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METHOD OF DISCRETE SOURCES FOR ELLIPTIC PROBLEMS ARISING IN SUPERCOLLIDER SIMULATION

An electrodynamical system of energy accumulation (accelerating section of a supercollider) is synthesized. Some model elliptic problems (boundary, inverse spectral, scattering) concerning synthesis of the optimal structure are considered. Numerical algorithms basing on the method of discrete sources and singular value decomposition technique are developed. Numerical examples are presented.

Introduction.

Electron (positron) accelerating structures are attractive to be fed with a wave flow converging onto the structure axis [1]. A proper structure might represent a periodic set of coaxial radial-corrugated discs which compose a Bragg reflection cavity [2] (Fig. 1). The principal problem is to optimize parameters of the electrodynamical system considered. The aim is to accumulate minimum of the RF energy in a paraxial domain under a given accelerating gradient (the value of the electronsynchronous harmonic of an electric field). The principal problem generates a lot of particular ones.

The accelerating gradient assumed being fixed, the RF energy accumulated within the channel is minimized, if the RF field is composed of only the electronsynchronous space harmonic and counter-propagating harmonic of the same amplitude. In other words, the field in the paraxial domain represents a homogeneous (in radial coordinate) standing wave (in longitudinal coordinate). This field is kept by the metallic surface, the equation of which (see (18) in Section 3) is obtained in [3]. This surface seems to be optimal.

Another "elementary" problem is the inverse spectral one: to find profiles of metallic discs (Fig. 1) providing zero eigenvalue for Helmholtz type operator under homogeneous boundary conditions. A constructive algorithm for this zero eigenvalue problem basing on the method of discrete sources (MDS) [4] together with singular value decomposition (SVD) technique [5] is presented in [6]. Several types of boundary profiles for amplifying waveguide channels are considered. Optimal parameters for these profiles are obtained.

Concordance of a feeding wave flow of minimum amplitude with a standing wave of a given amplitude in the paraxial domain is the next model problem. This coupling is provided by a system of two neighboring channels with asymmetric grooves (Bragg reflectors). Optimal parameters of neighboring grooves in the plane case are presented in [7]. There exists a numerical procedure permitting to correct these parameters for the cylindrical case.

In [8] a scattering problem for two joint plane waveguides of different widths is considered. Results concerning the same problem for a plane waveguide with protuberances will be published in near future. Both problems allow exact solutions, so they give understanding of MDS possibilities. This experience is useful for domains of rather arbitrary shapes.

This paper continues previous investigations. It deals with various elliptic problems (boundary, inverse spectral, scattering) arising in supercollider simulation.

1. Elliptic problems arising in supercollider simulation.

In this paper we present three model problems arising in supercollider simulation. The first is the classical boundary elliptic problem, the second is the inverse spectral one and the last is the scattering one. For 3D azimuth-symmetrical case all problems are governed by a scalar equation of Helmholtz type for the azimuth component of a magnetic field H_φ (wavelength λ being normalized to 2π, wave number k is equal to unity):

$$\Delta H_\varphi + \left(1 - \frac{1}{r^2}\right) H_\varphi = 0, \tag{1}$$

where $\Delta = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)$ is Laplacian in the case of azimuth symmetry, r and z are radial and axial (longitudinal) coordinates correspondingly.

In this case both the longitudinal component E_z of an electric field and the radial one E_r are expressed in terms of the azimuth component of a magnetic field H_φ:

$$E_z = \frac{\partial H_\varphi}{\partial r} + \frac{1}{r} H_\varphi, \tag{2}$$

$$E_r = -\frac{\partial H_\varphi}{\partial z}. \tag{3}$$

The model domain (Fig. 2) is defined by periodicity and symmetry of the structure (Fig. 1) in z coordinate: $0 \le z \le \pi$. This domain is an infinite strip (for the scattering problem) $0 \le r \le \infty$ or a finite rectangular (for other problems) $0 \le r \le r_2$ cut out by some curve

$$R(r, z) = 0 \tag{4}$$

which is the equation of a longitudinal cross-section of an elementary disc.

The boundary conditions reflect:

1) equality to zero of the tangential component of an electric field on a metallic surface (4)

$$\left(\text{rot } \overrightarrow{H_\varphi}\right)_\tau \equiv \frac{\partial R}{\partial z}\left(-\frac{\partial H_\varphi}{\partial z}\right) + \frac{\partial R}{\partial r}\left[\frac{1}{r}\frac{\partial (r\, H_\varphi)}{\partial r}\right] = 0; \tag{5}$$

2) periodicity and symmetry of the structure in z coordinate

$$\frac{\partial H_\varphi}{\partial z} = 0 \quad \text{at} \quad z = 0, \pi; \tag{6}$$

3) equality to zero of the azimuth component of a magnetic field on the structure axis

$$H_\varphi = 0 \quad \text{at} \quad r = 0. \tag{7}$$

One more boundary condition in r coordinate depends on the problem considered (see Fig. 2). For the classical boundary problem it is

$$H_\varphi = \begin{cases} +1, & 0 < z < z_1 \le \pi/2 \\ -1, & \pi/2 < z_2 < z \le \pi \end{cases} \quad \text{at} \quad r = r_2\,, \tag{8}$$

where z_1 and z_2 are the system parameters.

For the spectral problem it is a homogeneous condition

$$\frac{\partial H_\varphi}{\partial r} + \frac{1}{r} H_\varphi = 0 \quad \text{at} \quad r = r_2\,. \tag{9}$$

As for the scattering problem, the boundary condition is defined as follows. Input (from infinity) wave $H_{\varphi\,\mathrm{in}}$ is assumed to be a convergent cylindrical wave

$$H_{\varphi\,\mathrm{in}}(r) = \frac{i}{4} H_1^{(2)}(r). \tag{10}$$

The boundary condition (7) produces a regular part of a scattering solution — a divergent cylindrical wave

$$H_{\varphi\,\mathrm{reg}}(r) = \frac{i}{4} H_1^{(1)}(r). \tag{11}$$

Here, in (10)–(11), $H_1^{(1)}$, $H_1^{(2)}$ are the Hankel functions.

The boundary curve (4) generates an additional scattering wave $H_{\varphi\,\mathrm{add}}(r,z)$, finding of which is the essence of a scattering problem.

So, an unknown solution H_φ is sought as a sum of the input wave and the scattering one:

$$H_\varphi = H_{\varphi\,\mathrm{in}} + H_{\varphi\,\mathrm{scat}}, \tag{12}$$

while $H_{\varphi\,\mathrm{scat}}$ represents a sum of two components

$$H_{\varphi\,\mathrm{scat}} = H_{\varphi\,\mathrm{reg}} + H_{\varphi\,\mathrm{add}}. \tag{13}$$

For the sake of uniqueness, $H_{\varphi\,\mathrm{scat}}$ must be divergent in infinity, i. e., it satisfies the radiation condition

$$\frac{\partial H_{\varphi\,\mathrm{scat}}}{\partial r} \quad iH_{\varphi\,\mathrm{scat}} = O\left(\frac{1}{r}\right) \quad \text{at} \quad r = \infty. \tag{14}$$

Note, that for the scattering case it is necessary to use complex representation for electromagnetic fields.

2. Method of discrete sources together with singular value decomposition technique.

To find a nontrivial solution of Eq. (1) under corresponding boundary conditions, we use the MDS (method of discrete sources). An unknown wave H_φ is sought in the form

$$H_\varphi(r,z) = H_{\varphi\,\mathrm{in}} + H_{\varphi\,\mathrm{reg}} + \sum_{i=1}^{N} d_i\, G(r,z,\rho_i,\zeta_i), \tag{15}$$

where $H_{\varphi\,\mathrm{in}}$ and $H_{\varphi\,\mathrm{reg}}$ are known functions (see (10)–(11)), $G(r,z,\rho,\zeta)$ is the Green function of the operator (depending on the problem considered) in the envelope domain $\{0 \le z \le \pi, 0 \le r \le r_2, \infty\}$ (see Fig. 2); ρ_i, ζ_i are the source coordinates (outside of a domain, where a solution is sought); d_i are the amplitudes of sources; N is the number of sources used.

Substituting (15) into (5) for all collocation points (points, where the boundary condition (5) is verified) on the curve (4), we arrive at a set of linear algebraic equations (SLAE) for unknown values of source amplitudes d_i:

$$\begin{gathered}\sum_{i=1}^{N} d_i \left\{ \left[\tfrac{\partial G}{\partial r}(r_j,z_j,\rho_i,\zeta_i) + \tfrac{1}{r_j} G(r_j,z_j,\rho_i,\zeta_i) \right] \cos\beta_j - \right. \\ \left. - \tfrac{\partial G}{\partial z}(r_j,z_j,\rho_i,\zeta_i) \sin\beta_j \right\} = F(r)\cos\beta_j \quad j = \overline{1,N}\end{gathered} \tag{16}$$

where $\mathrm{tg}\,\beta = r'(s)/z'(s)$, $(r(s), z(s))$ is the parametric form presentation of the boundary curve (4); $F(r)$ is the function depending on the problem considered, e. g., $F(r) \equiv 0$ in the case of the spectral problem (boundary condition (9)).

A way of placement of source and collocation points is the problem of great importance. We do not pay attention here to this problem so far as principal recipes are presented in our previous publications [6], [8], [9]. We note only, that the primary idea is a uniform placement for collocation points on the boundary, while source points are placed near the boundary (outside of the domain considered) at the tops of isosceles triangles (other tops of triangles are collocation points on the boundary) (Fig. 3).

At first sight, for boundary and scattering problems there is no difference what numerical procedure for solving (16) is used. Originally we used LU method and had serious difficulties. Understanding had become later, after consideration of the spectral problem (boundary condition (9)). LU method is useless for a homogeneous SLAE (HSLAE). For this case there exists a very powerful technique known as singular value decomposition (SVD). It is based on the following theorem of linear algebra (we formulate this theorem for a real square matrix only): any matrix A can be written as the product of an orthogonal matrix U, a diagonal matrix W with positive or zero elements (the singular values), and the transpose of an orthogonal matrix V:

$$A = UWV^T, \qquad U^TU = V^TV = I. \tag{17}$$

The decomposition (17) can always be done, no matter how singular the matrix is, and it is unique up to making the same permutations of the columns of U and V and elements of W.

SVD allows to solve effectively HSLAE in the case that a matrix is singular. A solution is obtained immediately by means of SVD: any column of V whose corresponding element of W is zero yields a solution of HSLAE.

The reason for choosing of SVD technique for inhomogeneous SLAE is as follows. SVD gives a clear diagnosis of the situation: how close to singular the matrix A is (that is our case: we synthesize a resonator). Moreover, for the set of simultaneous equations $Ax = b$ SVD explicitly constructs orthonormal bases for the nullspace (subspace of x that is mapped to zero $Ax = 0$) and the range (subspace of b that can be "reached" by A, in the sense that there exists some x which is mapped there) of a matrix. These bases are the columns of V whose same numbered elements of W are zero and the columns of U whose same numbered elements of W are nonzero respectively. SVD diagnostics helps us to understand some strange solutions we obtain for boundary value problem (1)–(8) (see Section 3). That is why we prefer SVD technique for solving (16).

3. Numerical results.

Preliminary results concerning the inverse spectral problem (1)–(7), (9) are published in [6]. There presents a constructive algorithm for synthesis of resonance domains (to fit a boundary of a domain to such a shape that produces zero eigenvalue for Helmholtz type operator (1) under homogeneous boundary conditions). In particular, a resonance domain is formed by means of the special line [6]

$$r^2 = r_1^2 - 4 \ln \sin z, \tag{18}$$

where r_1 is the free parameter. This line forms an infinite resonator by itself, if $r_2 \to \infty$ (see Fig.2). To construct a resonator for a finite value of r_2, it is necessary to combine the line (18) with a piece of a straight line (the less r_2, the more the piece).

Having fixed the boundary contour $R(r, z) = 0$, let us set some placement of source and collocation points. Let this setting is characterized by the only parameter h which is a height (one and the same) of isosceles triangles (Fig. 3). Using the MDS technique (Section 2), we arrive at the HSLAE (16) ($F(r) \equiv 0$ in the homogeneous case) whose matrix is a discrete approximation of the boundary condition operator (5). SVD of this matrix produces singular values which are functions of r_2 and h. We need to find zeroes of these functions (as much as possible, but at least one). Here we describe results of our investigations in the case of z-symmetry.

A priori the inverse spectral problem (1)–(7), (9) must generate at least two zero eigenvalues providing a symmetric (in z coordinate) solution and a skew-symmetric one ($\cos z$ — the desirable standing wave). In fact, the discrete model (16)–(17) (MDS plus SVD) produces both "symmetric" and "skew-symmetric" eigenvectors (symmetric and skew-symmetric functions in (15)) corresponding to two smallest singular values of a matrix in Eq. (16) (note, if some singular value tends to zero, it becomes close to eigenvalue). Moreover, it is possible practically to equalize two smallest singular values by choosing the values of h and r_2 parameters. The more equalizing, the better situation is close to double zero eigenvalue as well to a resonance. These results are demonstrated in Fig. 4, where R is the resonance point corresponding to a double zero eigenvalue. In Fig. 5 corresponding structures of the axial component E_z of an electric field are shown.

Understanding of results for homogeneous problem (16) helps to explain some results for boundary value problem (1)–(8). We had tried to fit boundary profile to synthesize a resonator. The most strange result we had obtained was as follows.

In the case, when boundary profile is close to the resonance situation we observe strange points (A, B, C in Fig. 6) in structures of electromagnetic fields which attract all solutions (note, that numerical solutions depend on a source placement in the discrete model). SVD gives the clue. Using this technique we expand full solution into the sum of the "nullspace" solution (projection of the discrete solution onto the eigenvector corresponding to the zero eigenvalue) and the "range" one (projection onto the subspace corresponding to other nonzero eigenvalues). Zeroes of "nullspace" functions (Fig. 7) correspond to the strange points in Fig. 6. It is clear now, that the values of functions in Fig. 6 at the strange points are determined by the values of the "range" function (Fig. 7b) only which are nearly independent of a source placement in the resonance case.

Thus, we have got a constructive algorithm basing on the MDS and SVD techniques for synthesizing of resonance domains . The principal idea is to fit a boundary profile to a shape providing zero singular value in the matrix of HSLAE (16). Fig. 8 demonstrates the synthesized resonance profile and corresponding electromagnetic field structures.

Results concerning the scattering problem (1)–(7), (10)–(14) will be published in further papers.

4. Conclusion.

In this paper we show the way providing positive results in synthesis of resonance electrodynamical systems. Following this way we have found out the MDS and SVD to be powerful tools for solving of elliptic problems arising in supercollider simulation.

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Appendix. Figures.

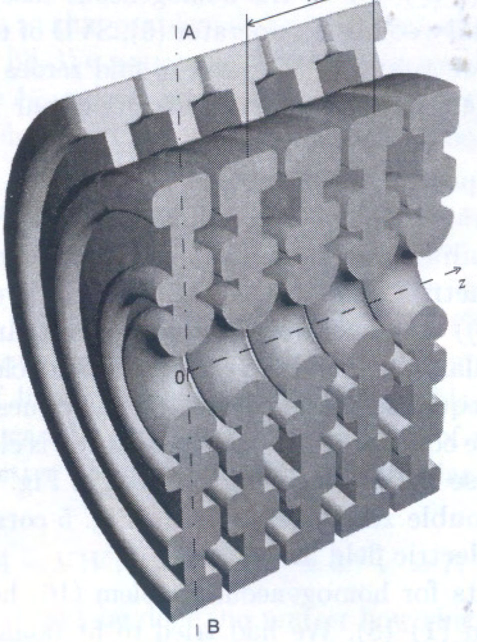

Figure 1. Accelerating channel (RF energy accumulation cavity).

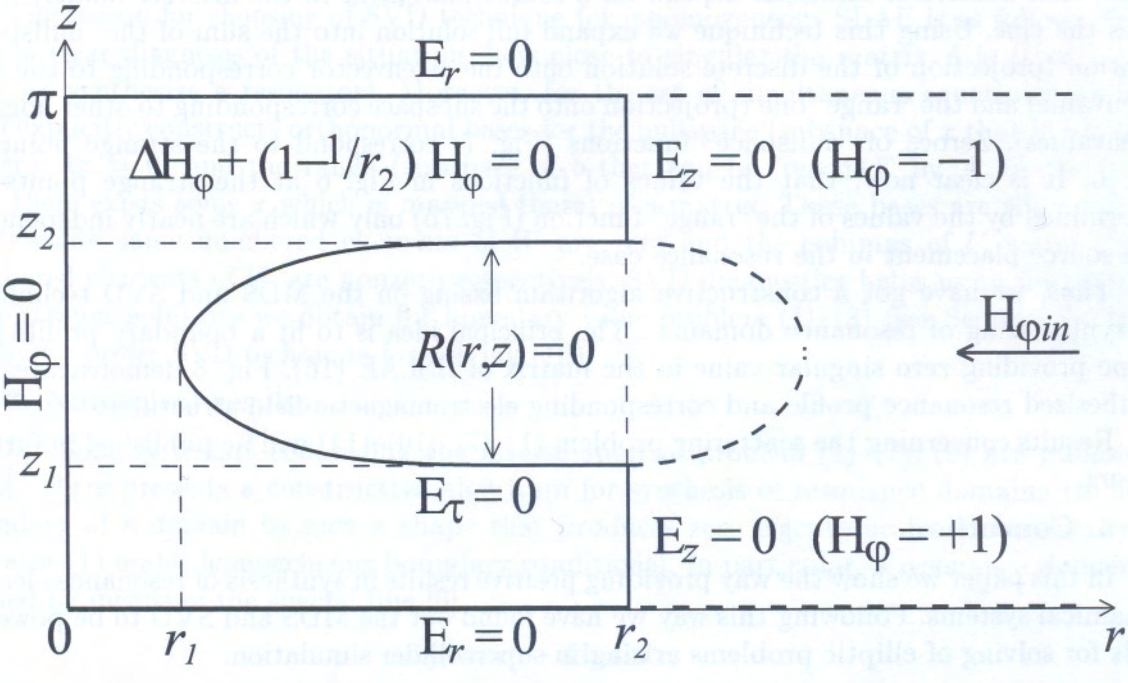

Figure 2. Model elliptic problems (boundary, spectral, scattering) arising in supercollider simulation.

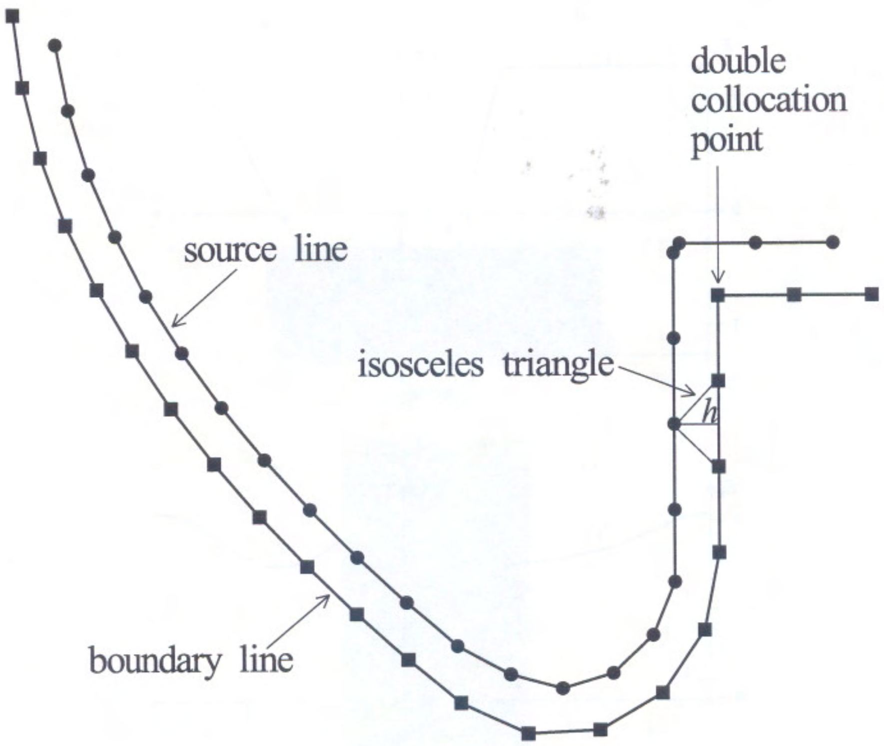

Figure 3. A typical placement of source and collocation points.

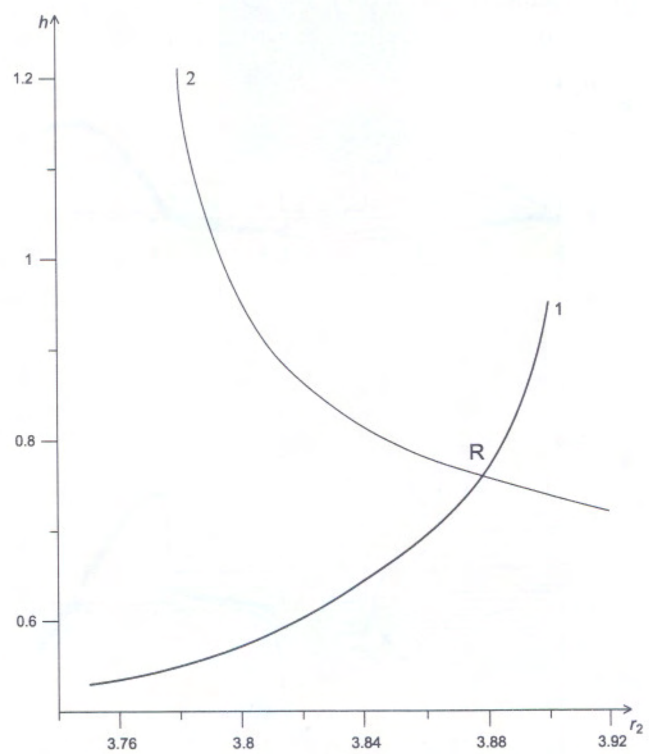

Figure 4. Symmetric (1) and skew-symmetric (2) lines for two smallest singular values of the matrix in Eq. (16).

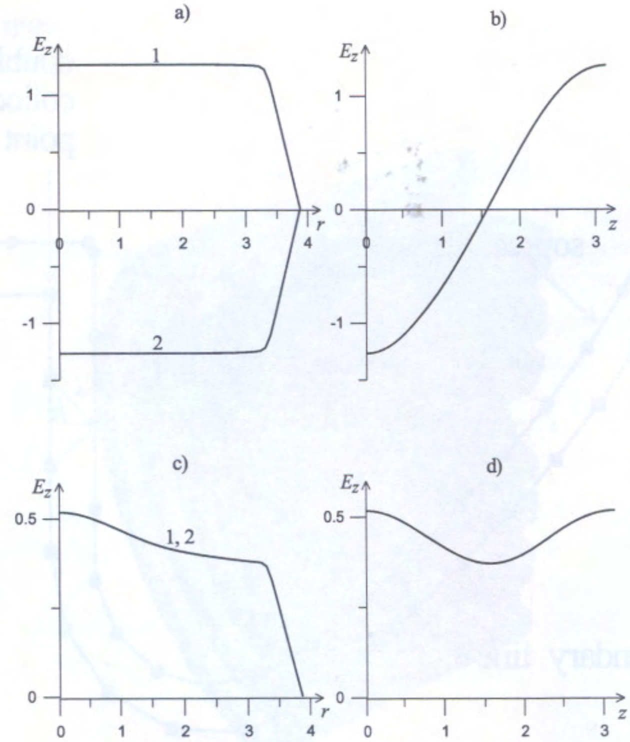

Figure 5. Axial component E_z of an electric field corresponding to zero eigenvalues:
a), b) — skew-symmetric case; c), d) — symmetric case;
a), c) — radial cross-section: 1 — at $z = 0$, 2 — at $z = \pi$;
b), d) — axial cross-section at $r = 1$.

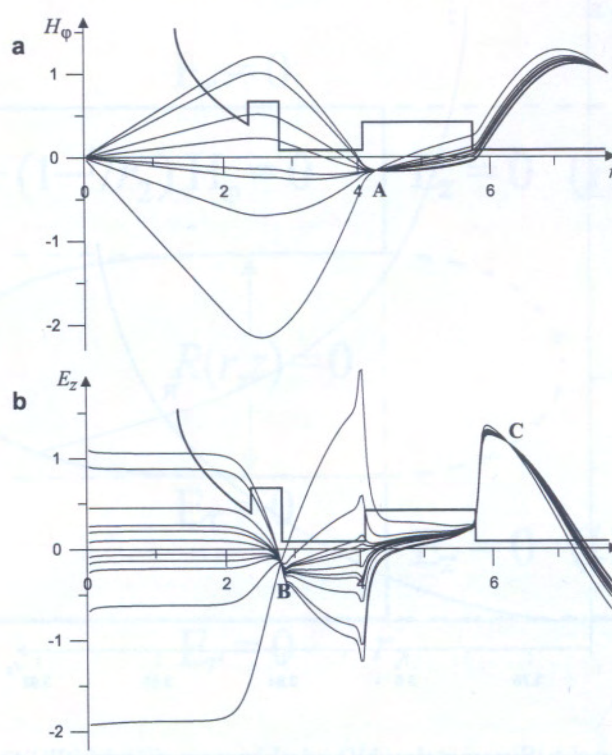

Figure 6. Numerical solutions at $z = 0$ for different source placements.

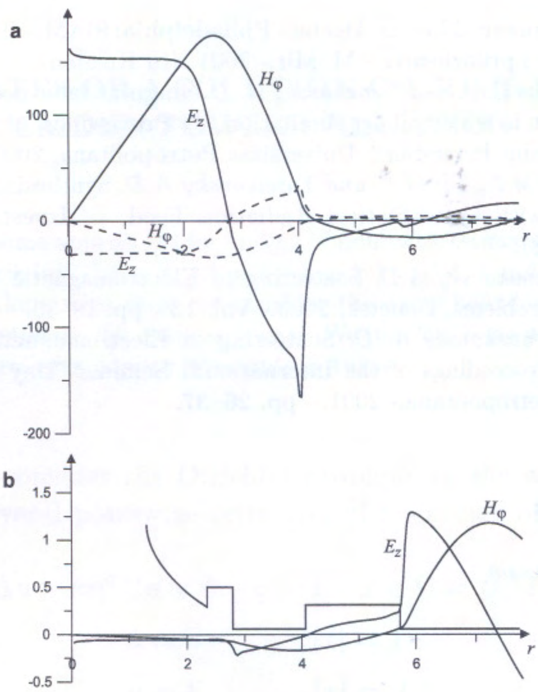

Figure 7. Resonance solutions at $z = 0$ (a — "nullspace" solutions for different source placements, b — "range" solution).

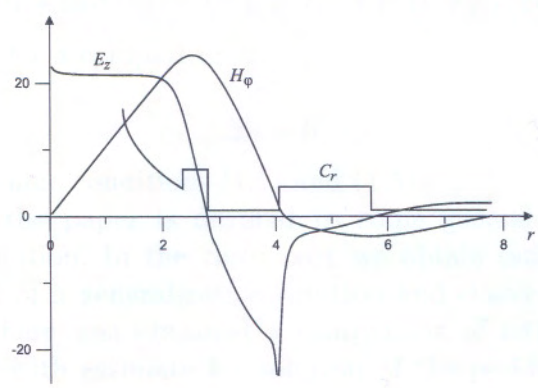

Figure 8. Synthesized disk profile with resonance property (C_r) and corresponding electromagnetic field structure.

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