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STRONGLY RADICAL SUPPLEMENTED MODULES

СИЛЬНО РАДИКАЛЬНО ДОПОВНЕНІ МОДУЛІ

Zöschinger studied modules whose radicals have supplements and called these modules *radical supplemented*. Motivated by this, we call a module *strongly radical supplemented* (briefly srs) if every submodule containing the radical has a supplement. We prove that every (finitely generated) left module is an srs-module if and only if the ring is left (semi)perfect. Over a local Dedekind domain, srs-modules and radical supplemented modules coincide. Over a no-local Dedekind domain, an srs-module is the sum of its torsion submodule and the radical submodule.

Зошінгер вивчав модулі, радикали яких мають доповнення, і назвав ці модулі *радикально-доповненими*. Мотивуючись цим, будемо називати модуль *сильно радикально доповненим* (або, скорочено, srs-модулем) якщо кожен підмодуль, що містить радикал, має доповнення. Доведено, що кожен (скінченнопороджений) лівий модуль є srs-модулем тоді і тільки тоді, коли кільце є лівим (напів)досконалим. Над локальною дедекіндовою областю srs-модулі та радикально доповнені модулі збігаються. Над нелокальною дедекіндовою областю srs-модуль є сумою свого підмодуля скруту і радикального підмодуля.

1. Introduction. Throughout, R is an associative ring with identity and all modules are unital left R-modules. Let M be an R-module. By $N \subseteq M$, we mean that N is a submodule of M. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$. A submodule S of M is called *small* (*in M*), denoted as $S \ll M$, if $M \neq S + L$ for every proper submodule L of M. By $\operatorname{Rad} M$ we denote the sum of all small submodules of M or, equivalently the intersection of all maximal submodules of M. A module M is called *supplemented* (see [1]), if every submodule N of M has a *supplement*, i.e., a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ (see [1]). An R-module M is said to be *radical supplemented* if $\operatorname{Rad} M$ has a supplement in M. Radical supplemented modules are studied by Zöschinger in [2] and [3]. Motivated by this definition, we call a module *strongly radical supplemented* if every submodule containing the radical has a supplement. srs-modules lies between radical supplemented modules and supplemented modules. Some examples are provided to show that these inclusions are proper.

In this paper, among other results, we prove that srs-modules are closed under factor modules and finite sums. Every left R-module is an srs-module if and only if R is left perfect. For modules with small radical the notions of supplemented and being srs-module coincide. This gives us, every finitely generated R-module is an srs-module if and only if R is semiperfect. Over a commutative non-local domain, we prove that every reduced srs-module M is of the form $M = T(M) + \operatorname{Rad} M$, where $T(M)$ is the torsion submodule of M. A commutative domain is h-local if and only if every finitely generated torsion module is an srs-module. Over a local Dedekind domain (i.e., over a DVR), a module is an srs-module if and only if it is radical supplemented. Over a non-local Dedekind domain an *srs*-module M is of the from $M = T(M) + \operatorname{Rad} M$.

2. Strongly radical supplemented modules. Firstly we show some properties of srs-modules.

Proposition 2.1. *Every homomorphic image of an srs-module is an srs-module.*

Proof. Let $L \subseteq N \subseteq M$ and $\operatorname{Rad}(M/L) \subseteq N/L$. Since $(\operatorname{Rad} M + L)/L \subseteq \subseteq \operatorname{Rad}(M/L)$, we have $\operatorname{Rad} M \subseteq N$. By assumption N has a supplement, say K, in M. Then by [1] (41.1(7)), $(K + L)/L$ is a supplement of N/L in M/L. Hence M/L is an srs-module.

Proposition 2.2. *If M is an srs-module, then $M/\operatorname{Rad} M$ is semisimple.*

Proof. By Proposition 2.1, $M/\operatorname{Rad} M$ is an srs-module. $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$, therefore $M/\operatorname{Rad} M$ is supplemented. By [1] (41.2(3)), $M/\operatorname{Rad} M$ is semisimple.

To prove that the finite sum of srs-modules is an srs-module, we use the following standard lemma (see [1] (41.2)).

Lemma 2.1. *Let M be an R-module and M_1, N be submodules of M with* $\operatorname{Rad} M \subseteq N$. *If M_1 is an srs-module and $M_1 + N$ has a supplement in M, then N has a supplement in M.*

Proof. Let L be a supplement of $M_1 + N$ in M. Since $\operatorname{Rad} M_1 \subseteq \operatorname{Rad} M \subseteq N$, we have $\operatorname{Rad} M_1 \subseteq (L + N) \cap M_1$. Then $(L + N) \cap M_1$ has a supplement, say K, in M_1 because M_1 is an srs-module. So

$$M = M_1 + N + L = K + [(L + N) \cap M_1] + N + L = (K + N) + L.$$

Since $N + K \subseteq N + M_1$, L is also a supplement of $N + K$ in M. Then by [4] (Lemma 1.3a), $K + L$ is a supplement of N in M.

Proposition 2.3. *Let $M = M_1 + M_2$, where M_1 and M_2 are srs-modules, then M is an srs-module.*

Proof. Suppose that $N \subseteq M$ with $\operatorname{Rad} M \subseteq N$. Clearly $M_1 + M_2 + N$ has the trivial supplement 0 in M, so by Lemma 2.1, $M_1 + N$ has a supplement in M. Applying the Lemma once more, we obtain a supplement for N in M.

***Corollary* 2.1.** *Every finite sum of srs-modules is an srs-module.*

Lemma 2.2. *Let M be a module with* $\operatorname{Rad} M = M$. *Then M is an srs-module.*

Proof. Clearly M has the trivial supplement 0 in M. Since $M = \operatorname{Rad} M$ is the unique submodule containing the radical, M is an srs-module.

Let M be an R-module. By $P(M)$ we denote the sum of all submodules V of M such that $\operatorname{Rad} V = V$.

***Corollary* 2.2.** *Let M be an R-module. Then $P(M)$ is an srs-module.*

Proof. For any module M, $\operatorname{Rad} P(M) = P(M)$. Then by Lemma 2.2, $P(M)$ is an srs-module.

The following example shows that srs-modules need not be supplemented.

***Example* 2.1.** Consider the $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$. Then M is an srs-module, because $\operatorname{Rad} \mathbb{Q} = \mathbb{Q}$. On the other hand, M is not supplemented by [4] (Theorem 3.1).

Proposition 2.4. *Let M be an R-module with* $\operatorname{Rad} M \ll M$. *Then M is supplemented if and only if M is an srs-module.*

Proof. One direction is clear. Suppose that M is an *srs*-module. Let N be a submodule of M. Then $N + \operatorname{Rad} M$ has a supplement, say L, in M. So $N + \operatorname{Rad} M + L = = M$ and $(N + \operatorname{Rad} M) \cap L \ll L$. Since $\operatorname{Rad} M \ll M$, we have $N + L = M$ and also

$N \cap L \subseteq (N + \operatorname{Rad} M) \cap L \ll L$, i.e., $N \cap L \ll L$. Hence N has a supplement L in M. Thus M is supplemented.

In [6], a ring R is called left max if every non-zero R-module has a maximal submodule. It is well known that R is a left max ring if and only if $\operatorname{Rad} M \ll M$ for every non-zero left R-module M. By using Proposition 2.4, we obtain the following corollary.

***Corollary* 2.3.** *Every srs-module over a left max ring is supplemented.*

Proposition 2.5. *Let M be an R-module. Suppose that $\operatorname{Rad} M$ is supplemented and M is an srs-module. Then M is supplemented.*

Proof. Let N be a submodule of M. By the hypothesis, $\operatorname{Rad} M + N$ has a supplement in M. Since $\operatorname{Rad} M$ is supplemented, N has a supplement in M by [1] (41.2). Hence M is supplemented.

A submodule $U \subseteq M$ is said to be *cofinite* if M/U is finitely generated. In [5], M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. It is also shown that M is cofinitely supplemented if and only if every maximal submodule of M has a supplement in M (see [5], Theorem 2.8). Since $\operatorname{Rad} M$ is contained in every maximal submodule of M, every *srs*-module is cofinitely supplemented. But the converse need not be true in general, as it is shown in the following example.

Firstly, we need the following lemma.

Lemma 2.3. *Let M be an R-module and U, $V \subseteq M$. If V is a supplement of U in M and $\operatorname{Rad} V \subseteq U$, then $\operatorname{Rad} V \ll V$.*

Proof. Suppose that $\operatorname{Rad} V + T = V$ for some $T \subseteq V$. Then

$$M = U + V = U + \operatorname{Rad} V + T = U + T.$$

Since V is a supplement and $T \subseteq V$, we have $T = V$. Hence $\operatorname{Rad} V \ll V$.

***Example* 2.2.** Let $\mathbb{Z}$ be the ring of integers and p be a prime in $\mathbb{Z}$. Consider the $\mathbb{Z}$-module, $M = \bigoplus_{n \geqslant 1} \mathbb{Z}_{p^n}$, where $\mathbb{Z}_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$. Then M is a torsion module and it is cofinitely supplemented by [5] (Corollary 4.7). To see that M is not an srs-module, consider the submodule pM of M. Since M/pM is a semisimple module, $\operatorname{Rad} M \subseteq pM$. We shall prove that pM has not a supplement in M. Suppose pM has a supplement, say N in M. Then $\operatorname{Rad} N \ll N$ by Lemma 2.3. Now since every element of M is annihilated by some power of p, the module M can be considered as a module over the local ring $\mathbb{Z}_{(p)}$. Then N is a bounded module by [5] (Lemma 2.1). Therefore $p^n N = 0$ for some $n \geqslant 1$. On the other hand, since N is a supplement of pM, we have $M = pM + N$, and so $p^n M = p^{n+1} M + p^n N = p^{n+1} M$. So that $p^n M$ is divisible module by [5] (Lemma 4.4). But M has no nonzero divisible submodule. Hence $p^n M = 0$, a contradiction. Therefore pM has not a supplement in M, i.e., M is not an srs-module.

Proposition 2.6. *Let R be any ring and M be an R-module. Suppose that $M/\operatorname{Rad} M$ is finitely generated. Then M is cofinitely supplemented if and only if it is an srs-module.*

Proof. Let M be an R-module and N be a submodule of M with $\operatorname{Rad} M \subseteq N$. Note that

$$[M/\operatorname{Rad} M]/[N/\operatorname{Rad} M] \cong M/N$$

is finitely generated and thus N is a cofinite submodule of M. Since M is cofinitely supplemented, N has a supplement in M. Therefore M is an srs-module. The converse is clear.

Now, we have the following implications on modules:

$$\text{supplemented} \Longrightarrow srs\text{-module} \Longrightarrow \text{cofinitely supplemented.}$$

Proposition 2.7. *Let M be an R-module and $\operatorname{Rad} M \subseteq U \subseteq M$. If V is a supplement of U in M, then $\operatorname{Rad} V \ll V$.*

Proof. Since $\operatorname{Rad} M \subseteq U$, we have $\operatorname{Rad} V \subseteq U$. Then $\operatorname{Rad} V \ll V$ by Lemma 2.3.

Recall from [6] that a submodule L of a module M is called a *Rad-supplement* of a submodule N of M in M if $N + L = M$ and $N \cap L \subseteq \operatorname{Rad} L$. Clearly every supplement submodule is a Rad-supplement.

***Corollary* 2.4.** *Let M be an R-module and $N \subseteq M$ such that $\operatorname{Rad} M \subseteq N$. Suppose that $N + L = M$ for some $L \subseteq M$. Then L is a supplement of N in M if and only if L is a Rad-supplement of N and $\operatorname{Rad} L \ll L$.*

In the following proposition, we characterize supplements of the radical of a module over semilocal rings.

Proposition 2.8. *Let R be a semilocal ring and M be an R-module. A submodule $N \subseteq M$ is a supplement of $\operatorname{Rad} M$ in M if and only if N is coatomic, M/N has no maximal submodules and $\operatorname{Rad} N = N \cap \operatorname{Rad} M$.*

Proof. ($\Rightarrow$) Let N be a supplement of $\operatorname{Rad} M$ in M. Then by [1] (41.1(5)), $\operatorname{Rad} N = N \cap \operatorname{Rad} M$. If $N = M$, then clearly $\operatorname{Rad} M \ll M$. Since R is semilocal, $M/\operatorname{Rad} M$ is semisimple. Therefore every proper submodule of M is contained in a maximal submodule, i.e., M is coatomic. Suppose that N is a proper submodule of M. If K is a maximal submodule of M with $N \subseteq K$, then $M = \operatorname{Rad} M + N \subseteq K$, a contradiction. So that N is not contained in any maximal submodule of M, i.e., M/N has no maximal submodules. By Proposition 2.7, we have $\operatorname{Rad} N \ll N$. Since $N/\operatorname{Rad} N$ is semisimple, N is coatomic.

($\Leftarrow$) Suppose that $N + \operatorname{Rad} M \neq M$. Then $(N + \operatorname{Rad} M)/\operatorname{Rad} M \subsetneqq M/\operatorname{Rad} M$. Since R is semilocal, $M/\operatorname{Rad} M$ is semisimple and so there exists a maximal submodule $K/\operatorname{Rad} M$ of $M/\operatorname{Rad} M$ such that $(N + \operatorname{Rad} M)/\operatorname{Rad} M \subseteq K/\operatorname{Rad} M$. So $N + \operatorname{Rad} M \subseteq K$, this implies $N \subseteq K$. Therefore K/N is a maximal submodule of M/N, a contradiction. So $N + \operatorname{Rad} M = M$. By the hypothesis, $N \cap \operatorname{Rad} M = \operatorname{Rad} N \ll N$. Hence N is a supplement of $\operatorname{Rad} M$ in M.

Now, we shall characterize the rings over which all (finitely generated) modules are srs-modules.

***Corollary* 2.5.** *For a ring R, the following statements are equivalent.*

(1) *R is semiperfect.*

(2) *${}_RR$ is an srs-module.*

(3) *Every finitely generated left R-module is an srs-module.*

Proof. For every finitely generated module M, we have $\operatorname{Rad} M \ll M$. On the other hand, by [1] (42.6), R is semiperfect if and only if every finitely generated R-module is supplemented. From this fact and Proposition 2.4, the implications $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ are clear.

***Corollary* 2.6.** *For a ring R, the following statements are equivalent.*

(1) *R is left perfect.*

(2) *The left R-module $R^{(\mathbb{N})}$ is an srs-module.*

(3) *Every left R-module is an srs-module.*

Proof. (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) By Proposition 2.1, ${}_RR$ is an srs-module. So R is semilocal by Proposition 2.2. Since $R^{(\mathbb{N})}$ is an srs-module, $\mathrm{Rad}\, R^{(\mathbb{N})}$ has a (weak) supplement in $R^{(\mathbb{N})}$. Therefore R is left perfect by [7] (Theorem 1).

The following is a slight modification of [4] (Lemma 1.3 (Folgerung)).

Proposition 2.9. *Let M be an R-module and K be a submodule of M. If K and M/K are srs-modules and K has a supplement L in P for every submodule P with $K \subseteq P \subseteq M$, then M is an srs-module.*

Proof. Let N be a submodule of M with $\mathrm{Rad}\, M \subseteq N$. It follows from [4] (Lemma 1.1(d)) that we can write $\mathrm{Rad}(M/K) = (\mathrm{Rad}\, M + K)/K \subseteq (N+K)/K$. Since M/K is an srs-module, $(N+K)/K$ has a supplement in M/K. That is, there exists a submodule V/K of M/K such that $(N+K)/K + V/K = M/K$ and $[(N+K)/K] \cap \cap [V/K] \ll V/K$. Since $K \subseteq V$, K has a supplement in V. Therefore $V = K + L$ and $K \cap L \ll L$ for some $L \subseteq V$. Now

$$M = N + V = N + (K + L) = (N + K) + L.$$

Suppose that $M = (N+K) + L'$ for some $L' \subseteq L$. Then $M/K = (N+K)/K + (L' + + K)/K$. But V/K is a supplement of $(N+K)/K$ in M/K and $(L'+K)/K \subseteq V/K$. By minimality of V/K, we obtain $(L'+K)/K = V/K$. It follows that $V = L' + K$. Since L is a supplement of K in V, we have $L' = L$. So L is a supplement of $N + K$ in M. By Lemma 2.1, N has a supplement in M. Hence M is an srs-module.

The following corollary is a direct consequence of Proposition 2.9.

***Corollary* 2.7.** *Let M be an R-module which contains an artinian submodule K. Then M is an srs-module if and only if M/K is an srs-module.*

Proof. One direction follows from Proposition 2.1. Conversely, suppose that M/K is an srs-module. By assumption, K is supplemented and so it is an srs-module. It follows from [3] that K has a supplement in every P with $K \subseteq P \subseteq M$. Therefore M is an srs-module by Proposition 2.9.

3. srs-Modules over Dedekind domains. Throughout this section, unless otherwise stated, we shall consider commutative rings. The following result is due to Zöschinger.

Lemma 3.1 [3] (Satz 3.1)**.** *For a module over a discrete valuation ring (DVR), the following statements are equivalent.*

(1) *M is radical supplemented,*

(2) *$M = T(M) \oplus X$, where the reduced part of $T(M)$ is bounded and $X/\mathrm{Rad}\, X$ is finitely generated,*

Now we shall prove that radical supplemented modules and srs-modules coincide over discrete valuation rings. Firstly we need the following lemma.

Lemma 3.2. *Let R be a local ring and M be an R-module. If $M/\mathrm{Rad}\, M$ is finitely generated, then M is an srs-module.*

Proof. Let N be a submodule of M such that $\mathrm{Rad}\, M \subseteq N$. Then M/N is finitely generated, and so $M = N + L$ for some finitely generated submodule L of M. Since ${}_RR$ is supplemented, L is also supplemented as it is finitely generated. So N has a supplement in M by Lemma 2.1.

Proposition 3.1. *Let R be a DVR and M be an R-module. Then M is an srs-module if and only if M is radical supplemented.*

Proof. One direction is clear. Suppose that M is radical supplemented. Then $M = T(M) \oplus X$ as in Lemma 3.1. Since $T(M)$ is bounded, it is supplemented by [4] (Theorem 2.4). By Lemma 3.2, X is an srs-module. Therefore M is an srs-module by Corollary 2.1.

Note that, by Example 2.2, Proposition 3.1 is not true in general for modules over Dedekind domains which are not DVR.

Proposition 3.2. *Let R be a non-local domain and M be a reduced R-module. If M is an srs-module, then $M = T(M) + \operatorname{Rad} M$.*

Proof. Suppose that $T(M) + \operatorname{Rad} M \neq M$. Since $\operatorname{Rad} M \subseteq T(M) + \operatorname{Rad} M$, $T(M) + \operatorname{Rad} M$ has a supplement, say L, in M. Then L has a maximal submodule K, because M is reduced. Let $K' = T(M) + \operatorname{Rad} M + K$. It is easy to see that K' is a maximal submodule of M. Then K' has a supplement V in M. By [1] (41.1(3)), V is local, and so $V \cong R/I$ for some nonzero $I \subseteq R$. Therefore V is torsion, and so $V \subseteq T(M)$. We get $M = K' + V = T(M) + \operatorname{Rad} M + K + V = T(M) + \operatorname{Rad} M + K = K'$, a contradiction. Hence $M = T(M) + \operatorname{Rad} M$.

Now we shall prove that, the converse of Proposition 3.2 is true, under a certain condition.

Proposition 3.3. *Let R be a domain and M be an R-module. Suppose that $M = T(M) + \operatorname{Rad} M$ and $T(M)$ is supplemented. Then M is an srs-module.*

Proof. Let N be a submodule of M such that $\operatorname{Rad} M \subseteq N$. Then $N = N \cap T(M) + \operatorname{Rad} M = T(N) + \operatorname{Rad} M$. Let L be a supplement of $T(N)$ in $T(M)$. Then $T(N) + L = T(M)$ and $T(N) \cap L \ll L$. It follows that $M = T(M) + \operatorname{Rad} M = T(N) + L + \operatorname{Rad} M \subseteq N + L$ and so $M = N + L$. Since L is torsion, $N \cap L = T(N) \cap L$. Therefore L is a supplement of N in M.

Let R be a Dedekind domain and M be an R-module. Since R is a dedekind domain, $P(M)$ is *the divisible part* of M. By [5] (Lemma 4.4), $P(M)$ is (divisible) injective and so there exists a submodule N of M such that $M = P(M) \oplus N$. Here N is called *the reduced part* of M. Note that $P(M) \subseteq \operatorname{Rad} M$. By Corollary 2.2, we know that $P(M)$ is an srs-module. Using these facts, we have the following result.

Proposition 3.4. *Let R be a Dedekind domain and M be an R-module. Then M is an srs-module if and only if the reduced part N of M is an srs-module.*

Proof. N is an srs-module as a homomorphic image of M by Proposition 2.1. The converse is by Proposition 2.3.

Proposition 3.5. *Let R be a non-local Dedekind domain and M be an srs-module. Then $M = T(M) + \operatorname{Rad} M$.*

Proof. Let $M = P(M) \oplus N$ with N reduced. Then N is an srs-module as a direct summand of M. By Proposition 3.2, $N = T(N) + \operatorname{Rad} N$. So that

$$M = P(M) \oplus N = P(M) + T(N) + \operatorname{Rad} N \subseteq T(M) + \operatorname{Rad} M.$$

Hence $M = T(M) + \operatorname{Rad} M$.

Recall from [5] that a commutative domain R is called *h-local* if every non-zero non-unit of R belongs to only finitely many maximal ideals and R/P is a local ring for every prime ideal P of R. It is also proved that a commutative domain R is h-local if and only if R/I is a semiperfect ring for every non-zero ideal I of R (see [5], Lemma 4.5). In [5], it is proved that, R is h-local if and only if every finitely generated torsion R-module is supplemented. Since for finitely generated modules supplemented modules and srs-modules coincide, we obtain the following .

Proposition 3.6. *Let R be a commutative domain. Then R is h-local if and only if every finitely generated torsion R-module is an srs-module.*

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