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DIRICHLET PROBLEM FOR POISSON EQUATIONS IN JORDAN DOMAINS

First, we study the Dirichlet problem for the Poisson equations $\triangle u(z) = g(z)$ with $g \in L^p$, $p > 1$, and continuous boundary data $\varphi : \partial D \to \mathbb{R}$ in arbitrary Jordan domains D in $\mathbb{C}$ and prove the existence of continuous solutions u of the problem in the class $W^{2,p}_{\rm loc}$. Moreover, $u \in W^{1,q}_{\rm loc}$ for some $q > 2$ and u is locally Hölder continuous. Furthermore, $u \in C^{1,\alpha}_{\rm loc}$ with $\alpha = (p-2)/p$ if $p > 2$. Then, on this basis and applying the Leray–Schauder approach, we obtain the similar results for the Dirichlet problem with continuous data in arbitrary Jordan domains to the quasilinear Poisson equations of the form $\triangle u(z) = h(z) \cdot f(u(z))$ with the same assumptions on h as for g above and continuous functions $f : \mathbb{R} \to \mathbb{R}$, either bounded or with nondecreasing $|f|$ of $|t|$ such that $f(t)/t \to 0$ as $t \to \infty$. We also give here applications to mathematical physics that are relevant to problems of diffusion with absorbtion, plasma and combustion. In addition, we consider the Dirichlet problem for the Poisson equations in the unit disk $\mathbb{D} \subset \mathbb{C}$ with arbitrary boundary data $\varphi : \partial\mathbb{D} \to \mathbb{R}$ that are measurable with respect to logarithmic capacity. Here we establish the existence of continuous nonclassical solutions u of the problem in terms of the angular limits in $\mathbb{D}$ a.e. on $\partial\mathbb{D}$ with respect to logarithmic capacity with the same local properties as above. Finally, we extend these results to almost smooth Jordan domains with qusihyperbolic boundary condition by Gehring–Martio.

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1. Introduction.

First of all, recall that the **Poisson kernel** is the $2\pi-$periodic function

$$P_r(\Theta) \ := \ \frac{1-r^2}{1-2r\cos\Theta + r^2} \ , \ r<1 \ , \ \Theta \in \mathbb{R} \ . \tag{1}$$

Here we will apply the notation of the **Poisson integral** in the unit disk $\mathbb{D}$:

$$\mathcal{P}_\varphi(z) \ := \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(\vartheta - t) \ \varphi(e^{it}) \ dt \ , \quad z = re^{i\vartheta}, \ r<1 \ , \ \vartheta \in \mathbb{R} \tag{2}$$

for arbitrary continuous functions $\varphi : \partial\mathbb{D} \to \mathbb{R}$. As known, $\mathcal{P}_\varphi$ is a harmonic function in $\mathbb{D}$ that is extended by continuity to $\overline{\mathbb{D}}$ with φ as its boundary data, see e.g. I.D.2 in [18].

Similarly, given a Jordan domain D in $\mathbb{C}$ and a continuous boundary function $\varphi : \partial D \to \mathbb{R}$, let us denote by $\mathcal{D}_\varphi$ the harmonic function in D that has the continuous extension to $\overline{D}$ with φ as its boundary data. As known, by the Lindelöf maximum principle, see e.g. Lemma 1.1 in [10], we have the uniqueness theorem for the bounded harmonic functions with continuous boundary data. By the Riemann theorem, see e.g. Theorem II.2.1 in [14], there is a conformal mapping $f : D \to \mathbb{D}$ that is extended to a homeomorphism $\tilde{f} : \overline{D} \to \overline{\mathbb{D}}$ by the Caratheodory theorem, see e.g. Theorem II.3.4 in [14]. Thus, the **Dirichlet operator** $\mathcal{D}_{\varphi}$ has the following useful representation

$$\mathcal{D}_{\varphi}(z) \ = \ \mathcal{P}_{\varphi\circ f_*^{-1}}(f(z)), \ z \in D, \qquad \text{where } f_* = \tilde{f}|_{\partial D} \ . \tag{3}$$

It is also known, see e.g. Corollary 1 in [16], that the **Newtonian potential**

$$N_g(z) \ := \ \frac{1}{2\pi}\int\limits_{\mathbb{C}} \log|z-w|\, g(w)\ dm(w) \tag{4}$$

of integrable functions $g : \mathbb{C} \to \mathbb{R}$ with compact support satisfies the **Poisson equation**

$$\triangle N_g \ = \ g \tag{5}$$

in the distributional sense, i.e.,

$$\int\limits_{\mathbb{C}} N_g(z)\, \triangle\psi(z)\, dm(z) \ = \ \int\limits_{\mathbb{C}} \psi(z)\, g(z)\, dm(z) \qquad \forall\ \psi \in C_0^{\infty}(\mathbb{C}) \ . \tag{6}$$

As usual, here $C_0^{\infty}(\mathbb{C})$ denotes the class of all infinitely differentiable functions $\psi : \mathbb{C} \to \mathbb{R}$ with compact support in $\mathbb{C}$, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator and $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$.

2. Dirichlet problem with continuous data.

By Theorem 2 in [16] we come to the following result on the existence, regularity and representation of solutions for the Dirichlet problem to the Poisson equation in arbitrary Jordan domains D in $\mathbb{C}$ where we assume that the charge density g is extended by zero outside of D.

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function and $g : D \to \mathbb{R}$ belong to the class $L^p(D)$ for $p > 1$. Then the function*

$$U \ := \ N_g \ - \ \mathcal{D}_{N_g^*} \ + \ \mathcal{D}_{\varphi} \ , \qquad N_g^* \ := \ N_g|_{\partial D} \ , \tag{7}$$

is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$, belongs to the class $W^{2,p}_{\rm loc}(D)$ and satisfies the Poisson equation $\triangle U = g$ a.e. in D. Moreover, $U \in W^{1,q}_{\rm loc}(D)$ for some $q > 2$ and U is locally Hölder continuous in D. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $g \in L^p(D)$ for $p > 2$.

Remark 1. Note also by the way that a generalized solution of the Dirichlet problem to the Poisson equation in the class $C(\overline{D}) \cap W^{1,2}_{\rm loc}(D)$ is unique at all, see e.g. Theorem 8.30 in [13], and (7) gives the effective representation of this unique solution.

The case of quasilinear Poisson equations is reduced to the case of the linear Poisson equations by the Leray–Schauder approach.

Theorem 2. *Let D be a Jordan domain in $\mathbb{C}$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function and $h : D \to \mathbb{R}$ be a function in the class $L^p(D)$ for $p > 1$. Suppose that a continuous function $f : \mathbb{R} \to \mathbb{R}$ has nondecreasing $|f|$ of $|t|$ and*

$$\lim_{t\to+\infty} \frac{f(t)}{t} \ = \ 0 \ . \tag{8}$$

Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$, $U|_D \in W^{2,p}_{\rm loc}$ such that

$$\triangle U(z) \ = \ h(z) \cdot f(U(z)) \qquad \text{for a.e. } z \in D \ . \tag{9}$$

Moreover, $U \in W^{1,q}_{\rm loc}(D)$ for some $q > 2$ and U is locally Hölder continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$.

In particular, the latter statement in Theorem 2 implies that $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha = (0,1)$ if h is bounded.

Proof. If $\|h\|_p = 0$ or $\|f\|_C = 0$, then the Dirichlet operator $\mathcal{D}_\varphi$ gives the desired solution of the Dirichlet problem for equation (9), see e.g. I.D.2 in [18]. Hence we may assume further that $\|h\|_p \neq 0$ and $\|f\|_C \neq 0$.

By Theorem 1 and the maximum principle for harmonic functions, we obtain the family of operators $F(g;\tau) : L^p(D) \to L^p(D)$, $\tau \in [0,1]$:

$$F(g;\tau) \ := \ \tau h \cdot f(N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi) \ , \ N_g^* := N_g|_{\partial D} \ , \qquad \forall\, \tau \in [0,1] \tag{10}$$

which satisfies all groups of hypothesis H1-H3 of Theorem 1 in [22].

H1). First of all, $F(g;\tau) \in L^p(D)$ for all $\tau \in [0,1]$ and $g \in L^p(D)$ because by Theorem 1 $f(N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi)$ is a continuous function and, moreover, by Theorem 1 in [16]

$$\|F(g;\tau)\|_p \ \leq \ \|h\|_p \ |f\,(\,2M\,\|g\|_p + \|\varphi\|_C)\,| \ < \ \infty \qquad \forall\, \tau \in [0,1] \ .$$

Thus, by Theorem 1 in combination with the Arzela–Ascoli theorem, see e.g. Theorem IV.6.7 in [6], the operators $F(g;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). By Theorem 1 in [16] and the maximum principle for harmonic functions, we have the estimate for solutions $g \in L^p$ of the equations $g = F(g;\tau)$:

$$\|g\|_p \ \leq \ \|h\|_p \ |f\,(\,2M\,\|g\|_p + \|\varphi\|_C)\,| \ \leq \ \|h\|_p \ |f(\,3M\,\|g\|_p)|$$

whenever $\|g\|_p \geq \|\varphi\|_C/M$, i.e. then it should be

$$\frac{|f(\,3M\,\|g\|_p)|}{3M\,\|g\|_p} \ \geq \ \frac{1}{3M\,\|h\|_p} \tag{11}$$

and hence $\|g\|_p$ should be bounded in view of condition (8).

Thus, by Theorem 1 in [22] there is a function $g \in L^p(D)$ such that $g = F(g;1)$ and, consequently, by our Theorem 1 the function $U := N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi$ gives the desired solution of the Dirichlet problem for the quasilinear Poisson equation (9). $\Box$

Remark 2. As it is clear from the proof, Theorem 2 is valid if f is an arbitrary continuous bounded function. Moreover, condition (8) can be replaced by the weaker

$$\limsup_{t\to+\infty} \frac{|f(t)|}{t} \ < \ \frac{1}{3M\|h\|_p} \tag{12}$$

where M is the constant from the estimate (14) of Theorem 1 in [16].

Theorem 2 together with Remark 2 can be applied to some physical problems. The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [5], p. 4, and, in detail, in [2]. A nonlinear system is obtained for the density u and the temperature T of the reactant. Upon eliminating T the system can be reduced to the equation

$$\triangle u \ = \ \lambda \cdot f(u) \tag{13}$$

with $h(z) \equiv \lambda > 0$ and, for isothermal reactions, $f(u) = u^q$ where $q > 0$ is called the order of the reaction. It turns out that the density of the reactant u may be zero in a subdomain called a **dead core.** A particularization of results in Chapter 1 of [5] shows that a dead core may exist just if and only if $0 < q < 1$ and λ is large enough, see also the corresponding examples in [15]. In this connection, the following statements may be of independent interest.

Corollary 1. *Let D be a Jordan domain in $\mathbb{C}$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function and let $h : D \to \mathbb{R}$ be a function in the class $L^p(D)$, $p > 1$. Then there exists a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ such that $u \in W^{2,p}_{\mathrm{loc}}(D)$ and*

$$\triangle\, u(z) \ = \ h(z) \cdot u^q(z)\ , \quad 0 \ < \ q \ < \ 1 \tag{14}$$

a.e. in D. Moreover, $u \in W^{1,\beta}_{\mathrm{loc}}(D)$ for some $\beta > 2$ and u is locally Hölder continuous in D. Furthermore, $u \in C^{1,\alpha}_{\mathrm{loc}}(D)$ with $\alpha = (p-2)/p$ if $p > 2$.

Corollary 2. *Let D be a Jordan domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ such that $u \in W^{2,p}_{\mathrm{loc}}(D)$ for all $p \geq 1$ and*

$$\triangle\, u(z) \ = \ u^q(z)\ , \quad 0 \ < \ q \ < \ 1\ , \tag{15}$$

a.e. in D. Moreover, $u \in C^{1,\alpha}_{\mathrm{loc}}(D)$ for all $\alpha \in (0,1)$.

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (13). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = +\infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [5]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \operatorname{sign} u$, we have that $u = |U|^Q \cdot \operatorname{sign} U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \operatorname{sign} u$, we come to the equation $\triangle U = |U|^q \cdot \operatorname{sign} U = \psi(U)$.

Corollary 3. *Let D be a Jordan domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$ such that $u \in W^{2,p}_{\rm loc}(D)$ for all $p \geq 1$ and*

$$\triangle U(z) \ = \ |U(z)|^{q-1}U(z) \ , \quad 0 \ < \ q \ < \ 1 \ , \tag{16}$$

a.e. in D. Moreover, $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha \in (0,1)$.

Finally, we recall that in the combustion theory, see e.g. [3], [24] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} \ = \ \frac{1}{\delta} \cdot \triangle u \ + \ e^u \ , \quad t \geq 0, \ z \in D, \tag{17}$$

takes a special place. Here $u \geq 0$ is the temperature of the medium and δ is a certain positive parameter.

We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (17), see [15]. Namely, the equation (9) is appeared here with $h \equiv \delta > 0$ and the function $f(u) = e^{-u}$ that is bounded as in Remark 2.

Corollary 4. *Let D be a Jordan domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$ such that $u \in W^{2,p}_{\rm loc}(D)$ for all $p \geq 1$ and*

$$\triangle U(z) \ = \ \delta \cdot e^{-U(z)} \ , \quad \delta > 0 \ , \tag{18}$$

a.e. in D. Moreover, $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha \in (0,1)$.

Due to the factorization theorem in [15], we plan to extend these results to semi–linear equations describing the corresponding physical phenomena in anisotropic and inhomogeneous media in arbitrary Jordan domains.

3. The definition and preliminary remarks on the logarithmic capacity.

Given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_E \log\left|\frac{1}{z-\zeta}\right| \, d\nu(\zeta) \tag{19}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the Borel set E is the quantity

$$C(E) = e^{-V} \ , \qquad V \ = \ \inf_{\nu} \ V_{\nu}(E) \ , \qquad V_{\nu}(E) \ = \ \sup_{z} \ U^{\nu}(z) \ . \tag{20}$$

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [23]:

$$C(E) \ = \ \tau(E) \ := \ \lim_{n\to\infty} \ V_n^{\frac{2}{n(n-1)}} \tag{21}$$

where V_n denotes the supremum of the product

$$V(z_1,\ldots,z_n) \ = \ \prod_{k<l}^{l=1,\ldots,n} \ |z_k - z_l| \tag{22}$$

taken over all collections of points $z_1,\ldots,z_n$ in the set E. Following Fékete, see [9], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E.

Remark 3. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is continuous by Hölder and, in particular, for quasiconformal mappings on compact sets, see e.g. Theorem II.4.3 in [21].

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [7], **inner** C_* and **outer** C^* **capacities** :

$$C_*(E) \ : \ = \ \sup_{F\subseteq E} \ C(E), \qquad C^*(E) \ : \ = \ \inf_{E\subseteq O} \ C(O) \tag{23}$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$ and infimum is taken over all open sets $O \subset \mathbb{C}$. A **set** $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if $C^*(E) = C_*(E)$, and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$.

A **function** $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{z \in E : \varphi(z) \in O\} \tag{24}$$

are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [23]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [23]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [7].

Remark 4. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [7]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of a sigma-compactum (union of countable collection of compact sets) and a set of logarithmic capacity zero. Thus, the measurability of functions with respect to logarithmic capacity is invariant under Hölder continuous change of variables.

It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. theorem II(7.4) in [27]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [7].

Dealing with measurable boundary functions $\varphi(\zeta)$ with respect to the logarithmic capacity, we will use the **abbreviation q.e. (quasi-everywhere)** on a set $E \subset \mathbb{C}$, if a property holds for all $\zeta \in E$ except its subset of zero logarithmic capacity, see [19].

4. Dirichlet problem with measurable data in the unit disk.

In the paper [8], it was proved as Theorem 3.1 the following analog of the known Luzin theorem in terms of logarithmic capacity, cf. e.g. Theorem VII(2.3) in [27].

Proposition 1. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ such that $\Phi'(x) = \varphi(x)$ q.e. on (a, b). Furthermore, the function Φ can be chosen such that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \le \varepsilon$ under arbitrary prescribed $\varepsilon > 0$ for all $x \in [a, b]$.*

Corollary 5. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : \partial\mathbb{D} \to \mathbb{R}$ such that $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$.*

The **Poisson–Stieltjes integral**

$$\Lambda_\Phi(z) \ := \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(\vartheta - t)\ d\Phi(e^{it}) \ , \quad z = re^{i\vartheta},\ r < 1\ ,\ \vartheta \in \mathbb{R} \tag{25}$$

is well-defined for arbitrary continuous functions $\Phi : \partial\mathbb{D} \to \mathbb{R}$, see e.g. Section 2 in [26].

Directly by the definition of the Riemann–Stieltjes integral and the Weierstrass type theorem for harmonic functions, see e.g. Theorem I.3.1 in [14], Λ_Φ is a harmonic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ because the function $P_r(\vartheta - t)$ is the real part of the analytic function

$$\mathcal{A}_\zeta(z) \ := \ \frac{\zeta + z}{\zeta - z}\ , \quad \zeta = e^{it}, \quad z = re^{i\vartheta}\ ,\ r < 1\ ,\ \vartheta \text{ and } t \in \mathbb{R}\ . \tag{26}$$

Next, by Theorem 1 in [26] we have the following useful conclusion.

Proposition 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity and $\Phi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function with $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$. Then Λ_Φ has the angular limit*

$$\lim_{z\to\zeta} \Lambda_\Phi(z) \ = \ \varphi(\zeta) \qquad q.e. \ on \ \partial\mathbb{D} \ . \tag{27}$$

Finally, by Theorem 2 in [16], Proposition 2 and the known Poisson formula, see e.g. I.D.2 in [18], we come to the following result on the existence, regularity and representation of solutions for the Dirichlet problem to the Poisson equation in the unit disk $\mathbb{D}$. We assume that the charge density g is extended by zero outside of $\mathbb{D}$ in the next theorem.

Theorem 3. *Let a function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to logarithmic capacity and let a continuous function Φ correspond to φ by Corollary 5. Suppose that a function $g : \mathbb{D} \to \mathbb{R}$ is in the class $L^p(\mathbb{D})$ for $p > 1$. Then the following function in $\mathbb{D}$*

$$U \ := \ N_g \ - \ \mathcal{P}_{N_g^*} \ + \ \Lambda_\Phi \ , \qquad\qquad N_g^* \ := \ N_g|_{\partial\mathbb{D}} \ , \tag{28}$$

belongs to the class $W^{2,p}_{\rm loc}(\mathbb{D})$, satisfies the Poisson equation $\triangle U = g$ a.e. in $\mathbb{D}$ and has the angular limit

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \qquad q.e. \ on \ \partial\mathbb{D} \ . \tag{29}$$

Moreover, $U \in W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$ and U is locally Hölder continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ with $\alpha = (p-2)/p$ if $g \in L^p(\mathbb{D})$ for $p > 2$.

Remark 5. Note that by the Luzin result, see also Theorem 3 in [26], the statement of Theorem 3 is valid in terms of the length measure as well as the harmonic measure on $\partial\mathbb{D}$. However, by the well–known Ahlfors–Beurling example, see [1], the sets of length zero as well as of harmonic measure zero are not invariant with respect to quasiconformal changes of variables. The latter circumstance does not make it is possible to apply the result in the future for the extension of the statement to generalizations of the Laplace equation in anisotropic and inhomogeneous media. Hence we prefer to use logarithmic capacity.

5. Dirichlet problem with measurable data in almost smooth domains.

We say that a Jordan curve Γ in $\mathbb{C}$ is **almost smooth** if Γ has a tangent q.e. Here it is said that a straight line L in $\mathbb{C}$ is **tangent** to Γ at a point $z_0 \in \Gamma$ if

$$\limsup_{z\to z_0, z\in\Gamma} \frac{{\rm dist}\,(z, L)}{|z - z_0|} \ = \ 0 \ . \tag{30}$$

In particular, Γ is almost smooth if Γ has a tangent at all its points except a countable set. The nature of such Jordan curves Γ is complicated enough because the countable set can be everywhere dense in Γ.

Now, given a domain D in $\mathbb{C}$, $k_D(z,z_0)$ denotes the **quasihyperbolic distance**,

$$k_D(z,z_0) \ := \ \inf_{\gamma} \int\limits_{\gamma} \frac{ds}{d(\zeta,\partial D)} \ , \tag{31}$$

introduced in the paper [12]. Here $d(\zeta,\partial D)$ denotes the Euclidean distance from the point $\zeta\in D$ to ∂D and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D.

Next, it is said that a domain D satisfies the **quasihyperbolic boundary condition** if

$$k_D(z,z_0) \ \le \ a\,\ln\frac{d(z_0,\partial D)}{d(z,\partial D)} + b \qquad \forall\ z\in D \tag{32}$$

for constants a and b and a point $z_0\in D$. The latter notion was introduced in [10] but, before it, was first applied in [4].

Remark 6. Given a Jordan domain D in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition. By the Riemann theorem, see e.g. Theorem II.2.1 in [14], there is a conformal mapping $f: D\to\mathbb{D}$ that is extended to a homeomorphism $\tilde f:\overline{D}\to\overline{\mathbb{D}}$ by the Caratheodory theorem, see e.g. Theorem II.3.4 in [14]. Moreover, $f_* := \tilde f|_{\partial D}$, as well as f_*^{-1}, is Hölder continuous by Corollary to Theorem 1 in [4]. Thus, by Remark 4 a function $\varphi:\partial D\to\mathbb{R}$ is measurable with respect to logarithmic capacity if and only if the function $\psi:=\varphi\circ f_*^{-1}:\partial\mathbb{D}\to\mathbb{R}$ is so. Set $\Phi:=\Psi\circ f_*$ where $\Psi:\partial\mathbb{D}\to\mathbb{R}$ is a continuous function corresponding to ψ by Corollary 5.

Proposition 3. *Let D be a Jordan domain in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition. Suppose that $\varphi:\partial D\to\mathbb{R}$ is measurable with respect to logarithmic capacity and $\Phi:\partial D\to\mathbb{R}$ is the continuous function corresponding to φ by Remark 6. Then the harmonic function $\mathcal{L}_{\Phi}(z):=\Lambda_{\Phi\circ f_*^{-1}}(f(z))$ has the angular limit φ q.e. on ∂D.*

Proof. Indeed, by Remark 6 and Proposition 2 there is the angular limit

$$\lim_{w\to\xi}\ \Lambda_{\Psi}(w) \ = \ \psi(\xi) \qquad \text{q.e. on } \partial\mathbb{D}\ . \tag{33}$$

By the Lindelöf theorem, see e.g. Theorem II.C.2 in [18], if ∂D has a tangent at a point ζ, then

$$\arg\,[\tilde f(\zeta)-\tilde f(z)] - \arg\,[\zeta - z] \to const \qquad \text{as } z\to\zeta\ .$$

After the change of variables $\xi:=\tilde f(\zeta)$ and $w:=\tilde f(z)$, we have that

$$\arg\,[\xi - w] - \arg\,[\tilde f^{-1}(\xi)-\tilde f^{-1}(w)] \to const \qquad \text{as } w\to\xi\ .$$

In other words, the conformal images of sectors in $\mathbb{D}$ with a vertex at ξ is asymptotically the same as sectors in D with a vertex at ζ. Thus, nontangential paths in $\mathbb{D}$ are transformed under $\tilde f^{-1}$ into nontangential paths in D.

Recall that firstly the almost smooth Jordan curve ∂D has a tangent q.e., secondly by Remark 6 the mappings f_* and f_*^{-1} are Hölder continuous, and thirdly by Remark 3 they transform sets of logarithmic capacity zero into sets of logarithmic capacity zero. Consequently, (33) implies the desired conclusion. □

Finally, by Theorem 2 in [16], Proposition 3 and the Poisson formula, we come to the following result on the existence, regularity and representation of solutions for the Dirichlet problem to the Poisson equation in the Jordan domains. We assume here that the charge density g is extended by zero outside of D in the next theorem.

Theorem 4. *Let D be a Jordan domain in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition, a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity and let a continuous function Φ correspond to φ by Remark 6. Suppose that a function $g : D \to \mathbb{R}$ is in the class $L^p(D)$ for $p > 1$. Then the following function in D*

$$U \ := \ N_g \ - \ \mathcal{D}_{N_g^*} \ + \ \mathcal{L}_\Phi \ , \qquad\qquad N_g^* \ := \ N_g|_{\partial D} \ , \tag{34}$$

belongs to the class $W^{2,p}_{\rm loc}(D)$, satisfies the Poisson equation $\triangle U = g$ a.e. in D and has the angular limit

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \qquad q.e. \ on \ \partial D \ . \tag{35}$$

Moreover, $U \in W^{1,q}_{\rm loc}(D)$ for some $q > 2$ and U is locally Hölder continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $g \in L^p(D)$ for $p > 2$.

Remark 7. Note that by the Luzin result, see also Theorem 3 in [26], the statement of Theorem 4 is valid in terms of the length measure on rectifiable ∂D. Indeed, by the Riesz theorem length $f_*^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}$ with $|E| = 0$, see e.g. Theorem II.C.1 and Theorems II.D.2 in [18]. Conversely, by the Lavrentiev theorem $|f_*(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ and length $\mathcal{E} = 0$, see [20], see also the point III.1.5 in [25].

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Задача Дирихле для уравнений Пуассона в жордановых областях.

Прежде всего, мы изучаем задачу Дирихле для уравнений Пуассона $\triangle u(z) = g(z)$ с $g \in L^p$, $p > 1$, и непрерывными граничными данными $\varphi : \partial D \to \mathbb{R}$ в произвольных жордановых областях $D \subset \mathbb{C}$ и доказываем существование непрерывных решений u этой задачи в классе $W^{2,p}_{\rm loc}$. Кроме того, $u \in W^{1,q}_{\rm loc}$ для некоторого $q > 2$ и u локально непрерывны по Гельдеру. Более того, $u \in C^{1,\alpha}_{\rm loc}$ с $\alpha = (p-2)/p$, если $p > 2$. Затем, на этой основе и применяя подход Лере–Шаудера, мы получаем аналогичные результаты для задачи Дирихле с непрерывными граничными данными в произвольных жордановых областях для квазилинейных уравнений Пуассона вида $\triangle u(z) = h(z) \cdot f(u(z))$ с теми же предположениями о функции h как выше для g и непрерывных функций $f : \mathbb{R} \to \mathbb{R}$, которые либо ограничены, либо с неубывающим $|f|$ от $|t|$, таких, что $f(t)/t \to 0$ при $t \to \infty$. Мы также приводим здесь приложения к математической физике, которые относятся к задачам диффузии с абсорбцией, плазме и горению. В дополнение, мы рассматриваем задачу Дирихле для уравнений Пуассона в единичном круге $\mathbb{D} \subset \mathbb{C}$ с произвольными граничными данными $\varphi : \partial\mathbb{D} \to \mathbb{R}$, которые измеримы относительно логарифмической емкости. Здесь мы устанавливаем существование неклассических решений этой проблемы в терминах угловых пределов в $\mathbb{D}$ п.в. на $\partial\mathbb{D}$ относительно логарифмической емкости с теми же локальными свойствами как и выше. Наконец, мы распространяем эти результаты на почти гладкие жордановы области D в $\mathbb{C}$ с квазигиперболическим граничным условием по Герингу–Мартио.

Ключевые слова: *задача Дирихле, квазилинейные уравнения Пуассона, логарифмический потенциал, логарифмическая емкость, угловые пределы.*

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Задача Діріхле для рівнянь Пуасона у жорданових областях.

Перш за все ми вивчаємо задачу Діріхле для рівнянь Пуасона $\triangle u(z) = g(z)$ с $g \in L^p$, $p > 1$, та неперервними граничними даними $\varphi : \partial D \to \mathbb{R}$ в довільних жорданових областях $D \subset \mathbb{C}$ та доводимо існування неперервних рішень u цієї задачі в класі $W^{2,p}_{\rm loc}$. Крім цього, $u \in W^{1,q}_{\rm loc}$ для деякого $q > 2$ та u локально неперервні за Гельдером. Більш того, $u \in C^{1,\alpha}_{\rm loc}$ з $\alpha = (p-2)/p$, якщо $p > 2$. Потім, на цій основі, застосовуючи підхід Лере–Шаудера, ми отримуємо аналогічні результати для задачі Діріхле з неперервними граничними даними в довільних жорданових областях для квазілінійних рівнянь Пуасона виду $\triangle u(z) = h(z) \cdot f(u(z))$ з тими же припущеннями про функції h як вище для g та неперервних функцій $f : \mathbb{R} \to \mathbb{R}$, які або обмежені, або з неспадним $|f|$ від $|t|$, таких, що $f(t)/t \to 0$ при $t \to \infty$. Ми також наводимо тут додатки до математичної фізики, які відносяться до задач дифузії з абсорбцією, плазмі та горінню. На додаток, ми розглядаємо задачу Діріхле для рівнянь Пуасона в одиничному колі $\mathbb{D} \subset \mathbb{C}$ з довільними граничними даними $\varphi : \partial\mathbb{D} \to \mathbb{R}$, які вимірні відносно логарифмічної ємності. Тут ми встановлюємо існування некласичних рішень цієї проблеми у термінах кутових границь у $\mathbb{D}$ п.в. на $\partial\mathbb{D}$ відносно логаріфмічної ємності з тими ж локальними властивостями як і вище. Нарешті, ми поширюємо ці результати на майже гладкі жорданові області D в $\mathbb{C}$ з квазігіперболічною граничною умовою за Герингом–Мартіо.

Ключові слова: *задача Діріхле, квазілінійні рівняння Пуасона, логарифмічна ємність, кутові межи.*

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