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A CLASS OF STRONG LIMIT THEOREMS FOR NONHOMOGENEOUS MARKOV CHAINS INDEXED BY A GENERALIZED BETHE TREE ON A GENERALIZED RANDOM SELECTION SYSTEM*

ПРО ОДИН КЛАС СИЛЬНИХ ГРАНИЧНИХ ТЕОРЕМ ДЛЯ НЕОДНОРІДНИХ МАРКОВСЬКИХ ЛАНЦЮЖКІВ, ЩО ПРОІНДЕКСОВАНІ УЗАГАЛЬНЕНИМ ДЕРЕВОМ БЕТЕ НА УЗАГАЛЬНЕНІЙ СИСТЕМІ ВИПАДКОВОГО ВИБОРУ

We study strong limit theorems for a bivariate function sequence of an nonhomogeneous Markov chain indexed by a generalized Bethe tree on a generalized random selection system by constructing a nonnegative martingale. As corollaries, we generalize results of Yang and Ye and obtain some limit theorems for frequencies of states, ordered couples of states, and the conditional expectation of a bivariate function on Cayley tree.

Вивчаються сильні граничні теореми для послідовності функцій двох змінних неоднорідного марковського ланцюжка, що проіндексований узагальненим деревом Бете на узагальненій системі випадкового вибору, шляхом побудови невід'ємного мартингала. Як наслідок, узагальнено результати Янга та Є і отримано деякі граничні теореми для частот станів, упорядкованих пар та умовного сподівання функції двох змінних на дереві Келі.

1. Introduction and definition. Let T be a tree which is infinite, connected and contains no circuits. Given any two vertices $x \neq y \in T$, there exists an unique path $x = x_1, x_2, \ldots, x_m = y$ from x to y with $x_1, x_2, \ldots, x_m$ distinct. The distance between x and y is defined to $m-1$, the number of edges in the path connecting x and y. To index the vertices on T, we first assign a vertex as the „root"and label it as O. A vertex is said to be on the nth level if the path linking it to the root has n edges. The root O is also said to be on the 0th level.

Definition 1. *Let T be a tree with root O, and let $\{N_n, n \geq 1\}$ be a sequence of positive integers. T is said to be a generalized Bethe tree or a generalized Cayley tree if each vertex on the nth level has N_{n+1} branches to the $n+1$th level. For example, when $N_1 = N + 1 \geq 2$ and $N_n = N$, $n \geq 2$, T is rooted Bethe tree (a homogeneous tree) $T_{B,N}$ on which each vertex has $N+1$ neighboring vertices ($T_{B,2}$ drawn in Figure), and when $N_n = N \geq 1$, $n \geq 1$, T is rooted Cayley tree $T_{C,N}$ on which each vertex has N branches to the next level.*

In the following, we always assume that T is a generalized Bethe tree and denote by $T^{(n)}$ the subgraph of T containing the vertices from level 0 (the root) to level n. We use (n,j), $1 \leq j \leq N_1 \ldots N_n$, $n \geq 1$, to denote the jth vertex at the nth level and denote by $|B|$ the number of vertices in the subgraph B, L_m^n the set of all vertices from level m to level n, L_n the set of all vertices on level n. It is easy to see that for $n \geq 1$,

$$|T^{(n)}| = \sum_{m=0}^{n} N_0 \ldots N_m = 1 + \sum_{m=1}^{n} N_1 \ldots N_m. \tag{1}$$

Let $S = \{s_0, s_1, s_2, \ldots\}$, $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and takes values in S, and $\mathcal{F}$ be the smallest Borel field containing all cylinder sets in

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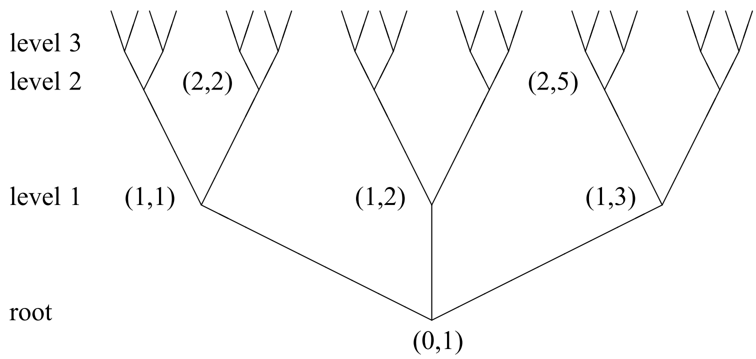

Bethe tree $T_{B,2}$

Ω. Let $X = \{X_t, t \in T\}$ be the coordinate stochastic process defined on the measurable space $(\Omega, \mathcal{F})$ (see [1, p. 412]); that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t), \quad t \in T. \tag{2}$$

Let μ be an arbitrary probability measure defined on $(\Omega, \mathcal{F})$. Denote

$$X^{T^{(n)}} \triangleq \{X_t, t \in T^{(n)}\}, \qquad \mu\big(X^{T^{(n)}} = x^{T^{(n)}}\big) = \mu\big(x^{T^{(n)}}\big), \tag{3}$$

where $\omega(t)$ is in fact the sample point function with respect to t, $X = \{X_t, t \in T\}$ is a stochastic process defined on the tree T, that is, $X = \{X_t, t \in T\}$ is a sequence of random variables defined on all the vertices of T $\big($i.e., $\{X_t, t \in T\} = \{X_{0,1}, X_{1,1}, X_{1,2}, \ldots, X_{1,N_0N_1}, X_{2,1}, \ldots, X_{2,N_0N_1N_2}, \ldots, X_{m,1}, \ldots, X_{m,N_0\ldots N_m}, \ldots\}\big)$. We denote by $x^{T^{(n)}}$ the realization of the stochastic process $X^{T^{(n)}}$. $X^{T^{(n)}}$ stands for the sequence of the random variables defined on all the vertices from the root to level n on the tree T. $x_{0,1}$ is the realization of $X_{0,1}$ which is the random variable defined on the root.

Now we give a definition of Markov chains field on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see [2]).

Definition 2. *Let $\{P_n = P_n(j|i),\ i, j \in S,\ n \geq 1\}$ be stochastic matrices on S^2, $p = (p(s_0), p(s_1), p(s_2), \ldots)$ be a distribution on S, and μ_P be a measure on $(\Omega, \mathcal{F})$. If*

$$\mu_P(x_{0,1}) = p(x_{0,1}), \tag{4}$$

$$\mu_P(x^{T^{(n)}}) = p(x_{0,1}) \prod_{m=0}^{n-1} \prod_{i=1}^{N_0\ldots N_m} \prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} P_{m+1}(x_{m+1,j}|x_{m,i}), \quad n \geq 1. \tag{5}$$

Then μ_P will be called a Markov chains field on the tree T determined by the stochastic matrices P_n and the distribution p.

The tree model have recently drawn the increasing interest from specialists in physics, probability and information theory. For the early studies on Markov chains fields on trees see Spitzer [3]. Benjamini and Peres [4] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [5] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Pemantle [6] proved a mixing property and a weak law of large numbers for a PPG-invariant and ergodic random field on a homogeneous tree. Ye and Berger [7], by using Pemantle's result and a combinatorial approach, have studied the Shannon – McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang [8, 9] and Liu [1] have studied strong laws of large numbers for the frequency of occurrence of states for Markov chains field on the Bethe tree and the generalized Bethe tree. Yang and Ye [2] have discussed the strong limit theorems for nonhomogeneous Markov chain indexed by the homogeneous tree. Shi and Yang [10] have investigated a limit property of random transition probability for a nonhomogeneous Markov chain indexed by a tree.

In this paper, we study a class of strong limit theorems for a bivariate function sequence for nonhomogeneous Markov chains field indexed by the generalized Bethe tree on the the generalized random selection system by constructing a nonnegative martingale. As corollaries, we generalize Yang and Ye's results (see [2, 8]) and obtain some limit theorems for frequencies of states, ordered couples of states, the harmonic mean of the transition probabilities of the nonhomogeneous Markov chain and the conditional expectation on Cayley tree.

Definition 3. *Let* $\{f_{m,i}(x_{0,1}, x_{1,1}, \ldots, x_{1,N_0N_1}, \ldots, x_{m,1}, \ldots, x_{m,i-1}), 0 \le m \le \le n, 1 \le i \le N_0 \ldots N_m\}$ *be a series of real-valued functions defined on* $S^{T^{(n)}}$, $n = 1, 2, \ldots,$ *which take values in an arbitrary interval* $[a, b]$ $(a, b \in R)$. *Denote*

$$Y_0 = f_{0,1} = 1,$$

$$Y_{m,i} = f_{m,i}(X_{0,1}, X_{1,1}, \ldots, X_{1,N_0N_1}, \ldots, X_{m,1}, \ldots, X_{m,i-1}),$$

$$1 \le m \le n, \quad 2 \le i \le N_0 \ldots N_m,$$

$$Y_{m+1,1} = f_{m+1,1}(X_{0,1}, X_{1,1}, \ldots, X_{1,N_0N_1}, \ldots, X_{m,1}, \ldots, X_{m,N_0 \ldots N_m}),$$

$$1 \le m \le n - 1. \tag{6}$$

We call $\{Y_{m,i},\ 0 \le m \le n, 1 \le i \le N_0 \ldots N_m\}$ *as the generalized random selection system on the generalized Bethe trees* (*the traditional random selection system takes values in the set* $\{0, 1\}$).

We first explain the conception of the traditional random selection, which is the crucial part of the gambling system. We give a set of real-valued functions $f_n(x_1, \ldots, x_n)$ defined on S^n, $n = 1, 2, \ldots,$ which will be called the random selection function if they take values in a two-valued set $\{0, 1\}$. Then let

$$Y_1 = y \quad (y \text{ is an arbitrary real number}),$$

$$Y_{n+1} = f_n(X_1, \ldots, X_n), \quad n \ge 1,$$

where $\{Y_n, n \ge 1\}$ be called as the gambling system (the random selection system). Let $\delta_i(j)$ be the Kronecker delta function on S, that is for $i, j \in S$

$$\delta_i(j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

In order to explain the real meaning of the notion of the random selection, we consider the traditional gambling model. Let $\{X_n, n \geq 0\}$ be a nonhomogeneous Markov chain, and $\{g_n(x,y), n \geq 1\}$ be a real-valued function sequence defined on S^2. Interpret X_n as the result of the n th trial, the type of which may change at each step. Let $\mu_n = Y_n g_n(X_{n-1}, X_n)$ denote the gain of the bettor at the n th trial, where Y_n represents the bet size, $g_n(X_{n-1}, X_n)$ is determined by the gambling rules, and $\{Y_n, n \geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n \geq 1\}$ by the results of the last trial. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on X_{n-1} as $n \geq 1$, and b_0 is a constant. Thus $\sum_{k=1}^{n} Y_k g_k(X_{k-1}, X_k)$ represents the total gain in the first n trials, $\sum_{k=1}^{n} b_k$ the accumulated entrance fees, and $\sum_{k=1}^{n} \left[Y_k g_k(X_{k-1}, X_k) - b_k\right]$ the accumulated net gain. Motivated by the classical definition of „fairness"of game of chance (see Kolmogorov [11]), we introduce the following definition:

Definition 4. *The game is said to be fair, if for almost all* $\omega \in \{\omega\colon \sum_{k=1}^{\infty} Y_k = \infty\}$, *the accumulated net gain in the first* n *trial is to be of smaller order of magnitude than the accumulated stake* $\sum_{k=1}^{n} Y_k$ *as* n *tends to infinity, that is*

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \sum_{k=1}^{n} [Y_k g_k(X_{k-1}, X_k) - b_k] = 0 \quad \textit{a.s. on} \quad \left\{\omega\colon \sum_{k=1}^{\infty} Y_k = \infty\right\}.$$

We generalize the traditional gambling system to the case of the nonhomogeneous Markov chain indexed by the generalized Bethe tree, and obtain the following conclusion:

2. Main results.

Theorem 1. *Let* $X = \{X_t, t \in T\}$ *be a nonhomogeneous Markov chain indexed by the generalized Bethe tree with the initial distribution and the transition matrices defined as Definition* 2. *Let* $\{g_n(x,y), n \geq 1\}$ *be a series of real-valued functions defined on* S^2. *Let* $\{a_n, n \geq 0\}$ *be a nonnegative stochastic sequence, denote* $\alpha > 0$,

$$F_n(\omega) = \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j}), \tag{7}$$

$$G_n(\omega) = \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} \mathrm{E}\left[Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j}) | X_{m,i}\right], \tag{8}$$

$$H_n(\omega) = \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} \mathrm{E}\left[\exp\{\alpha \left|Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})\right|\} | X_{m,i}\right]. \tag{9}$$

Put

$$D = \left\{\omega\colon \lim_{n\to\infty} a_n = \infty,\ \limsup_{n\to\infty} \frac{1}{a_n} H_n(\omega) < \infty\right\}, \tag{10}$$

then

$$\lim_{n\to\infty} \frac{1}{a_n}[F_n(\omega) - G_n(\omega)] = 0 \quad \mu_P\text{-a.s.}, \quad \omega \in D. \tag{11}$$

Proof. Consider the probability measure space $(\Omega, \mathcal{F}, \mu_{\mathcal{P}})$, letting λ be an arbitrary constant, we construct

$$T_n(\lambda, \omega) =$$

$$= \frac{e^{\lambda\left[\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})\right]}}{\prod_{m=0}^{n-1}\prod_{i=1}^{N_0\ldots N_m}\prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}[e^{\lambda Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})}|X_{m,i}]}, \tag{12}$$

$$n \geq 1.$$

Noting that $X = \{X_t, t \in T\}$ satisfies (5), we have

$$\mathrm{P}(X^{L_n} = x^{L_n}|X^{T^{(n-1)}} = x^{T^{(n-1)}}) = \frac{\mu_P(x^{T^{(n)}})}{\mu_P(x^{T^{(n-1)}})} =$$

$$= \prod_{i=1}^{N_0\ldots N_{n-1}} \prod_{j=N_n(i-1)+1}^{N_n i} P_n(x_{n,j}|x_{n-1,i}). \tag{13}$$

Denoting $\mathcal{F}_n = \sigma(X^{T^{(n)}})$, by (12), (13) and Markov's property, we have

$$\mathrm{E}\left[T_n(\lambda, \omega)|\mathcal{F}_{n-1}\right] =$$

$$= T_{n-1}(\lambda, \omega) \frac{\mathrm{E}\left[e^{\lambda\left[\sum_{i=1}^{N_0\ldots N_{n-1}}\sum_{j=N_n(i-1)+1}^{N_n i} Y_{n-1,i} g_n(X_{n-1,i}, X_{n,j})\right]}|X^{T^{(n-1)}}\right]}{\prod_{i=1}^{N_0\ldots N_{n-1}}\prod_{j=N_n(i-1)+1}^{N_n i} \mathrm{E}[e^{\lambda Y_{n-1,i} g_n(X_{n-1,i}, X_{n,j})}|X_{n-1,i}]} =$$

$$= \frac{T_{n-1}(\lambda, \omega) \sum\limits_{x^{L_n} \in S^{L_n}} e^{\lambda\left[\sum\limits_{i=1}^{N_0\ldots N_{n-1}}\sum\limits_{j=N_n(i-1)+1}^{N_n i} Y_{n-1,i} g_n(X_{n-1,i}, x_{n,j})\right]}}{\prod\limits_{i=1}^{N_0\ldots N_{n-1}}\prod\limits_{j=N_n(i-1)+1}^{N_n i} \mathrm{E}[e^{\lambda Y_{n-1,i} g_n(X_{n-1,i}, X_{n,j})}|X_{n-1,i}]} \times$$

$$\times \prod_{i=1}^{N_0\ldots N_{n-1}} \prod_{j=N_n(i-1)+1}^{N_n i} P_n(x_{n,j}|X_{n-1,i}) =$$

$$= T_{n-1}(\lambda,\omega)\frac{\sum\limits_{x^{L_n}\in S^{L_n}}\prod\limits_{i=1}^{N_0\ldots N_{n-1}}\prod\limits_{j=N_n(i-1)+1}^{N_n i} e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},x_{n,j})}P_n(x_{n,j}|X_{n-1,i})}{\prod\limits_{i=1}^{N_0\ldots N_{n-1}}\prod\limits_{j=N_n(i-1)+1}^{N_n i}\mathrm{E}[e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},X_{n,j})}|X_{n-1,i}]} =$$

$$= T_{n-1}(\lambda,\omega)\frac{\prod\limits_{i=1}^{N_0\ldots N_{n-1}}\prod\limits_{j=N_n(i-1)+1}^{N_n i}\sum\limits_{x_{n,j}\in S} e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},x_{n,j})}P_n(x_{n,j}|X_{n-1,i})}{\prod\limits_{i=1}^{N_0\ldots N_{n-1}}\prod\limits_{j=N_n(i-1)+1}^{N_n i}\mathrm{E}[e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},X_{n,j})}|X_{n-1,i}]} =$$

$$= T_{n-1}(\lambda,\omega)\frac{\prod_{i=1}^{N_0\ldots N_{n-1}}\prod_{j=N_n(i-1)+1}^{N_n i}\mathrm{E}[e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},X_{n,j})}|X_{n-1,i}]}{\prod_{i=1}^{N_0\ldots N_{n-1}}\prod_{j=N_n(i-1)+1}^{N_n i}\mathrm{E}[e^{\lambda Y_{n-1,i}g_n(X_{n-1,i},X_{n,j})}|X_{n-1,i}]} =$$

$$= T_{n-1}(\lambda,\omega). \tag{14}$$

Therefore, $\{T_n(\lambda,\omega),\mathcal{F}_n, n\geq 1\}$ is a nonnegative martingale. By Doob's martingale convergence theorem, we obtain

$$\lim_{n\to\infty} T_n(\lambda,\omega) = T_\infty(\lambda,\omega) < \infty \quad \mu_P\text{-a.s.} \tag{15}$$

By the first equation $\lim\limits_{n\to\infty} a_n = \infty$ of (10) and (15) we have

$$\limsup_{n\to\infty}\frac{1}{a_n}\ln T_n(\lambda,\omega)\leq 0 \quad \mu_P\text{-a.s.},\quad \omega\in D. \tag{16}$$

By (7), (12) and (16), we obtain

$$\limsup_{n\to\infty}\frac{1}{a_n}\left\{\lambda F_n(\omega)-\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\ln\mathrm{E}[e^{\lambda Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})}|X_{m,i}]\right\}\leq$$

$$\leq 0 \quad \mu_P\text{-a.s.},\quad \omega\in D. \tag{17}$$

By (8), (17) and the inequalities $\ln x\leq x-1 (x>0)$, $e^x-1-x\leq (1/2)x^2e^{|x|}$, noticing that

$$\max\left\{x^2e^{-hx}, x\geq 0\right\} = \frac{4e^{-2}}{h^2},\quad h>0,$$

letting $0<|\lambda|<\alpha$, we have

$$\limsup_{n\to\infty}\frac{1}{a_n}\lambda\{F_n(\omega)-G_n(\omega)\}\leq$$

$$\leq \limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\Big\{\ln\mathrm{E}[e^{\lambda Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})}|X_{m,i}]-$$

$$-\mathrm{E}[\lambda Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})|X_{m,i}]\Big\}\leq$$

$$\leq\limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\Big\{\mathrm{E}[e^{\lambda Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})}|X_{m,i}]-$$

$$-1-\mathrm{E}[\lambda Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})|X_{m,i}]\Big\}\leq$$

$$\leq\limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\mathrm{E}\left[\frac{\lambda^2}{2}Y_{m,i}^2g_{m+1}^2(X_{m,i},X_{m+1,j})\times\right.$$

$$\left.\times e^{|\lambda||Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})|}|X_{m,i}\right]=$$

$$=\frac{\lambda^2}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\mathrm{E}\Big[Y_{m,i}^2g_{m+1}^2(X_{m,i},X_{m+1,j})\times$$

$$\times e^{(|\lambda|-\alpha)|Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})|}e^{\alpha|Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})}|X_{m,i}\Big]\leq$$

$$\leq\ \frac{\lambda^2}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\mathrm{E}\left[\frac{e^{\alpha|Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})|}4e^{-2}}{(|\lambda|-\alpha)^2}\Big|X_{m,i}\right]$$

$$\mu_P\text{-a.s.},\quad \omega\in D. \tag{18}$$

Taking $0<\lambda<\alpha$, dividing two sides of (18) by λ, we arrive at

$$\limsup_{n\to\infty}\frac{1}{a_n}\{F_n(\omega)-G_n(\omega)\}\leq\frac{2\lambda e^{-2}}{(\lambda-\alpha)^2}\times$$

$$\times\limsup_{n\to\infty}\frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i}\mathrm{E}[e^{\alpha|Y_{m,i}g_{m+1}(X_{m,i},X_{m+1,j})}|X_{m,i}]<\infty$$

$$\mu_P\text{-a.s.},\quad \omega\in D. \tag{19}$$

Since $2\lambda e^{-2}\big/(\lambda-\alpha)^2\to 0$ as $\lambda\to +0$, by (19) we obtain

$$\limsup_{n\to\infty}\frac{1}{a_n}\{F_n(\omega)-G_n(\omega)\}\leq 0\quad \mu_P\text{-a.s.},\quad \omega\in D. \tag{20}$$

Taking $-\alpha<\lambda<0$, dividing two sides of (18) by λ, we have

$$\liminf_{n\to\infty}\frac{1}{a_n}\big\{F_n(\omega)-G_n(\omega)\big\}\geq\frac{2\lambda e^{-2}}{(\lambda+\alpha)^2}\limsup_{n\to\infty}\frac{1}{a_n}H_n(\omega)\quad \mu_P\text{-a.s.},\quad \omega\in D. \tag{21}$$

Since $2\lambda e^{-2}/(\lambda+\alpha)^2 \to 0$ as $\lambda \to -0$, by (21) we obtain

$$\liminf_{n\to\infty} \frac{1}{a_n}\{F_n(\omega) - G_n(\omega)\} \geq 0 \quad \mu_P\text{-a.s.}, \quad \omega \in D. \tag{22}$$

Therefore, it follows from (20) and (22) that (11) holds.

Theorem 2. *Let $X = \{X_t, t \in T\}$ be a nonhomogeneous Markov chain indexed by the generalized Bethe tree with the initial distribution and the transition matrices defined as Definition* 2. *Let $\{g_n(x,y),\, n \geq 1\}$, $\{a_n, n \geq 0\}$, $F_n(\omega)$ and $G_n(\omega)$ be all defined as Theorem* 1. *Denote $\alpha > 0$,*

$$B_n(\omega) = \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}\Big[Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha|Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|} | X_{m,i}\Big]. \tag{23}$$

Put

$$L(\omega) = \left\{\omega\colon \lim_{n\to\infty} a_n = \infty,\ \limsup_{n\to\infty} \frac{1}{a_n} B_n(\omega) < \infty\right\}, \tag{24}$$

then

$$\lim_{n\to\infty} \frac{1}{a_n}\big[F_n(\omega) - G_n(\omega)\big] = 0 \quad \mu_P\text{-a.s.}, \quad \omega \in L(\omega). \tag{25}$$

Proof. By the third inequality of (18) in the proof of Theorem 1, taking $0 < |\lambda| < \alpha$, we arrive at

$$\limsup_{n\to\infty} \frac{1}{a_n} \lambda\{F_n(\omega) - G_n(\omega)\} \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}\left[\frac{\lambda^2}{2} Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times\right.$$

$$\left. \times e^{|\lambda||Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|} | X_{m,i}\right] \leq$$

$$\leq \frac{\lambda^2}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}\Big[Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha|Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|} | X_{m,i}\Big] < \infty$$

$$\mu_P\text{-a.s.}, \quad \omega \in D. \tag{26}$$

Take $0 < \lambda < \alpha$, dividing two sides of (26) by λ, we have

$$\limsup_{n\to\infty} \frac{1}{a_n}\{F_n(\omega) - G_n(\omega)\} \leq$$

$$\leq \frac{\lambda}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} \mathrm{E}\Big[Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha |Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|} |X_{m,i}\Big] < \infty \quad \mu_P\text{-a.s.}, \quad \omega \in D. \tag{27}$$

Letting $\lambda \to +0$, we have by (27) that

$$\limsup_{n\to\infty} \frac{1}{a_n} \{F_n(\omega) - G_n(\omega)\} \leq 0 \quad \mu_P\text{-a.s.}, \quad \omega \in D. \tag{28}$$

Taking $-\alpha < \lambda < 0$ in (26), we similarly obtain

$$\liminf_{n\to\infty} \frac{1}{a_n} \{F_n(\omega) - G_n(\omega)\} \geq$$

$$\geq \frac{\lambda}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0 \ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1} i} \mathrm{E}\Big[Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha |Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|} |X_{m,i}\Big] \quad \mu_P\text{-a.s.}, \quad \omega \in D.$$

Letting $\lambda \to -0$, we have

$$\liminf_{n\to\infty} \frac{1}{a_n} \{F_n(\omega) - G_n(\omega)\} \geq 0 \quad \mu_P\text{-a.s.}, \quad \omega \in D. \tag{29}$$

It follows from (28) and (29) that (25) holds.

***Corollary* 1** [2]. *Let $X = \{X_t, t \in T\}$ be a nonhomogeneous Markov chain indexed by a homogeneous tree $T_{B,N}$. Let $\{g_n(x,y), n \geq 1\}$, $\{a_n, n \geq 0\}$ be defined as Theorem* 1. *Denote $\alpha > 0$,*

$$G_n(\omega) = \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\Big[g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha |g_{m+1}(X_{m,i}, X_{m+1,j})|} |X_{m,i}\Big]. \tag{30}$$

Put

$$J(\omega) = \left\{\omega\colon \lim_{n\to\infty} a_n = \infty,\ \limsup_{n\to\infty} \frac{1}{a_n} G_n(\omega) < \infty\right\}, \tag{31}$$

then

$$\lim_{n} \frac{1}{a_n} \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \Big\{g_{m+1}(X_{m,i}, X_{m+1,j}) -$$

$$- \mathrm{E}[g_{m+1}(X_{m,i}, X_{m+1,j}) | X_{m,i}]\Big\} = 0 \quad \mu_P\text{-}a.s., \quad \omega \in J(\omega). \tag{32}$$

Proof. Letting $N_0 = 1,\ N_1 = N + 1,\ N_n = N (n \geq 2)$, $Y_{m,i} \equiv 1$ in Theorem 2, (30), (31) and (32) follow from (23), (24) and (25).

Remark. The corollary is Theorem 1 of Yang and Ye (see [2]). Letting $Y_{m,i} = 1$ in Theorem 1, it can be seen that the condition (9), (10) weakens the condition (30), (31) of Theorem 1 in the paper of Yang and Ye. Correspondingly the conclusion is strengthened.

***Corollary* 2** [8]. *Let $X = \{X_t, t \in T\}$ be a nonhomogeneous Markov chain indexed by the homogeneous tree. Let $g(x, y)$ be a function defined on S^2 taking values in $\{0, 1\}$, $\{a_n, n \geq 0\}$ be defined as Theorem 1. Put*

$$G(\omega) = \Bigg\{\omega\colon \lim_{n\to\infty} a_n = \infty,$$

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[g(X_{m,i}, X_{m+1,j})|X_{m,i}\right] < \infty\Bigg\}, \tag{33}$$

then

$$\lim_{n} \frac{1}{a_n} \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \Big\{g(X_{m,i}, X_{m+1,j}) -$$

$$- \mathrm{E}[g(X_{m,i}, X_{m+1,j})|X_{m,i}]\Big\} = 0 \quad \mu_P\text{-}a.s., \quad \omega \in G(\omega). \tag{34}$$

Proof. Letting $g_n(x, y) = g(x, y)$, $n \geq 1$, $Y_{m,i} \equiv 1$ in Theorem 2, by (23), (33) and the definition of $g(x, y)$, we have

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\Big[Y_{m,i}^2 g_{m+1}^2(X_{m,i}, X_{m+1,j}) \times$$

$$\times e^{\alpha|Y_{m,i} g_{m+1}(X_{m,i}, X_{m+1,j})|}|X_{m,i}\Big] \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{a_n} e^{\alpha} \sum_{m=1}^{n-1} \sum_{i=1}^{(N+1)N^{m-1}} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[g(X_{m,i}, X_{m+1,j})|X_{m,i}\right] < \infty. \tag{35}$$

Hence $G(\omega) \subset J(\omega)$, (34) follows from Theorem 2.

***Corollary* 3.** *Let $S = \{1, 2, \ldots, N\}$, and*

$$\beta_n = \min\{P_n(y|x), x, y \in S\}, \quad n \geq 1. \tag{36}$$

If there exists $\alpha > 0$, such that

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} e^{\frac{\alpha}{\beta_{m+1}}} \prod_{j=0}^{m+1} N_j = M < \infty, \tag{37}$$

then the harmonic mean of the transition probabilities $\{P_{m+1}(X_{m+1,j}|X_{m,i}), 0 \leq m \leq \leq n-1, 1 \leq i \leq N_0 \ldots N_m, N_{m+1}(i-1)+1 \leq j \leq N_{m+1}i\}$ for the nonhomogeneous

Markov chain indexed by the generalized bethe tree converges to N^{-1} a.s., that is

$$\lim_{n\to\infty} \frac{|T^{(n)}|}{\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} P_{m+1}(X_{m+1,j}|X_{m,i})^{-1}} = \frac{1}{N} \quad \mu_P\text{-a.s.} \tag{38}$$

Proof. Letting $a_n(\omega) = |T^{(n)}|$, $g_{m+1}(X_{m,i}, X_{m+1,j}) = P_{m+1}(X_{m+1,j}|X_{m,i})^{-1}$, $Y_{m,i} \equiv 1$ in Theorem 1, by (9), (10), (36) and (37) we have

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}\left[\exp\{\alpha\,|Y_{m,i}g_{m+1}(X_{m,i}, X_{m+1,j})|\}|X_{m,i}\right] =$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}[\exp\{\alpha\left|P_{m+1}(X_{m+1,j}|X_{m,i})^{-1}\right|\}|X_{m,i}] \le$$

$$\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}[e^{\frac{\alpha}{\beta_{m+1}}}|X_{m,i}] =$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} N_0 \ldots N_m N_{m+1} e^{\frac{\alpha}{\beta_{m+1}}} =$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} e^{\frac{\alpha}{\beta_{m+1}}} \prod_{j=0}^{m+1} N_j = M < \infty. \tag{39}$$

By (10) and (39) we obtain $D = \Omega$. Noticing that

$$\mathrm{E}\left[g_{m+1}(X_{m,i}, X_{m+1,j})|X_{m,i}\right] = \mathrm{E}\left[P_{m+1}(X_{m+1,j}|X_{m,i})^{-1}|X_{m,i}\right] =$$

$$= \sum_{x_{m+1,j}\in S} P_{m+1}(x_{m+1,j}|X_{m,i})^{-1} P_{m+1}(x_{m+1,j}|X_{m,i}) = N. \tag{40}$$

By (11) and (40), we arrive at

$$\lim_{n\to\infty} \frac{1}{a_n}[F_n(\omega) - G_n(\omega)] =$$

$$= \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} [P_{m+1}(X_{m+1,j}|X_{m,i})^{-1} - N] =$$

$$= \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} P_{m+1}(X_{m+1,j}|X_{m,i})^{-1} -$$

$$- \lim_{n\to\infty} \frac{|T^{(n)}| - 1}{|T^{(n)}|} N =$$

$$= \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} P_{m+1}(X_{m+1,j}|X_{m,i})^{-1} - N = 0.$$

Hence, (38) follows from the above equation.

Remark. The corollary is a generalization of Theorem 1 of Shi and Yang (see [10]).

3. Derivation results. In the Definition 2, if for all n,

$$P_n = P = (P(y|x)) \quad \forall x, y \in S. \tag{41}$$

$X = \{X_\sigma, \sigma \in T\}$ will be also called S-valued homogeneous Markov chain indexed by a generalized Bethe tree. At the moment, we have

$$\mu_P(x_{0,1}) = p(x_{0,1}), \tag{42}$$

$$\mu_P(x^{T^{(n)}}) = p(x_{0,1}) \prod_{m=0}^{n-1} \prod_{i=1}^{N_0\ldots N_m} \prod_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} P(x_{m+1,j}|x_{m,i}), \quad n \geq 1. \tag{43}$$

Theorem 3. *Let $X = \{X_t, t \in T\}$ be a homogeneous Markov chain indexed by the generalized Bethe tree, $g(x,y)$, $F_n(\omega)$ and $G_n(\omega)$ be defined as before. $\{Y_{m,i},\ 0 \leq \leq m \leq n,\ 1 \leq i \leq N_0 \ldots N_m\}$ take values in a real-valued interval $[a,b]$, where $a, b \in R$. Denote $M = \max\{|a|, |b|\}$, if*

$$\sum_{l\in S}\sum_{k\in S} \exp\{\alpha M\,|g(k,l)|\} P(l|k) < \infty. \tag{44}$$

Then

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|}[F_n(\omega) - G_n(\omega)] = 0 \quad \mu_P\text{-a.s.} \tag{45}$$

Proof. Letting $a_n = |T^{(n)}|$ in Theorem 1, by (10) we have

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \mathrm{E}\left[\exp\{\alpha\,|Y_{m,i}g(X_{m,i}, X_{m+1,j})|\}|X_{m,i}\right] =$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \sum_{x_{m+1,j}\in S} \exp\{\alpha\,|Y_{m,i}g(X_{m,i}, x_{m+1,j})|\}\times$$

$$\times P(x_{m+1,j}|X_{m,i}) =$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \sum_{l\in S}\sum_{k\in S} \delta_k(X_{m,i}) \exp\{\alpha\,|Y_{m,i}g(k,l)|\}\times$$

$$\times P(l|k) \leq \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N_0\ldots N_m} \sum_{j=N_{m+1}(i-1)+1}^{N_{m+1}i} \sum_{l\in S}\sum_{k\in S} \exp\{\alpha M\,|g(k,l)|\}\times$$

$$\times P(l|k) \leq \sum_{l\in S}\sum_{k\in S} \limsup_{n\to\infty} \frac{|T^{(n)}|-1}{|T^{(n)}|} \exp\{\alpha M\,|g(k,l)|\} P(l|k) =$$

$$= \sum_{l\in S}\sum_{k\in S} \exp\{\alpha M\,|g(k,l)|\} P(l|k). \tag{46}$$

By (44) and (46), we have $D=\Omega$. Therefore, (45) follows from Theorem 1.

***Corollary* 4** [2]. *Let $X=\{X_t, t\in T\}$ be a homogeneous Markov chain indexed by a Cayley tree $T_{C.N}$, $S_n(k,l)$ be the number of couple (k,j) in the set of random couple $\{(X_{m,i}, X_{m+1,j}), 0\leq m\leq n-1, 1\leq i\leq N^m, N(i-1)+1\leq j\leq Ni\}$, $S_n(k)$ be the number of k in the set of random variables $X=\{X_t, t\in T^{(n)}\}$. Then*

$$\lim_{n\to\infty}\left[\frac{S_n(k,l)}{|T^{(n)}|} - \frac{S_{n-1}(k)}{|T^{(n-1)}|}P(l|k)\right] = 0 \quad \mu_P\text{-a.s.} \tag{47}$$

Proof. Letting $a_n=|T^{(n)}|$, $g_n(x,y)=g(x,y)=I_k(x)I_j(y)$, $n\geq 1$, $N_0=1$, $N_n=N$, $n\geq 1$, $Y_{m,i}\equiv 1$ in Theorem 1, we have by (10) that

$$\limsup_{n} \frac{1}{a_n}\sum_{m=0}^{n-1}\sum_{i=1}^{N_0\ldots N_m}\sum_{j=N_{m+!}(i-1)+1}^{N_{m+!}i} \mathrm{E}\left[\exp\{\alpha\,|Y_{m,i}g(X_{m,i},X_{m+1,j})|\}|X_{m,i}\right] =$$

$$= \limsup_{n} \frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[\exp\{\alpha\,|I_k(X_{m,i})I_j(X_{m+1,j})|\}|X_{m,i}\right] \leq$$

$$\leq \limsup_{n} \frac{|T^{(n)}|-1}{|T^{(n)}|} e^{\alpha} = e^{\alpha} < \infty. \tag{48}$$

Hence it implies that $D=\Omega$. By (7) and (8), we obtain

$$F_n(\omega) = \sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni} I_k(X_{m,i})I_l(X_{m+1,j}) = S_n(k,l), \tag{49}$$

$$G_n(\omega) = \sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[I_k(X_{m,i})I_l(X_{m+1,j})|X_{m,i}\right] =$$

$$= \sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni} I_k(X_{m,i})P(l|X_{m,i}) =$$

$$= \sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni} I_k(X_{m,i})P(l|k) =$$

$$= NS_{n-1}(k)P(l|k). \tag{50}$$

By (49), (50) and (11), noticing $\lim_{n\to\infty}|T^{(n)}|/|T^{(n-1)}|=N$, we have

$$\lim_{n\to\infty}\frac{1}{a_n}[F_n(\omega)-G_n(\omega)]=$$

$$=\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\big[S_n(k,l)-NS_{n-1}(k)P(l|k)\big]=$$

$$=\lim_{n\to\infty}\left[\frac{1}{|T^{(n)}|}S_n(k,l)-\frac{1}{|T^{(n-1)}|}S_{n-1}(k)P(l|k)\right]=0. \tag{51}$$

Hence (47) follows from (51) directly.

Lemma 1 [9]. *Let $X^{T^{(n)}}=\{X_t,t\in T^{(n)}\}$ be a homogeneous Markov chain indexed by a Cayley tree $T_{C,N}$ which takes values in the finite alphabet set $S=\{1,2,\ldots,N\}$ with the initial distribution $p=(p(1),p(2)\ldots,p(N))$ and transition matrix* (41), *assume that the matrix* (41) *is ergodic. Let $S_n(k,\omega)$ be the number of k in $X^{T^{(n)}}=\{X_t,t\in T^{(n)}\}$. Then for all $k\in S$,*

$$\lim_n\frac{S_n(k,\omega)}{|T^{(n)}|}=\pi(k)\quad \mu_P\text{-a.s.}, \tag{52}$$

where $\pi=(\pi(1),\ldots,\pi(N))$ is the stationary distribution determined by P.

Theorem 4. *Under the hypothesis of Lemma* 1,

$$\lim_n\frac{1}{|T^{(n)}|}\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni}\mathrm{E}\big[\exp\{\alpha\,|g(X_{m,i},X_{m+1,j})|\}|X_{m,i}\big]=$$

$$=\sum_{k\in S}\sum_{l\in S}\pi(k)\exp\{\alpha|g(k,l)|\}P(l|k)\quad \mu_P\text{-a.s.} \tag{53}$$

Proof. By (52) and the definition of $S_n(k)$, we have

$$\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni}\mathrm{E}\big[\exp\{\alpha\,|g(X_{m,i},X_{m+1,j})|\}|X_{m,i}\big]=$$

$$=\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni}\sum_{x_{m+1,j}\in S}\exp\{\alpha|g(X_{m,i},x_{m+1,j})|\}P(x_{m+1,j}|X_{m,i})=$$

$$=\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni}\sum_{l\in S}\sum_{k\in S}\delta_k(X_{m,i})\exp\{\alpha|g(k,l)|\}P(l|k)=$$

$$=\sum_{l\in S}\sum_{k\in S}\exp\{\alpha|g(k,l)|\}P(l|k)\sum_{m=0}^{n-1}\sum_{i=1}^{N^m}\sum_{j=N(i-1)+1}^{Ni}\delta_k(X_{m,i})=$$

$$=\sum_{l\in S}\sum_{k\in S}\exp\{\alpha|g(k,l)|\}P(l|k)NS_{n-1}(k). \tag{54}$$

By (52) and (54), noticing that $\lim\limits_{n\to\infty}|T^{(n)}|/|T^{(n-1)}|=N$, we obtain

$$\lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[\exp\{\alpha\,|g(X_{m,i}, X_{m+1,j})|\}|X_{m,i}\right] =$$

$$= \lim_n \sum_{l\in S}\sum_{k\in S} \exp\{\alpha|g(k,l)|\}P(l|k)\frac{NS_{n-1}(k)}{|T^{(n)}|} =$$

$$= \lim_n \sum_{l\in S}\sum_{k\in S} \exp\{\alpha|g(k,l)|\}P(l|k)\frac{S_{n-1}(k)}{|T^{(n-1)}|} =$$

$$= \sum_{l\in S}\sum_{k\in S} \pi(k)\exp\{\alpha|g(k,l)|\}P(l|k) \quad \mu_P\text{-a.s.} \tag{55}$$

Theorem 5. *Let $X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}$ be a homogeneous Markov chain indexed by a Cayley tree $T_{C,N}$, we have*

$$\lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}\left[\exp\{\alpha\,|g(X_{m,i}, X_{m+1,j})|\}\right] =$$

$$= \sum_{k\in S}\sum_{l\in S} p(k)\exp\{\alpha\,|g(k,l)|\}P(l|k) \quad \mu_P\text{-}a.s. \tag{56}$$

Proof. In virtue of properties of Markov chain, we obtain

$$\lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \mathrm{E}[\exp\{\alpha\,|g(X_{m,i}, X_{m+1,j})|\}] =$$

$$= \lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \sum_{x_{m,i}\in S} \sum_{x_{m+1,j}\in S} \exp\{\alpha\,|g(x_{m,i}, x_{m+1,j})|\}\times$$

$$\times P(x_{m,i}, x_{m+1,j}) =$$

$$= \lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \sum_{x_{m,i}\in S} \sum_{x_{m+1,j}\in S} p(x_{m,i})\exp\{\alpha\,|g(x_{m,i}, x_{m+1,j})|\}\times$$

$$\times P(x_{m+1,j} \mid x_{m,i}) =$$

$$= \lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} \sum_{k\in S}\sum_{l\in S} p(k)\exp\{\alpha\,|g(k,l)|\}P(l|k) =$$

$$= \sum_{k\in S}\sum_{l\in S} \lim_n \frac{1}{|T^{(n)}|} \sum_{m=0}^{n-1} \sum_{i=1}^{N^m} \sum_{j=N(i-1)+1}^{Ni} p(k)\exp\{\alpha\,|g(k,l)|\}P(l|k) =$$

$$= \sum_{k\in S}\sum_{l\in S} p(k)\exp\{\alpha\,|g(k,l)|\}P(l|k) \quad \mu_P\text{-a.s.} \tag{57}$$

Hence (56) follows from (57). We have accomplished the proof.

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