems (13) and (27) forms the limiting spectral problem for problem (1). Let us write the corresponding operator spectral problem for the limit problem. Consider a Hilbert vector space  $\mathcal{V}_0 := L_2(\Omega_0) \times L_2(D)$  with the scalar product

$$(\bar{u}, \bar{v})_{\mathcal{V}_0} = \int_{\Omega_0} u^{(1)}v^{(1)} dx + \int_D u^{(2)}v^{(2)} dx^0, \quad \forall \bar{u} = (u^{(1)}, u^{(2)}), \bar{v} = (v^{(1)}, v^{(2)}) \in \mathcal{V}_0.$$

By  $\mathcal{H}_0$  denote the following anisotropic Sobolev vector-space  $H^1(\Omega_0) \times \{u \in L_2(D) : \exists \partial_{x_2} u \in L_2(D), u|_{I_h} = 0\}$  with the scalar product

$$(\bar{u}, \bar{v})_{\mathcal{H}_0} = (u^{(1)}v^{(1)})_{H^1(\Omega_0)} + \int_D (\partial_{x_2} u^{(2)} \partial_{x_2} v^{(2)} + u^{(2)}v^{(2)}) dx^0, \quad \forall \bar{u}, \bar{v} \in \mathcal{H}_0.$$

Define the linear operator  $A_0 : \mathcal{H}_0 \mapsto \mathcal{H}_0$  by the formula

$$(A_0 \bar{u}, \bar{v})_{\mathcal{H}_0} = (\bar{u}, \bar{v})_{\mathcal{V}_0}, \quad \forall \bar{u}, \bar{v} \in \mathcal{H}_0.$$

It is easy to verify that  $A_0$  is bounded, self-adjoint and positive. Since the imbedding  $\mathcal{H}_0 \subset \mathcal{V}_0$  is not compact, the operator  $A_0$  is non-compact.

The spectral problem  $A_0 \bar{v} = (\mu + 1)^{-1} \bar{v}$  in  $\mathcal{H}_0$  is equivalent to the spectral problem

$$\begin{aligned} -\Delta v^{(1)}(x) &= \mu v^{(1)}(x), \quad x \in \Omega_0, \quad \partial_\nu v^{(1)}(x) = 0, \quad x \in \partial\Omega_0; \\ -\partial_{x_2}^2 v^{(2)}(x^0) &= \mu v^{(2)}(x^0), \quad x^0 \in D, \quad v^{(2)}(x_1, 0) = \partial_{x_2} v^{(2)}(x_1, l) = 0, \quad x_1 \in I_h, \end{aligned} \quad (30)$$

whose spectrum is the union  $\sigma_{\Omega_0} \cup \sigma_D$  (the spectra of problems (13) and (27)). In this paper we assume that  $\sigma_{\Omega_0} \cap \sigma_D = \emptyset$ .

**2.5. Junction-layer problems.** Take the thin cylinder  $G_0(\varepsilon)$  and introduce the 3-dimensional "rapid" variables  $\xi = \varepsilon^{-1}x$ . After passage to  $\varepsilon = 0$ , in the coordinates  $\xi$  the set  $G_j(\varepsilon)$  transforms to the semi-infinite cylinder  $\omega \times (0, +\infty)$ , and the domain  $\Omega_0$  to the half-space  $\{\xi : \xi_2 < 0\}$ . The periodic location of the thin cylinders suggests that the functions of junction-layer type should be taken 1-periodic relative to  $\xi_1$ . Therefore, we consider the

union  $\Pi$  of the semi-cylinder  $\Pi_+ = \{\xi : \xi' = (\xi_1, \xi_3) \in \omega, \xi_2 \geq 0\}$  and the half-layer  $\Pi_- = \{\xi : |\xi_1| < 1/2, \xi_2 < 0\}$  as the basic domain, where problems for the junction layer will be posed. Let us investigate some properties of the solutions to the following junction-layer problem

$$\begin{aligned} -\Delta_\xi Z(\xi) &= F(\xi), & \xi \in \Pi, \\ \partial_{\nu(\xi)} Z(\xi) &= B(\xi), & \xi \in S = \partial\Pi_+ \setminus \omega, \\ \partial_{\xi_2} Z(\xi', 0) &= 0, & \xi' \in \mathbb{R}^2 \setminus \bar{\omega}, \quad \xi_1 \in (-\frac{1}{2}, \frac{1}{2}), \\ \partial_{\xi_1}^k Z(-\frac{1}{2}, \xi_2, \xi_3) &= \partial_{\xi_1}^k Z(+\frac{1}{2}, \xi_2, \xi_3), & \xi_2 < 0, \quad k = 0, 1, \end{aligned} \quad (31)$$

where  $\nu(\xi) = (\nu_1(\xi'), \nu_2(\xi'))$  is the outward normal derivative on  $S$ . To formulate the existence theorem for problem (31) we introduce the "energy" space  $\mathcal{H}(\Pi)$ , which is the closure of the space  $C_{0,\sharp}^\infty(\bar{\Pi})$  by the norm

$$\|u\|_{\mathcal{H}} = (\|\nabla_\xi u\|_{L_2(\Pi)}^2 + \|d^{-1}u\|_{L_2(\Pi)}^2)^{1/2},$$

where  $d$  is a smooth weight function, positive in  $\bar{\Pi}$  and it is equal to  $\rho \ln \rho$  for  $\rho = (\xi_2^2 + \xi_3^2)^{1/2} > 2$  and  $\xi_2 < 0$ , and to  $\xi_2$  for  $\xi_2 > 2$ ;  $C_{0,\sharp}^\infty(\bar{\Pi})$  is a space of smooth function, which are finite with respect to  $\xi_2, \xi_3$  and satisfy the last periodic condition in (31).

A function  $Z \in \mathcal{H}(\Pi)$  is called a weak solution to problem (31) if for all functions  $v \in \mathcal{H}$  the following integral identity holds:  $\int_\Pi \nabla_\xi Z \cdot \nabla_\xi v \, d\xi = \int_\Pi Fv \, d\xi + \int_S Bv \, d\sigma_\xi$ .

LEMMA 1. *If  $dF \in L_2(\Pi)$ ,  $dB \in L_2(S)$ , and*

$$\int_\Pi F(\xi) \, d\xi + \int_S B(\xi) \, d\sigma_\xi = 0, \quad (32)$$

*then there exists a weak solution  $Z \in \mathcal{H}(\Pi)$  to problem (31). The solution  $Z$  is defined up to an additive constant.*

The proof of this lemma repeats the proof of Theorem 1 in [10], Lemma 4.1 ([11]), and Lemma 3.1 ([7]), where similar problems were considered.

REMARK 2. Due to the symmetry of  $\Pi$ , there exists a unique weak solution to problem (31), which is even or odd relative to  $\xi_1$  if so are  $F$  and  $B$  (see Remark 3.2 in [7]).

Similarly as in [10, 5, 11, 8], we establish the asymptotic properties of solutions to problem (31).

LEMMA 2. *If  $F \in C_0^\infty(\bar{\Pi})$ ,  $B \in C^\infty(\bar{S})$ ,  $B(\xi) = 0$  for  $\xi_2 \geq R_0 > 0$ , and condition (32) holds, then there exists the unique solution  $Z \in \mathcal{H}(\Pi)$  to problem (31), which admits the differentiated asymptotic representation*

$$Z(\xi) = \begin{cases} \mathcal{O}(\exp(-\delta_1 \xi_2)) & \text{as } \xi_2 \rightarrow +\infty \quad (\delta_1 > 0), \\ a_1 + \mathcal{O}(\rho^{-1}) & \text{as } \rho \rightarrow \infty. \end{cases} \quad (33)$$

*If the functions  $F$  and  $B$  are odd with respect to  $\xi_1$ , then the solution  $Z \in \mathcal{H}(\Pi)$  decays exponentially at infinity.*

In the next sections, we will see that the leading terms of (11) have the form

$$\begin{aligned} \varepsilon^{\alpha-1} v_{\alpha-1}^+(x_1, 0) + \varepsilon^{\alpha-\frac{1}{2}} v_{\alpha-\frac{1}{2}}^+(x_1, 0) + \varepsilon^\alpha \left( V_\alpha(x_1, 0) + W_1(\xi) \partial_{x_1} v_{\alpha-1}^+(x_1, 0) + \right. \\ \left. + |\omega| W_2(\xi) \partial_{x_2} v_{\alpha-1}^+(x_1, 0) \right) \Big|_{\xi=\varepsilon^{-1}x} + \dots, \quad (\alpha = \frac{1}{2}, \text{ or } \alpha = 1). \end{aligned} \quad (34)$$

Substituting (34) in problem (1) and collecting the coefficients of the same power of  $\varepsilon$ , we arrive problems for the functions  $W_1$  and  $W_2$ . So, the function  $W_2$  must be a non-trivial solution of the homogeneous problem (31), the function  $W_1$  must be a solution of problem (31) with right-hand sides  $F(\xi) \equiv 0$ ,  $B(\xi) = -\nu_1(\xi')$ .

**COROLLARY 1.** *The homogeneous problem (31) has nontrivial solution  $W_2$ , which does not belong to  $\mathcal{H}(\Pi)$ ; this solution has the differential asymptotics*

$$W_2(\xi) = \begin{cases} |\omega|^{-1} \xi_2 + \mathcal{O}(\exp(-\delta_2 \xi_2)) & \text{as } \xi_2 \rightarrow +\infty, \\ -\pi^{-1} \ln \rho + c_\omega + \mathcal{O}(\rho^{-1}) & \text{as } \rho \rightarrow \infty \quad (\xi_2 < 0), \end{cases} \quad (35)$$

and is even relative to  $\xi_1$ ; moreover,  $\int_\omega W_2(\xi', 0) d\xi' = 0$ . Here the constant  $\delta_2 > 0$ .

*Problem (31) with right-hand sides  $F(\xi) \equiv 0$ ,  $B(\xi) = -\nu_1(\xi')$  has nontrivial solution  $W_1$ , which does not belong to  $\mathcal{H}(\Pi)$ ; this solution has the differential asymptotics*

$$W_1(\xi) = \begin{cases} -\xi_1 + \mathcal{O}(\exp(-\delta_1 \xi_2)) & \text{as } \xi_2 \rightarrow +\infty, \\ \mathcal{O}(\exp(-\delta_1 \rho)) & \text{as } \rho \rightarrow \infty \quad (\xi_2 < 0), \end{cases} \quad (36)$$

and is odd relative to  $\xi_1$  ( $\delta_1 > 0$ ).

*Proof.* Such a nontrivial solution to the homogeneous problem can be found in the form

$$W_2(\xi) = -\pi^{-1} \chi_0(\rho) \ln \rho + |\omega|^{-1} \chi_0(\xi_2) \xi_2 + Z_2(\xi),$$

where  $\chi_0(t)$ ,  $t \in \mathbb{R}$ , is a smooth cut-off function equal to 0 on  $(-\infty, 1]$  and to 1 on  $[2, +\infty)$ ;  $Z_2$  is the solution to problem (31) with right-hand sides

$$F(\xi) = -\pi^{-1} [\Delta, \chi_0](\ln \rho) + |\omega|^{-1} [\Delta, \chi_0](\xi_2), \quad B(\xi) \equiv 0,$$

here  $[A, B] = AB - BA$  is the commutator of the operator  $A$  and  $B$ . The existence of  $Z_2$  follows from Lemma 1 ( $F$  has compact support and  $\int_\Pi F(\xi) d\xi = 0$ ). It remains to observe that  $F$  is even relative to  $\xi_1$  and to apply Remark 2 and Lemmas 2. The absence of a constant term in the asymptotics of  $W_2$  as  $\xi_2 \rightarrow +\infty$  leads to the zero mean over the cross-section of the cylinder  $\Pi_+$ .

Analogously we prove the second part. The solution  $W_1$  with asymptotics (36) is sought in the form  $W_1(\xi) = -\chi_0(\xi_2) \xi_1 + Z_1(\xi)$ , where  $Z_1$  is the solution to problem (31) with right-hand sides  $F(\xi) = -\chi_0''(\xi_2) \xi_1$ ,  $B(\xi) = -(1 - \chi_0(\xi_2)) \nu_1(\xi')$ .

**2.6.** A singular solution of the Neumann problem in  $\Omega_0$ . Substituting (10), (12) in (1) and collecting the coefficients of the powers of  $\varepsilon$ , we arrive the following problem for the leading term  $v_0^-$

$$\begin{aligned} -\Delta_x v_0^-(x) &= \tau_0 v_0^-(x), & x \in \Omega_0, \\ \partial_\nu v_0^-(x) &= 0, & x \in \partial\Omega_0 \setminus I_h. \end{aligned} \quad (37)$$



It is obvious that the eigenvalues (14) of problem (13) and the corresponding eigenfunctions, which is smooth in  $\overline{\Omega_0}$ , satisfy (37).

The method of matched asymptotic expansions that we use here requires that the solution of problem (37) and (31) admit similar representations (respectively, near the segment  $I_h = \{x : |x_1| < h, x_2 = x_3 = 0\}$  and near infinity). Since a logarithmic component is present in the solution to the homogeneous problem (31) (see (35)), we are forced to deal with a solution to (37) that has singularities on  $I_h$ . The basic principles of constructing such solutions were formulated in [2, 3, 10]; here we use the approach suggested in [1]. From results of this paper it follows the following lemma.

LEMMA 3. Let  $\tau_0$  be an eigenvalue of the Neumann problem (13) with the multiplicity  $q$  (the case  $q = 0$  does not exclude),  $\Phi_1, \dots, \Phi_q$  are the corresponding eigenfunctions orthonormalized in  $L_2(\Omega_0)$ . Let  $b$  be a function from  $W_\infty^1(I_h)$  such that  $(b, \Phi_i|_{I_h})_{L_2(I_h)} = 0$ .

Then there exists a solution to problem (37) with the following asymptotics

$$v_0^-(x) := \mathbb{U}(x; b) = -\pi^{-1}b(x_1) \ln r + (Jb)(x_1) + \sum_{i=1}^q \alpha_i \Phi_i|_{I_h}(x_1) + \mathcal{O}(r(1 + |\ln r|)), \quad r = \sqrt{x_1^2 + x_3^2} \rightarrow 0, \quad (38)$$

where  $J$  is the integral operator defined by the formula

$$(Jb)(x_1) = \int_{-h}^h (b(s) - b(x_1))G(x_1, 0, 0; s; \lambda_0) ds + \sum_{\pm} \pm \int_0^{\pm h} (b(s) - b(x_1))G(x_1, 0; \pm 2h - s; \lambda_0) ds + b(x_1)\{(\pi)^{-1} \ln 2 - g_-^0(x_1 - 0, x_1) - g_+^0(x_1 + 0, x_1)\}. \quad (39)$$

In (39),  $G$  is the  $4h$ -periodic in  $x_1$  solution to the problem

$$\begin{aligned} -\Delta_x G(x; s; \tau_0) - \tau_0 G(x; s; \tau_0) &= -\sum_{i=1}^q \Phi_i(s) \Phi_i(x), & x \in \Omega_0^*, \\ \partial_\nu G(x; s; \tau_0) &= 0, & x \in \Gamma_0^* \setminus \{s\}, \\ (G(\cdot; s; \tau_0), \Phi_i)_{\Omega_0^*} &= 0, \quad i = 1, \dots, q, & s \in I_{2h}, \\ G(x; s; \tau_0) &= \frac{1}{2\pi|x-s|} + \mathcal{O}(|\ln|x-s||), & x \rightarrow s, \end{aligned} \quad (40)$$

where the symbol  $s$  denotes both the point  $(s, 0, 0) \in I_{2h}$  and its coordinate on the segment  $I_{2h}$ ;  $g$  is a  $4h$ -periodic primitive for the function  $I_{2h} \ni x_1 \rightarrow G(x_1, 0, 0; s; \tau_0)$ . Due to the last condition in (40),  $g$  can be represented in the form

$$g(x_1, s) = \pm(2\pi)^{-1} \ln|x_1 - s| \pm g_\pm^0(x_1, s) \quad \text{for } \pm x_1 > \pm s.$$

The properties of the operator (39) were studied in [1]; we discuss them briefly. Let  $H_{\ln}^s(I_h)$  denote the space of restrictions to  $I_h$  of the  $4h$ -periodic functions belonging to the Hörmander space  $H_{\ln}^s(I_{2h})$  with the weight function  $\mu_s(\xi) = (1 + \ln|\xi| + |\ln|\xi||)^s$ . In other words, the norm in  $H_{\ln}^s(I_{2h})$  is glued from the norms

$$\|\gamma\|_{H_{\ln}^s(I_{2h})(\mathbb{R})} = \left( \int_{\mathbb{R}} \mu_s^2(\xi) |\mathcal{F}\gamma(\xi)|^2 d\xi \right)^{1/2},$$

where  $\mathcal{F}\gamma$  denotes the Fourier transform of  $\gamma$ . The embedding  $H_{\ln}^s(I_{2h}) \subset L_2(I_{2h})$  is compact for  $s > 0$ . From [1] it follows that operator  $J : H_{\ln}^{\frac{1}{2}}(I_h) \mapsto H_{\ln}^{-\frac{1}{2}}(I_h) := H_{\ln}^{\frac{1}{2}}(I_h)^*$  is continuous, symmetric (as an operator in  $L_2(I_h)$  with the domain of definition  $H_{\ln}^{\frac{1}{2}}(I_h)$ ), discrete (for  $\lambda_0 > 0$  large enough its resolvent is a compact operator in  $L_2(I_h)$ ), the eigenvalues of  $J$  form a sequence

$$\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_p \dots \rightarrow -\infty, \quad (41)$$

and corresponding eigenfunctions  $\{b_p : p \in \mathbb{N}\}$  belong to  $C^\infty(I_h)$  and can be orthonormalized in  $L_2(I_h)$ .

### 3. Asymptotic approximations and estimates.

**3.1. The case  $\tau_0 \in \sigma_{\Omega_0} \setminus \sigma_D$ .** Let  $\tau_0$  be an eigenvalue of problem (13) with multiplicity  $q$ ;  $\Phi_1, \dots, \Phi_q$  are the corresponding eigenfunctions orthonormalized in  $L_2(\Omega_0)$ . In this case the first term in (15) is a linear combination

$$v_0^-(x) = \sum_{i=1}^q a_i^{(0)} \Phi_i(x), \quad x \in \Omega_0, \quad (42)$$

where the constants  $a_i^{(0)}$ ,  $i = 1, \dots, q$ , will be defined further.

Applying the method of matched asymptotic expansions to the leading terms of (10) and (11) and to the leading terms of (11) and (9), we obtain that  $v_{-\frac{1}{2}}^+(x_1, 0) = 0$  and  $w_0(x_1, \xi) = v_0^-(x_1, 0, 0) = v_0^+(x_1, 0)$ . Since  $\tau_0$  is not an eigenvalue of problem (27), then  $v_{-\frac{1}{2}}^+$  must be trivial. As a result,  $v_0^+$  is the unique solution to the following problem

$$\begin{aligned} -\partial_x^2 v_0^+(x_1, x_2) &= \tau_0 v_0^+(x_1, x_2), \quad (x_1, x_2) \in G_0, \\ v_0^+(x_1, 0) &= v_0^-(x_1, 0, 0), \quad \partial_{x_2} v_0^+(x_1, l) = 0, \quad x_1 \in I_h. \end{aligned} \quad (43)$$

Obviously,

$$v_0^+(x_1, x_2) = \frac{v_0^-(x_1, 0, 0)}{\cos(\sqrt{\tau_0} l)} \cos \sqrt{\tau_0}(l - x_2), \quad (x_1, x_2) \in G_0. \quad (44)$$

Matching the next following terms of (9) - (11), we get that all terms in (9), (10), (11) and (12) at  $k = 2p$ ,  $p \in \mathbb{N}$ , are trivial and these asymptotic expansions are expansions with respect to the nonnegative integral powers of  $\varepsilon$ .

For the function  $v_1^-$ , we obtain the following relations

$$-\Delta_x v_1^- - \tau_0 v_1^- = \tau_1 v_0^-, \quad x \in \Omega_0; \quad \partial_\nu v_1^- = 0, \quad x \in \partial\Omega_0 \setminus I_h, \quad (45)$$

and because of (34) ( $\alpha = 1$ ), (35), (36), and (44),  $v_1^-$  must have the logarithmic singularity

$$\begin{aligned} v_1^-(x) &= -\pi^{-1} |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \gamma_0(x_1) \ln r + \\ &+ |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \gamma_0(x_1) (\pi^{-1} \ln \varepsilon + c_\omega) + V_1(x_1, 0) \quad \text{as } r \rightarrow 0, \end{aligned} \quad (46)$$

where  $\gamma_0(x_1) = v_0^-(x_1, 0, 0) = v_0^+(x_1, 0) = \sum_{i=1}^q a_i^{(0)} \Phi_i|_{I_h}$ ,  $x_1 \in I_h$ .

Comparing problem (45) with problem (37) and using the representation (38) with  $\alpha_i = 0$ ,  $i = 1, \dots, q$ , we can state that there exists such a singular solution

$$v_1^-(x) = |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \mathcal{U}(x; \gamma_0), \quad x \in \Omega_0, \quad (47)$$

to problem (46) if

$$\tau_1 \bar{a}^{(0)} = -2 |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \mathbb{M} \bar{a}^{(0)}, \quad (48)$$

where  $\bar{a}^{(0)} = (a_1^{(0)}, \dots, a_q^{(0)})$  and  $\mathbb{M} = \{(\Phi_i, \Phi_j)_{I_h}, \quad i, j = 1, \dots, q\}$  is the Gram matrix, which is symmetric and nonnegative.

From the spectral problem (48) we define  $\tau_1$  in (12) and the constants  $\{a_i^{(0)}\}$  in (42). The spectral problem (48) has  $q$  eigenvalues and we assume that they are simple, i.e.,  $\tau_1^{(1)} < \dots < \tau_1^{(q)}$ ; the corresponding eigenvectors  $\bar{a}_i^{(0)} = (a_{i1}^{(0)}, \dots, a_{iq}^{(0)}) \quad i = 1, \dots, q$ , can be orthonormalized by the following way  $\bar{a}_i^{(0)} \cdot \bar{a}_j^{(0)} = \delta_{ij}, \quad i, j = 1, \dots, q$ . So, let  $\tau_1$  be an eigenvalue of (48),  $\bar{a}^{(0)} = (a_1^{(0)}, \dots, a_q^{(0)})$  is the corresponding normalized eigenvector. Then the function  $v_0^-$  is defined and  $\|v_0^-\|_{L_2(\Omega_0)} = 1$ .

The singular solution  $v_1^-$  has the asymptotics

$$v_1^-(x) \sim |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \left( (J\gamma_0)(x_1) - \pi^{-1} \gamma_0(x_1) \ln r + \mathcal{O}(r(1 + \ln r)) \right), \quad r \rightarrow 0$$

(see Lemma 3). Comparing this asymptotics with (46) and taking into account the matching principle, we deduce the second boundary condition

$$V_1(x_1, 0) = |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \left( (J\gamma_0)(x_1) - \gamma_0(x_1) (\pi^{-1} \ln \varepsilon + c_\omega) \right)$$

for the equation (26) at  $k = 1$  and unique determine the function  $V_1$ .

As result, we have defined the leading terms of the outer expansions (9), (10), the inner expansion (34) ( $\alpha = 1$ ), and the expansion (12) and can construct a global asymptotic approximation  $U(\cdot, \varepsilon)$  belonging to  $H^1(\Omega_\varepsilon)$ :

$$\begin{aligned} U(x, \varepsilon) &= (1 - \chi(r/\varepsilon)) (v_0^-(x) + \varepsilon v_1^-(x)) + \\ &+ \chi(r) \left( v_0^+(x_1, 0) + \varepsilon (V_1(x_1, 0) + W_1(x/\varepsilon) \partial_1 v_0^+(x_1, 0) + |\omega| W_2(x/\varepsilon) \partial_2 v_0^+(x_1, 0)) \right) - \\ &- \chi(r) \left( 1 - \chi\left(\frac{r}{\varepsilon}\right) \right) \left( \gamma_0(x_1) + \varepsilon |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0} l) \left( (J\gamma_0)(x_1) - \frac{1}{\pi} \gamma_0 \ln r \right) \right), \quad x \in \Omega_0, \quad (49) \end{aligned}$$

$$\begin{aligned} U(x, \varepsilon) &= v_0^+(x_1, x_2) + \varepsilon \left( V_1(x_1, x_2) + Y_1(x_1/\varepsilon) \partial_1 v_0^+(x_1, x_2) + \right. \\ &\left. + \chi(x_2) \sum_{i=1}^2 (|\omega|^{\delta_{i,2}} W_i\left(\frac{x}{\varepsilon}\right) - Y_1\left(\frac{x_1}{\varepsilon}\right) \delta_{i,1} - \varepsilon^{-1} x_2 \delta_{i,2}) \partial_i v_0^+(x_1, 0) \right), \quad x \in G_\varepsilon, \quad (50) \end{aligned}$$

where  $\chi$  is a smooth cut-off function such that  $\chi(t) = 1$  if  $|t| < R_0 := 4^{-1} \min\{l, b, \min \varphi\}$ , and  $\chi(t) = 0$  if  $|t| > 2R_0$ ;  $Y_1(t) = -t + [t + \frac{1}{2}]$  ( $[t]$  is the integral part of  $t$ ).

Putting  $U(\cdot, \varepsilon)$  and  $\tau_0 + \varepsilon \tau_1$  in problem (1) instead of  $u(\cdot, \varepsilon)$  and  $\lambda(\varepsilon)$  respectively and estimating the residuals, we deduce the inequality

$$\|U(\cdot, \varepsilon) - (1 + \tau_0 + \varepsilon \tau_1) A_\varepsilon U(\cdot, \varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq c \varepsilon^{3/2}, \quad (51)$$

where the constant  $c$  is independent of  $\varepsilon$ . The first part of Lemma 12 in [12] yields the estimate

$$\min_{m \in \mathbb{N}} \left| \frac{1}{1 + \tau_0 + \varepsilon \tau_1} - \frac{1}{1 + \lambda_m(\varepsilon)} \right| \leq \frac{\|A_\varepsilon U - (1 + \tau_0 + \varepsilon \tau_1)^{-1} U\|_\varepsilon}{\|U\|_\varepsilon} \leq c \varepsilon^{3/2}, \quad (52)$$

which partly justifies the asymptotics constructed above for the solutions of the spectral problem (1).

**3.2. The case  $\tau_0 \in \sigma_D \setminus \sigma_{\Omega_0}$ .** In the previous subsections everything has been prepared in order to determine terms of the asymptotic expansions for the eigenvalue  $\lambda(\varepsilon)$  and eigenfunction  $u(\cdot, \varepsilon)$  of the original perturbed problem (1) for this case. But, as distinct from before considered case, the outer and inner asymptotic expansions for the eigenfunction are expansions with respect to the scale  $\{\varepsilon^{k-\frac{1}{2}} : k \in \mathbb{N}_0\}$ , and the eigenvalue  $\lambda(\varepsilon)$  is expanded in an integer power series about  $\varepsilon$ .

Let  $\tau_0$  be an eigenvalue of problem (27). Then  $v_{-\frac{1}{2}}^+(x^0) = \gamma_{-\frac{1}{2}}(x_1)\varphi(x_2)$ ,  $x^0 \in D$ , the corresponding eigenfunction, where  $\varphi$  is defined by formula (29). The  $\gamma_{-\frac{1}{2}}$  is involved in the determination of the first term  $v_{\frac{1}{2}}^-$  of the outer expansion (10), namely,

$$v_{\frac{1}{2}}^-(x) = |\omega| \sqrt{\tau_0} \mathbb{U}(x; \gamma_{-\frac{1}{2}}), \quad x \in \Omega_0,$$

where  $\mathbb{U}$  is defined by (38).

For the function  $V_{\frac{1}{2}}$  we obtain the following problem

$$\begin{aligned} -\partial_{x_2}^2 V_{\frac{1}{2}}(x^0) - \tau_0 V_{\frac{1}{2}}(x^0) &= \tau_1 v_{-\frac{1}{2}}^+(x^0), \quad x_2 \in (0, l); \\ V_{\frac{1}{2}}(x_1, 0) &= |\omega| \sqrt{\tau_0} \left( (J\gamma_{-\frac{1}{2}})(x_1) - \gamma_{-\frac{1}{2}}(x_1) (\pi^{-1} \ln \varepsilon + c_\omega) \right), \quad \partial_{x_2} V_{\frac{1}{2}}(x_1, l) = 0. \end{aligned}$$

The solvability condition for this problem reads as follows

$$\tau_1 \gamma_{-\frac{1}{2}}(x_1) = -\frac{2|\omega|}{l} \tau_0 \left( (J\gamma_{-\frac{1}{2}})(x_1) - \gamma_{-\frac{1}{2}}(x_1) (\pi^{-1} \ln \varepsilon + c_\omega) \right), \quad x_1 \in I_h. \quad (53)$$

Relation (53) is the spectral problem for the operator  $J : H_{\ln}^{\frac{1}{2}}(I_h) \mapsto H_{\ln}^{-\frac{1}{2}}(I_h)$ , which is continuous, symmetric, and discrete (see (39)). Thus  $\gamma_{-\frac{1}{2}} = b_k$  and

$$\tau_1 = \tau_1^{(k)} = 2|\omega| l^{-1} \tau_0 (\pi^{-1} \ln \varepsilon + c_\omega - \Lambda_k), \quad k \in \mathbb{N}, \quad (54)$$

where  $\Lambda_k$  is an eigenvalue and  $b_k$  is the corresponding eigenfunction of the operator  $J$ .

Now, similarly as in § 3.1, we construct a global asymptotic approximation  $U(\cdot, \varepsilon)$  belonging to  $H^1(\Omega_\varepsilon)$  for the eigenfunction  $u(\cdot, \varepsilon)$ :

$$\begin{aligned} U(x, \varepsilon) &= \varepsilon^{\frac{1}{2}} \left( (1 - \chi(r/\varepsilon)) v_{\frac{1}{2}}^-(x) + \chi(r) (V_{\frac{1}{2}}(x_1, 0) + \sqrt{\tau_0} |\omega| b_k(x_1) W_2(\xi)) - \right. \\ &\quad \left. - \chi(r) (1 - \chi(r/\varepsilon)) |\omega| \sqrt{\tau_0} b_k(x_1) (\Lambda_k - \pi^{-1} \ln r) \right), \quad x \in \Omega_0; \quad (55) \end{aligned}$$

$$\begin{aligned} U(x, \varepsilon) &= \varepsilon^{-\frac{1}{2}} v_{-\frac{1}{2}}^+(x_1, x_2) + \varepsilon^{\frac{1}{2}} \left( V_{\frac{1}{2}}(x_1, x_2) + Y_1\left(\frac{x_1}{\varepsilon}\right) \partial_1 v_{-\frac{1}{2}}^+(x_1, x_2) + \right. \\ &\quad \left. + \chi(x_2) \sqrt{\tau_0} b_k(x_1) (|\omega| W_2(x/\varepsilon) - \varepsilon^{-1} x_2) \right), \quad x \in G_\varepsilon, \quad (56) \end{aligned}$$

and prove estimates similarly to the estimates (51) and (52).

**3.3. Justification.** To justify the constructed asymptotic approximations we use the scheme proposed in [9]. The basic spaces and the corresponding operators are specified in subsections 1.2 and 2.4. Special extension operator is constructed similarly as in [8]. Conditions C5 and C6, in fact, has been verified in the previous subsections: the result of the action of the operator  $R_\varepsilon$  is the construction of the approximation function  $U$  on the basis of an eigenfunction of the limit spectral problem (30). Applying this scheme to problems (1) and (30), we get the following theorems.

**THEOREM 1.** *Let  $\tau_0 \in \sigma_{\Omega_0} \setminus \sigma_D$  and its multiplicity is equal to  $q$ ;  $\tau_1^{(i)}$ ,  $i = 1, \dots, q$ , are eigenvalues of (48). Then there exist exactly  $q$  eigenvalues of problem (1) with the following asymptotics*

$$\lambda^{(i)}(\varepsilon) \sim \tau_0 + \varepsilon\tau_1^{(i)} + \mathcal{O}(\varepsilon^{3/2}), \quad i = 1, \dots, q.$$

*For the corresponding eigenfunctions we have the following estimates*

$$\|u^{(i)}(\cdot, \varepsilon) - U^{(i)}(\cdot, \varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq C_i \varepsilon^{3/2}, \quad i = 1, \dots, q,$$

*where  $U^{(i)}$  is defined by (49) and (50).*

**THEOREM 2.** *Let  $\tau_0 \in \sigma_D \setminus \sigma_{\Omega_0}$ ;  $\tau_1^{(k)}(\ln \varepsilon)$ ,  $k \in \mathbb{N}$ , are defined by (54). Then for any there exists an eigenvalues  $\lambda^{(k)}(\varepsilon)$  of problem (1) with the following asymptotics*

$$\lambda^{(k)}(\varepsilon) \sim \tau_0 + \varepsilon\tau_1^{(k)}(\ln \varepsilon) + \mathcal{O}(\varepsilon^{3/2}), \quad k \in \mathbb{N}.$$

*Let  $\Lambda_k$  be a simple eigenvalue of the integral operator  $J$ . Then*

$$\|u^{(k)}(\cdot, \varepsilon) - U^{(k)}(\cdot, \varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq C_k \varepsilon^{3/2},$$

*where  $U^{(k)}$  is defined by (55) and (56).*

**THEOREM 3. (THE HAUSDORFF CONVERGENCE)** *Only the points of the spectrum of problem (30) are accumulation points for the spectrum of problem (1) as  $\varepsilon \rightarrow 0$ .*

The eigenvalues  $\{\lambda_n(\varepsilon)\}$  at fixed indices  $n$ , are usually called *low eigenvalues*; the corresponding proper-oscillations are called *low frequency oscillations*.

**DEFINITION 1.** The value  $\mathcal{T} := \sup_{n \in \mathbb{N}} \overline{\lim}_{\varepsilon \rightarrow 0} \lambda_n(\varepsilon)$  is called the threshold of low eigenvalues of problem (1).

**THEOREM 4. (LOW-FREQUENCY CONVERGENCE)** *The threshold  $\mathcal{T} = \pi^2/4l^2$ .*

*Let  $\mu_0 < \mu_1 \leq \dots \leq \mu_{m_0}$  be eigenvalues of problem (13), which are less than  $\mathcal{T}$ , and  $\mu_{m_0+1} \geq \mathcal{T}$ . Then  $\forall n = 1, 2, \dots, m_0$*

$$\lambda_n(\varepsilon) \rightarrow \mu_n \quad \text{as } \varepsilon \rightarrow 0.$$

*For any  $n \geq m_0$*

$$\lambda_n(\varepsilon) \rightarrow \mathcal{T} \quad \text{as } \varepsilon \rightarrow 0.$$

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