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ON WEAKLY s-NORMAL SUBGROUPS OF FINITE GROUPS*

ПРО СЛАБКО s-НОРМАЛЬНІ ПІДГРУПИ СКІНЧЕННИХ ГРУП

Assume that G is a finite group and H is a subgroup of G. We say that H is s-permutably imbedded in G if, for every prime number p that divides $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G; a subgroup H is s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; a subgroup H is weakly s-normal in G if there are a subnormal subgroup T of G and a subgroup H_* of H such that $G = HT$ and $H \cap T \leq H_*$, where H_* is a subgroup of H that is either s-permutably imbedded or s-semipermutable in G. We investigate the influence of weakly s-normal subgroups on the structure of finite groups. Some recent results are generalized and unified.

Нехай G — скінченна група, а H — підгрупа G. Будемо говорити, що H є s-переставно вкладеною в G, якщо для будь-якого простого числа p, що ділить $|H|$, силовська p-підгрупа H є також силовською p-підгрупою деякої s-переставної підгрупи G; H є s-напівпереставною в G, якщо $HG_p = G_pH$ для будь-якої силовської p-підгрупи G_p групи G із $(p, |H|) = 1$; H є слабко s-нормальною в G, якщо існують субнормальна підгрупа T групи G і підгрупа H_* підгрупи H такі, що $G = HT$ і $H \cap T \leq H_*$, де H_* — підгрупа H, що є або s-переставно вкладеною, або s-напівпереставною в G. Досліджено вплив слабко s-нормальних підгруп на будову скінченних груп. Узагальнено та уніфіковано деякі нещодавні результати.

1. Introduction. All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. G always denotes a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ *a formation* provided that (i) if $G \in \mathcal{F}$ and $H \triangleleft G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for all normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (ref. [1, p. 713], Satz 8.6).

Two subgroups H and K of G are said to be *permutable* if $HK = KH$. A subgroup H of G is said to be *s-permutable* (or *s-quasinormal, π-quasinormal*) [2] in G if H permutes with every Sylow subgroup of G; H is said *c-normal* [3] in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. More recently, Skiba in [4] introduces the following concept, which covers both s-permutability and c-normality:

Definition 1.1. *Let H be a subgroup of G. H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G.*

From [5], we know that a subgroup H of G is said to be *s-permutably embedded* in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. In [6], we give a new concept which covers properly both s-permutably embedding property and Skiba's weakly s-permutability.

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Definition 1.2. *Let H be a subgroup of G. We say that H is weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.*

In another direction, a subgroup H of G is said to be *s-semipermutable* [7] in G if H permutes with every Sylow p-subgroup G_p of G with $(|H|, p) = 1$. It is easy to give concrete examples to show that s-semipermutablity and s-permutably embedding property are not equivalent. Here, we introduce a new concept which covers properly both s-semipermutability and weakly s-permutably embedding property.

Definition 1.3. *Let H be a subgroup of G. We say that H is weakly s-normal in G if there are a subnormal subgroup T of G and a subgroup H_* of H such that $G = HT$ and $H \cap T \leq H_*$, where H_* is a subgroup of H which is either s-permutably embedded or s-semipermutable in G.*

Remark. Obviously, weakly s-permutably embedding property (or s-semipermutability) implies weakly s-normality by the definitions. The converse does not hold in general.

Examples. **1.** Suppose that $G = A_5$, the alternative group of degree 5. Then A_4 is weakly s-normal in G, but not weakly s-permutably embedded in G.

2. Suppose that $G = S_4$, the symmetric group of degree 4. Take $H = \langle (34) \rangle$. Then H is weakly s-normal in G, but not s-semipermutable in G.

In the literature, authors usually put the assumptions on either the minimal subgroups (and cyclic subgroups of order 4 when $p = 2$) or the maximal subgroups of some kinds of subgroups of G when investigating the structure of G, such as in [7–13, 16–21] etc. In the nice paper [4], Skiba provided a unified viewpoint for a series of similar problems.

For the sake of convenience of statement, we introduce the following notation.

Let P be a p-subgroup of G for some $p \in \pi(G)$. We say that P satisfies $(*)$ ($(*)'$, $(\triangle)$, $(\diamondsuit_1)$, $(\diamondsuit_2)$, $(\diamondsuit_3)$, $(\diamondsuit_4)$, respectively) in G if

$(*)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $|H| = 2|D|$ (if P is a non-abelian 2-group and $|P\colon D| > 2$) are weakly s-permutable in G.

$(*)'$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutable in G. When P is a non-abelian 2-group and $|P\colon D| > 2$, in addition, the subgroup H of P is weakly s-permutable in G if $|H| = 2|D|$ and $\exp(H) > 2$.

$(\triangle)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-permutably embedded in G. When $p = 2$ and $|P\colon D| > 2$, in addition, the subgroup H of P is weakly s-permutably embedded in G if $|H| = 2|D|$ and $\exp(H) > 2$.

$(\diamondsuit_1)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are weakly s-normal in G. When P is a non-abelian 2-group and $|P\colon D| > 2$, in addition, H is weakly s-normal in G if $|H| = 2|D|$ and $\exp(H) > 2$.

$(\diamondsuit_2)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are either s-permutably embedded or s-semipermutable in G. When P is a non-abelian 2-group and $|P\colon D| > 2$, in addition, the subgroup H of P is either s-permutably embedded or s-semipermutable in G if $|H| = 2|D|$ and $\exp(H) > 2$.

$(\diamondsuit_3)$: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are s-semipermutable in G. When P is a non-abelian 2-group and

$|P\colon D| > 2$, in addition, the subgroup H of P is s-semipermutable in G if $|H| = 2|D|$ and $\exp(H) > 2$.

($\diamondsuit_4$): P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are either s-semipermutable or c-normal in G. When P is a non-abelian 2-group and $|P\colon D| > 2$, in addition, the subgroup H of P is either s-semipermutable or c-normal in G if $|H| = 2|D|$ and $\exp(H) > 2$.

The following is the main result of [4].

Theorem 1.1 ([4], Theorem 1.3)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ satisfies $(*)$ in G. Then $G \in \mathcal{F}$.*

Scrutinizing the proof of [4] (Theorem 1.3), we can find that the following theorem holds:

Theorem 1.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ satisfies $(*)'$ in G. Then $G \in \mathcal{F}$.*

In [6], Theorem 1.2 was extended as follows.

Theorem 1.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. If every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\triangle$ in G, then $G \in \mathcal{F}$.*

In [22], there holds the following result.

Theorem 1.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\diamondsuit_3$ in G, then $G \in \mathcal{F}$.*

In [23], Theorem 1.4 was extended as follows.

Theorem 1.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\diamondsuit_4$ in G, then $G \in \mathcal{F}$.*

In this paper, the main purpose is to generalize results mentioned above as Theorem 3.4. Theorem 3.2 related to p-nilpotency of groups is a main step in the proof of Theorem 3.4.

2. Preliminaries.

Lemma 2.1. *Suppose that H is an s-semipermutable subgroup of G. Then*

(a) *If $H \le K \le G$, then H is s-semipermutable in K.*

(b) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(c) *If $H \le O_p(G)$, then H is s-permutable in G.*

Proof. By [7].

Lemma 2.2 ([5], Lemma 1)**.** *Suppose that U is s-permutably embedded in a group G, and that $H \le G$ and $N \trianglelefteq G$.*

(a) *If $U \le H$, then U is s-permutably embedded in H.*

(b) *UN is s-permutably embedded in G and UN/N is s-permutably embedded in G/N.*

Lemma 2.3 ([21], Lemma 2.3)**.** *Suppose that H is s-permutable in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.4 ([21], Lemma 2.4)**.** *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Now we give some basic properties of weakly s-normality.

Lemma 2.5. *Let U be a weakly s-normal subgroup of G and N a normal subgroup of G. Then*

(a) *If $U \leq H \leq G$, then U is weakly s-normal in H.*

(b) *Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly s-normal in G/N.*

(c) *Suppose that U is a p-group for some prime p and N is a p'-subgroup. Then UN/N is weakly s-normal in G/N.*

(d) *Suppose that U is a p-group for some prime p and U is neither s-semipermutable nor s-permutably embedded in G. Then G has a normal subgroup M such that $|G\colon M| = p$ and $G = MU$.*

(e) *If $U \leq O_p(G)$ for some prime p, then U is weakly s-permutable in G.*

Proof. By the hypotheses, there are a subnormal subgroup T of G and a subgroup U_* of U such that $G = UT$ and $U \cap T \leq U_*$, where U_* is a subgroup of U which is either s-permutably embedded or s-semipermutable in G.

(a) $H = U(H \cap T)$. Obviously $H \cap T$ is subnormal in H and $U \cap (H \cap T) = U \cap T \leq U_*$. By Lemmas 2.1 and 2.2, we know that U_* is either s-permutably embedded or s-semipermutable in H. Hence U is weakly s-normal in H.

(b) $G/N = (U/N)(TN/N)$. Obviously TN/N is subnormal in G/N and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leq U_*N/N$. By Lemmas 2.1 and 2.2, we know that U_*N/N is either s-permutably embedded or s-semipermutable in G/N. Hence U/N is weakly s-normal in G/N.

(c) It is easy to see that $N \leq T$ and $G/N = (UN/N)(T/N)$. We have T/N is subnormal in G/N and $(UN/N) \cap (T/N) = (U \cap T)N/N \leq U_*N/N$. By Lemmas 2.1 and 2.2, we know that U_*N/N is either s-permutably embedded or s-semipermutable in G/N. Hence U/N is weakly s-normal in G/N.

(d) If $T = G$, then $U = U \cap T \leq U_* \leq U$. Thus $U = U_*$ is either s-semipermutable or s-permutably embedded in G, contrary to the hypotheses. Consequently, T is a proper subgroup of G. Hence G has a proper normal subgroup K such that $T \leq K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $|G\colon M| = p$ and $G = MU$.

(e) By Lemmas 2.1(c) and 2.4 and the definitions.

Lemma 2.6 ([14], A, 1.2)**.** *Let U, V and W be subgroups of a group G. Then the following statements are equivalent.*

(a) $U \cap VW = (U \cap V)(U \cap W)$.

(b) $UV \cap UW = U(V \cap W)$.

Lemma 2.7 ([1], VI, 4.10)**.** *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either A or B is contained in a proper normal subgroup of G.*

Lemma 2.8 ([1], III, 5.2 and IV, 5.4)**.** *Suppose that p is a prime and G is a minimal non-p-nilpotent group, i.e., G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent. Then*

(a) *G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a non-normal cyclic q-subgroup of G for some prime $q \neq p$.*

(b) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(c) *The exponent of P is p or* 4.

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [15] (X, 13). We would like to give the following basic facts we will use in our proof.

Lemma 2.9 ([15], X, 13). *Let G be a group and M a subgroup of G.*

(a) *If M is normal in G, then $F^*(M) \leq F^*(G)$.*

(b) $F^*(G) \neq 1$ *if* $G \neq 1$; *in fact,* $F^*(G)/F(G) = \operatorname{soc}(F(G)C_G(F(G))/F(G))$.

(c) $F^*(F^*(G)) = F^*(G) \geq F(G)$; *if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

3. Main results.

Theorem 3.1. *Let G be a group and $P = G_p$ a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are weakly s-normal in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and G is a counter-example with minimal order. We will derive a contradiction in several steps.

Step 1. G has a unique minimal normal subgroup N such that G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N, we will show that G/N satisfies the hypotheses of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P \cap N = P_1 \cap N$ is a Sylow subgroup of N. By the hypotheses, there are a subnormal subgroup K_1 of G and a subgroup $(P_1)_*$ of P_1 such that $G = P_1K_1$ and $P_1 \cap K_1 \leq (P_1)_*$, where $(P_1)_*$ is a subgroup of P_1 which is either s-permutably embedded or s-semipermutable in G. Then $G/N = M/N \cdot K_1N/N = P_1N/N \cdot K_1N/N$. It is easy to see that K_1N/N is subnormal in G/N. Since $(|N\colon P_1 \cap N|, |N\colon K_1 \cap N|) = 1$, $(P_1 \cap N)(K_1 \cap N) = N = N \cap G = N \cap (P_1K_1)$. By Lemma 2.6, $(P_1N) \cap (K_1N) = (P_1 \cap K_1)N$. It follows from Lemmas 2.1 and 2.2 that $(P_1N/N) \cap (K_1N/N) = (P_1 \cap K_1)N/N \leq (P_1)_*N/N$, $(P_1)_*N/N$ is either s-permutably embedded or s-semipermutable in G/N. Hence M/N is weakly s-normal in G/N. Therefore G/N satisfies the hypotheses of the theorem. The choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

Step 2. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by Step 1. By Lemma 2.5(c), G/N satisfies the hypotheses, hence G/N is p-nilpotent. Now the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction.

Step 3. $O_p(G) = 1$ and $G = PN$. Therefore G is not solvable and N is a direct product of isomorphic non-abelian simple groups.

If $O_p(G) \neq 1$, Step 1 yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = NP_1$. By the hypotheses, there are a subnormal subgroup T of G and a subgroup $(P_1)_*$ of P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_*$, where $(P_1)_*$ is a subgroup of P_1 which is either s-permutably embedded or s-semipermutable in G. Since $N \leq O^p(G) \leq T$ by Step 1, we have $P_1 \cap N = (P_1)_* \cap N$.

If $(P_1)_*$ is s-semipermutable in G, then, for any Sylow q-subgroup G_q of G, $q \neq p$, there holds

$$[P_1 \cap N, G_q] \leq N \cap (P_1)_*G_q = N \cap (P_1)_* = N \cap P_1.$$

Obviously, $P_1 \cap N$ is normalized by P. Therefore $P_1 \cap N$ is normal in G. The minimality of N implies that $P_1 \cap N = 1$. Hence N is of order p. Thus G is p-nilpotent, a contradiction. Hence P_1 is s-permutably embedded in G. Then we get a contradiction with the same argument in the Step 3 of the proof of [6] (Theorem 3.1).

If $PN < G$, then PN is p-nilpotent. Hence N is p-nilpotent. Therefore $N = N_p \leq \leq O_p(G) = 1$ by Step 2, a contradiction. Hence $G = PN$.

By Step 2, we can see that G is not solvable and N is a direct product of isomorphic non-abelian simple groups. Thus Step 3 holds.

Step 4. The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by Tate's theorem [1, p. 431] (Satz 4.7), contrary to Step 3. Consequently, there is a maximal subgroup P_1 of P such that $P = = (N \cap P)P_1$. Since P_1 is weakly s-normal in G, by the hypotheses, there are a subnormal subgroup T of G and a subgroup $(P_1)_*$ of P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_*$, where $(P_1)_*$ is a subgroup of P_1 which is either s-permutably embedded or s-semipermutable in G.

Suppose that $(P_1)_*$ is s-semipermutable in G. Since $G = PN$, any Sylow q-subgroup N_q of N is a Sylow q-subgroup of G, where $q \neq p$. We have $(P_1)_*N_q \leq G$, and thus $(P_1)_*N_q \cap N$ is a proper subgroup of N since N is nonsolvable. Then $N \cap (P_1)_*N_q = ((P_1)_* \cap N)N_q < N$. Applying Lemma 2.7, we know that N has a proper normal subgroup M such that either $(P_1)_* \cap N \leq M$ or $N_q \leq M$. Since M is proper in N, by [1] (I, Satz 9.12(b)), M contains no Sylow subgroups of N. Thus $(P_1)_* \cap N \leq M$. Noticing that $P_1 \cap N = (P_1)_* \cap N \leq P_1 \cap M$, we have

$$|N/M|_p = \frac{|N|_p}{|M|_p} = |P \cap N \colon P \cap M| \leq |P \cap N \colon P_1 \cap N| \leq |P \colon P_1| = p.$$

Hence N/M is p-nilpotent by [1] (IV, Satz 2.8), but this is a contradiction.

Hence P_1 is s-permutably embedded in G. Now we get the final contradiction with the same argument in the Step 4 of the proof of [6] (Theorem 3.1).

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P satisfies $\diamondsuit_1$ in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and G is a counter-example with minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Lemma 2.5(c) guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|P \colon D| > p$.

By Theorem 3.1.

Step 3. G has no subgroup of index p.

Suppose that G has a subgroup M such that $|G \colon M| = p$. Then $M \lhd G$. By Step 2 together with induction, M is p-nilpotent, consequently, G is p-nilpotent, a contradiction.

Step 4. $|D| > p$.

Assume that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 2.8(a), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \in \mathrm{Syl}_q(G_1)$, $p \neq q$. Denote $\Phi = \Phi(P_1)$. Let X/Φ be a subgroup of P_1/Φ of order p, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 2.8(c). By the hypotheses, L is weakly s-normal in G, thus in G_1 by Lemma 2.5(a). Since $L \leq P_1 = O_p(G_1)$, by Lemma 2.5(e), L is weakly s-permutable in G_1. Since G_1 is a minimal non-p-

nilpotent subgroup, G_1 has no subgroup of index p. Thus, by [4] (Lemma 2.10(5)), L is s-permutable in G_1. Then $X/\Phi = L\Phi/\Phi$ is s-permutable in G_1/Φ. [4] (Lemma 2.11) implies that $|P_1/\Phi| = p$ since P_1/Φ is minimal normal in G_1/Φ. It follows immediately that P_1 is cyclic. Hence G_1 is p-nilpotent by [1] (Lemma 2.11), contrary to the choice of G_1.

Step 5. P satisfies $\diamondsuit_2$ in G.

Assume that $H \leq P$ such that $|H| = |D|$ and H is neither s-permutably embedded nor s-semipermutable in G. By Lemma 2.5(d), there is a normal subgroup M of G such that $|G\colon M| = p$, contrary to Step 3.

Step 6. If N is minimal normal in G contained in P, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.5(e) and [4] (Lemma 2.11), N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 7. If N is minimal normal in G contained in P, then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemmas 2.1(b) and 2.2. Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 6. We will show that every cyclic subgroup of P/N of order p or order 4 (when P/N is a non-abelian 2-group) is either s-permutably embedded or s-semipermutable in G/N. Let $K \leq P$ with $|K/N| = p$. By Step 4, N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is either s-permutably embedded or s-semipermutable in G by the hypotheses and Step 5, $K/N = LN/N$ is either s-permutably embedded or s-semipermutable in G/N by Lemmas 2.1(b) and 2.2. If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N| = 2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. Since $X/N = LN/N \cong L/(L \cap N)$ is cyclic of order 4, by the hypotheses and Step 5, L is either s-permutably embedded or s-semipermutable in G. By Lemmas 2.1 and 2.2, $X/N = LN/N$ is either s-permutably embedded or s-semipermutable in G/N. Hence P/N satisfies $\diamondsuit_2$ in G/N. By the minimal choice of G, G/N is p-nilpotent.

Step 8. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 7, G/N is p-nilpotent. This means that G has a subgroup of index p, contrary to Step 3.

Step 9. Each minimal normal subgroup of G is not p-nilpotent, $G = LP$ for any minimal normal subgroup L of G.

For any minimal normal subgroup L of G, if L is p-nilpotent, by the fact that $L_{p'}\,\mathrm{char}\, L \lhd G$, we have $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Then $L \leq O_p(G) = 1$ by Step 8, a contradiction. If LP is proper in G, by induction, LP is p-nilpotent, and so L is p-nilpotent, a contradiction. Thus $G = LP$ for any minimal normal subgroup L of G.

Step 10. G is a non-abelian simple group.

Take a minimal normal subgroup L of G. If $L < G$, by Step 9, $G = LP$. Then G has a subgroup of index p, contrary to Step 3. Thus $G = L$ is simple.

Step 11. The final contradiction.

Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup of G with $q \neq p$. If H is s-semipermutable in G, then $HQ^g = Q^g H$ for any $g \in G$ by the hypotheses and Step 5. Since G is simple by Step 10, $G = HQ$ by Lemma 2.7, a contradiction. Hence H is s-permutably embedded in G. So H is a Sylow subgroup of some subnormal subgroup of G. But the subnormal subgroups of G are exactly G and 1, whereas H is a Sylow p-subgroup of neither of them, the final contradiction.

This completes the proof.

***Corollary* 3.1.** *Suppose that G is a group. If every non-cyclic Sylow subgroup of G satisfies $\diamondsuit_1$ in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of E satisfies $\diamondsuit_1$ in G. Then $G \in \mathcal{F}$.*

Proof. Set $p \in \pi(E)$. Suppose that P is a Sylow p-subgroup of E. Since P satisfies $\diamondsuit_1$ in G by hypotheses, P satisfies $\diamondsuit_1$ in E by Lemma 2.5(a). Applying Corollary 3.1, we have E has a Sylow tower of supersolvable type. Let q be the largest prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is weakly s-permutable in G by Lemma 2.5(e). Hence Q satisfies $(*)'$ in G. Since $F^*(Q) = Q$ by Lemma 2.9, we get $G \in \mathcal{F}$ by applying Theorem 1.2.

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $\diamondsuit_1$ in G. Then $G \in \mathcal{F}$.*

Proof. Assume that this theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. By Lemma 2.5(a) the hypothesis is still true for $(F^*(E), F^*(E))$, and so $F^*(E)$ is supersolvable by Theorem 3.3. Hence $F^*(E) = F(E)$, by Lemma 2.9(c). Thus every non-cyclic Sylow subgroup of $F^*(E)$ satisfies $(*)'$ in G. Hence $G \in \mathcal{F}$, by Theorem 1.2.

This completes the proof of the theorem.

4. Some applications. From the definition of weakly s-normal subgroup, we can see that [4] (Corollaries 5.1 – 5.24) and [6] (Corollaries 4.1 – 4.14) are corollaries of our Theorems 3.3 and 3.4. Furthermore, we have the following corollaries.

***Corollary* 4.1.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of E are either s-permutably embedded or s-semipermutable or c-normal in G.*

***Corollary* 4.2.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are either s-semipermutable or c-normal in G.*

***Corollary* 4.3.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of E are either s-permutably embedded or c-normal in G.*

***Corollary* 4.4.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of E are either s-permutably embedded or s-semipermutable in G.*

***Corollary* 4.5** ([19], Theorem 1)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of E are s-semipermutable in G.*

***Corollary* 4.6.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order* 4 *of $F^*(E)$ are either s-permutably embedded or s-semipermutable or c-normal in G.*

***Corollary* 4.7.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(E)$ are either s-permutably embedded or s-semipermutable in G.*

***Corollary* 4.8.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order* 4 *of $F^*(E)$ are either s-permutably embedded or c-normal in G.*

***Corollary* 4.9.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order* 4 *of $F^*(E)$ are either s-semipermutable or c-normal in G.*

***Corollary* 4.10** ([19], Theorem 1)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a solvable normal subgroup E such that $G/E \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F(E)$ are s-semipermutable in G.*

***Corollary* 4.11** ([19], Theorem 3)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a solvable normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order* 4 *of $F(E)$ are s-semipermutable in G.*

***Corollary* 4.12.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a solvable normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of $F(E)$ of prime order are either s-permutably embedded or s-semipermutable in G and the Sylow* 2*-subgroups of $F(E)$ are abelian.*

***Corollary* 4.13** ([19], Theorem 6)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a solvable normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of $F(E)$ of prime order are s-semipermutable in G and the Sylow* 2*-subgroups of $F(E)$ are abelian.*

Theorem 3.2 is also interesting. Using a similar way, we can generalize it as follows.

Theorem 4.1. *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P satisfies $\diamondsuit_1$ in G, then G is p-nilpotent.*

***Corollary* 4.14.** *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either s-permutably embedded or s-semipermutable or c-normal in G, then G is p-nilpotent.*

***Corollary* 4.15.** *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with*

$(|G|, p-1) = 1$. *If every maximal subgroup of* P *is either* s*-permutably embedded or* s*-semipermutable in* G, *then* G *is* p*-nilpotent.*

***Corollary* 4.16.** *Let* G *be a group,* H *a normal subgroup of* G *such that* G/H *is* p*-nilpotent and* P *a Sylow* p*-subgroup of* H, *where* p *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *If* P *satisfies* $(*)'$ *in* G, *then* G *is* p*-nilpotent.*

***Corollary* 4.17** ([17], Theorem 3.3)**.** *Let* G *be a group,* H *a normal subgroup of* G *such that* G/H *is* p*-nilpotent and* P *a Sylow* p*-subgroup of* H, *where* p *is the minimal prime dividing the order of* G. *If every maximal subgroup of* P *is* s*-semipermutable in* G, *then* G *is* p*-nilpotent.*

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