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RELATIVELY THIN AND SPARSE SUBSETS OF GROUPS

ВІДНОСНО ТОНКІ ТА РОЗРІДЖЕНІ ПІДМНОЖИНИ ГРУП

Let G be a group with the identity e, $\mathcal{I}$ be a left-invariant ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G. A subset A of G is called $\mathcal{I}$-thin if $gA \cap A \in \mathcal{I}$ for every $g \in G \setminus \{e\}$. A subset A of G is called $\mathcal{I}$-sparse if, for every infinite subset S of G, there exists a finite subset $F \subset S$ such that $\bigcap_{g\in F} gA \in \mathcal{F}$. An ideal $\mathcal{I}$ is said to be thin-complete (sparse-complete) if every $\mathcal{I}$-thin ($\mathcal{I}$-sparse) subset of G belongs to $\mathcal{I}$. We define and describe the thin-completion and the sparse-completion of an ideal in $\mathcal{P}_G$.

Припустимо, що G — група з одиницею e, $\mathcal{I}$ — інваріантний зліва ідеал в булевій алгебрі $\mathcal{P}_G$ всіх підмножин групи G. Підмножина A групи G називається $\mathcal{I}$-тонкою, якщо $gA \cap A \in \mathcal{I}$ для кожного $g \in G \setminus \{e\}$. Підмножина A групи G називається $\mathcal{I}$-розрідженою, якщо для кожної нескінченної множини S групи G існує скінченна підмножина $F \subset S$ така, що $\bigcap_{g\in F} gA \in \mathcal{F}$. Говорять, що ідеал $\mathcal{I}$ тонко-повний (розріджено-повний), якщо кожна $\mathcal{I}$-тонка ($\mathcal{I}$-розріджена) множина групи G належить $\mathcal{I}$. Визначено та описано тонке та розріджене доповнення ідеалу в $\mathcal{P}_G$.

Let G be a group with the identity e, $\mathcal{P}_G$ be the Boolean algebra of all subsets of G. A family $\mathcal{F}$ of subsets of G is called

left-invariant if $gF \in \mathcal{F}$ for all $g \in G$ and $F \in \mathcal{F}$;

downward closed if $E \subseteq F$ and $F \in \mathcal{F}$ implies $E \in \mathcal{F}$;

additive if $E \cup F$ for all subsets $E, F \in \mathcal{F}$;

an *ideal* if $\mathcal{F}$ is downward closed and additive.

The family $\mathcal{F}_G$ of all finite subsets of G is a left-invariant ideal of $\mathcal{P}_G$.

Given a left-invariant ideal $\mathcal{I}$ in $\mathcal{P}_G$, we classify the subsets of G by their size with respect to $\mathcal{I}$.

A subset $A \subseteq G$ is said to be

$\mathcal{I}$-large if there exist $F \in \mathcal{F}_G$ and $I \in \mathcal{I}$ such that $G = FA \cup I$;

$\mathcal{I}$-small if $L \setminus A$ is $\mathcal{I}$-large for every $\mathcal{I}$-large subset L;

$\mathcal{I}$-thick if $L \cap A \neq \varnothing$ for every $\mathcal{I}$-large subset L;

$\mathcal{I}$-thin if $A \cap gA \in \mathcal{I}$ for every $g \in G \setminus \{e\}$.

$\mathcal{I}$-sparse if each infinite set $S \subset G$ contains a finite subset $F \subset S$ with $\bigcap_{g\in F} gA \in \mathcal{I}$.

For the smallest ideal $\mathcal{I}_\varnothing = \{\varnothing\}$, $\mathcal{I}_\varnothing$-large, $\mathcal{I}_\varnothing$-small and $\mathcal{I}_\varnothing$-thick sets turn into large, small and thick subsets which have been intensively studied last time (see the survey [1]). On the other hand, $\mathcal{F}_G$-thin and $\mathcal{F}_G$-sparse subsets are known as thin and sparse sets, see [2]. The $\mathcal{I}$-large and $\mathcal{I}$-small subsets appeared in [3]. For every left-invariant ideal $\mathcal{I}$, the family $\mathcal{S}(\mathcal{I})$ of all $\mathcal{I}$-small subsets of G is a left-invariant ideal containing $\mathcal{I}$.

The paper consists of two sections. In the first section we study the thin-extension $\tau(\mathcal{I})$ and the thin-completion $\tau^*(\mathcal{I})$ of the ideal $\mathcal{I}$ in $\mathcal{P}_G$. In the second section we study the sparse-extension $\sigma(\mathcal{I})$ and the sparse-completion $\sigma^*(\mathcal{I})$ of $\mathcal{I}$.

1. Relatively thin subsets of groups.

Proposition 1. *For a subset A of a group G and $\mathcal{I}$ be a left-invariant ideal in $\mathcal{P}_G$, the following statements hold*:

(1) *A is $\mathcal{I}$-small if and only if $G \setminus FA$ is $\mathcal{I}$-large for every $F \in \mathcal{F}_G$*;

(2) *A is $\mathcal{I}$-thick if and only if for any $F \in \mathcal{F}_G$ and $I \in \mathcal{I}$ there exists $x \in G$ such that $Fx \subseteq A \setminus I$*;

(3) *A is not $\mathcal{I}$-small if and only if there exists $F \in \mathcal{F}_G$ such that FA is $\mathcal{I}$-thick;*

(4) *if A is $\mathcal{I}$-thick, then for every $F \in \mathcal{F}_G$ the set $\{g \in A\colon Fg \subseteq A\}$ is $\mathcal{I}$-thick.*

Proof. 1. Theorem 2.1 from [3].

2. We suppose that A is $\mathcal{I}$-thick and take $F \in \mathcal{F}_G$ and $I \in \mathcal{I}$. If $Fx \nsubseteq A \setminus I$ for every $x \in G$, then $G = F^{-1}(G \setminus (A \setminus I))$ and thus the set $L = G \setminus (A \setminus I)$ is $\mathcal{I}$-large and so is the set $L \setminus I$. Since $A \cap (L \setminus I) = L \cap (A \setminus I) = \varnothing$, the set A is not $\mathcal{I}$-thick, which is a contradiction.

If A is not $\mathcal{I}$-thick, then $L \cap A = \varnothing$ for some $\mathcal{I}$-large subset L. Find $I \in \mathcal{I}$ and $F \in \mathcal{F}_G$ such that $G = F(L \cup I)$. Then for each $x \in G$ the set $F^{-1}x$ meets $L \cup I$ and hence cannot lie in $A \setminus I \subset G \setminus (L \cup I)$.

3. By (1), A is not $\mathcal{I}$-small if and only if there exists $F \in \mathcal{F}_G$ such that $G \setminus FA$ is not $\mathcal{I}$-large. On the other hand, $G \setminus FA$ is not $\mathcal{I}$-large if and only if for each $\mathcal{I}$-large set $L \subset G$ we get $L \not\subset G \setminus FA$, which is equivalent to $L \cap FA \neq \varnothing$.

4. We fix $F \in \mathcal{F}_G$, $e \in F$ and put $B = \{g \in A\colon Fg \subseteq A\}$. Then we take an arbitrary $H \in \mathcal{F}_G$, $e \in H$. Given any $I \in \mathcal{I}$, there exists $a \in A \setminus I$ such that $FHa \subseteq A \setminus I$. By the definition of B, $Ha \subseteq B$ so $Ha \subseteq B \setminus I$ and B is $\mathcal{I}$-thick.

Proposition 1 is proved.

Let $\mathcal{F}$ be a left-invariant downward closed family of subsets of a group G. A subset $A \subseteq G$ is called *$\mathcal{F}$-thin* if $gA \cap A \in \mathcal{F}$ for every $g \in G \setminus \{e\}$. The family of all $\mathcal{F}$-thin subsets of G is denoted by $\tau(\mathcal{F})$. The definition implies that $\tau(\mathcal{F})$ is left-invariant, downward closed and $\mathcal{F} \subseteq \tau(\mathcal{F})$. If $\mathcal{F} = \tau(\mathcal{F})$, then the family $\mathcal{F}$ is called *thin-complete*. The intersection $\tau^*(\mathcal{F})$ of all thin-complete families that contain $\mathcal{F}$ is called the *thin-completion* of $\mathcal{F}$. The thin-completion $\tau^*(\mathcal{F})$ contains the subfamilies $\tau^\alpha(\mathcal{F})$ defined by transfinite induction:

$$\tau^0(\mathcal{F}) = \mathcal{F} \quad \text{and} \quad \tau^\alpha(\mathcal{F}) = \tau(\tau^{<\alpha}(\mathcal{F})), \quad \text{where} \quad \tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta<\alpha} \tau^\beta(\mathcal{F})$$

for each ordinal α.

The families $\tau^n(\mathcal{F})$ for $n \in \omega$ admit a simple characterization:

Proposition 2. *Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant downward closed family of subsets of a group G and $n \in \omega$. A subset $A \subset G$ belongs to the family $\tau^n(\mathcal{F})$ if and only if*

$$\bigcap_{i_0,\ldots,i_n \in \{0,1\}} g_0^{i_0} \ldots g_n^{i_n} A \in \mathcal{F}$$

for any elements $g_0, \ldots, g_n \in G \setminus \{e\}$.

Proof. For $n = 0$, the statement follows from the left-invariance of $\mathcal{F}$. Assume that the proposition is true for some $n \in \omega$. Then $A \in \tau^{n+1}(\mathcal{F})$ if and only if $A \cap g_{n+1}A \in \tau^n(\mathcal{F})$ for each $g_{n+1} \in G$, $g_{n+1} \neq e$. By the inductive hypothesis

$$A \cap g_{n+1}A \in \tau^n(\mathcal{F}) \;\Leftrightarrow \bigcap_{i_0,\ldots,i_n \in \{0,1\}} g_0^{i_0} \ldots g_n^{i_n}(A \cap g_{n+1}A) \in \mathcal{F}$$

for all $g_0, \ldots, g_n \in G \setminus \{e\}$, which is equivalent to

$$\bigcap_{i_0,\ldots,i_{n+1} \in \{0,1\}} g_0^{i_0} \ldots g_{n+1}^{i_{n+1}} A \in \mathcal{F}.$$

Proposition 2 is proved.

***Remark* 1.** In [4] T. Banakh and N. Lyaskovska (answering a problem posed in a preliminary version of this paper) proved that a subset A of a group G belongs to the family $\tau^*(\mathcal{F})$ if and only if for each sequence $(g_n)_{n\in\omega} \in (G \setminus \{e\})^\omega$ there is $n \in \omega$ such that

$$\bigcap_{i_0,\ldots,i_n\in\{0,1\}} g_0^{i_0}\ldots g_n^{i_n} A \in \mathcal{F}.$$

Next, we describe the structure of the thin-completion $\tau^*(\mathcal{F})$.

Proposition 3. *If G is a group of cardinality $\kappa = |G|$ and $\mathcal{F} \subset \mathcal{P}_G$ is a left-invariant downward closed family of subsets of G, then*

$$\tau^*(\mathcal{F}) = \bigcup_{\alpha<\kappa^+} \tau^\alpha(\mathcal{F}).$$

Proof. Clearly, $\tau^{<\kappa^+}(\mathcal{F}) \subseteq \tau^*(\mathcal{F})$. So, it suffices to show that each set $A \in \tau^{\kappa^+}(\mathcal{F})$ belongs to $\tau^{<\kappa^+}(\mathcal{F})$.

First we consider the case of infinite cardinal $\kappa = |G|$. For any $A \in \tau^{\kappa^+}(\mathcal{F})$ and $x \in G \setminus \{e\}$, we get $A \cap xA \in \tau^{<\kappa^+}(\mathcal{F})$ and hence $A \cap xA \in \tau^{\alpha_x}(\mathcal{F})$ for some ordinal $\alpha_x < \kappa^+$. Let $\alpha = \sup\{\alpha_x \colon x \in G \setminus \{e\}\} < \kappa^+$ and observe that $A \cap xA \in \tau^{\alpha_x}(\mathcal{F}) \subset \tau^\alpha(\mathcal{F})$ for all $x \in G \setminus \{e\}$ and thus $A \in \tau^{\alpha+1}(\mathcal{F}) \subset \tau^{<\kappa^+}(\mathcal{F})$.

Now consider the case of finite κ. In this case $\tau^{<\kappa^+}(\mathcal{F}) = \tau^\kappa(\mathcal{F})$. By Proposition 2, the inclusion $A \in \tau^\kappa(\mathcal{F})$ will follow as soon as we check that for any elements $g_0, \ldots, g_\kappa \in G \setminus \{e\}$

$$\bigcap_{i_0,\ldots,i_\kappa\in\{0,1\}} g_0^{i_0}\ldots g_\kappa^{i_\kappa} A \in \mathcal{F}.$$

Define a sequence of subsets $(C_n)_{n=0}^\kappa$ letting $C_0 = \{e, g_0\}$ and $C_{n+1} = C_n \cdot \{e, g_{n+1}\}$ for $n < \kappa$. Since $C_\kappa \setminus C_0 = \bigcup_{1\le n\le\kappa} C_n \setminus C_{n-1}$ and $|C_\kappa \setminus C_0| \le |G \setminus C_0| = \kappa - 2$, there is a positive number $n \le \kappa$ such that $C_{n-1} = C_n$. For this number we get

$$C_n \cdot \{e, g_n\} = C_{n-1} \cdot \{e, g_n\} = C_n = C_{n-1}.$$

Now consider the sequence $h_0, \ldots, h_\kappa, h_{\kappa^+}$ defined by $h_i = g_i$ if $i \le n$ and $h_i = g_{i-1}$ if $n < i \le \kappa^+$. This sequence induces a sequence of sets $(D_i)_{i\le\kappa^+}$ defined inductively by $D_0 = \{e, h_0\}$ and $D_i = D_{i-1} \cdot \{e, h_i\}$ for $0 < i \le \kappa^+$. It follows that $D_i = C_i$ for $i \le n$ and $D_i = C_{i-1}$ for $n < i \le \kappa^+$. In particular, $D_{\kappa^+} = C_\kappa$. Since $A \in \tau^{\kappa^+}(\mathcal{F})$, we get the required inclusion

$$\bigcap_{i_0,\ldots,i_\kappa\in\{0,1\}} g_0^{i_0}\ldots g_\kappa^{i_\kappa} A = \bigcap_{g\in C_\kappa} gA = \bigcap_{g\in D_{\kappa^+}} gA = \bigcap_{i_0,\ldots,i_{\kappa^+}\in\{0,1\}} h_0^{i_0}\ldots h_{\kappa^+}^{i_{\kappa^+}} A \in \mathcal{F}$$

implying $A \in \tau^\kappa(\mathcal{F}) = \tau^{<\kappa^+}(\mathcal{F})$.

Proposition 3 is proved.

***Remark* 2.** In general the ordinal κ^+ in Proposition 3 cannot be replaced by a smaller ordinal: by [4], for a group G containing an element of infinite order, we get $\tau^*(\mathcal{F}_G) \ne \tau^\alpha(\mathcal{F}_G)$ for each countable ordinal α.

In Boolean groups the situation is totally different. By a *Boolean group* we understand a group G such that $x^2 = e$ for all $x \in G$. Let $[G]^{\le n} = \{A \subset G \colon |A| \le n\}$.

Theorem 1. *For a group G, the following statements hold*:

(1) *G is Boolean if and only if $\tau^*(\mathcal{I}_\varnothing) = \tau(\mathcal{I}_\varnothing) = [G]^{\le 1}$*;

(2) *if* G *is Boolean, then* $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$;

(3) *if* G *is infinite and* $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$, *then* G *is Boolean;*

(4) *if* $\mathcal{I}$ *is a left invariant ideal,* G *has no elements of order* 2 *and* $T_1, T_2 \in \tau(\mathcal{I})$, *then* $T_1 \cup T_2 \in \tau^2(\mathcal{I})$.

Proof. 1. For every group G, $\tau(\mathcal{I}_\varnothing) = [G]^{\leq 1}$. Let G be a Boolean group, $A \in \mathcal{P}(G)$, $|A| > 1$, $a, b \in A$, $a \neq b$, $g = ab^{-1}$. Then $\{a, b\} \subseteq gA \cap A$ so $A \notin \tau([G]^{\leq 1})$ and $\tau^*(\mathcal{I}_\varnothing) = [G]^{\leq 1}$.

On the other hand, assume that G has an element a such that $a^2 \neq e$. We put $A = \{e, a\}$ and note that $|gA \cap A| \leqslant 1$ for every $g \in G$, $g \neq e$. It follows that $A \in \tau([G]^{\leq 1})$ so $\tau^*(\mathcal{I}_\varnothing) \neq [G]^{\leq 1}$.

2. We take an arbitrary subset $A \in \tau^2(\mathcal{F}_G)$. By Proposition 2(2),

$$A \cap gA \cap fA \cap fgA \in \mathcal{F}_G$$

for all $f, g \in G \setminus \{e\}$. We put $f = g$ and get $A \cap gA \in \mathcal{F}_G$ so $A \in \tau(\mathcal{F}_G)$ and $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$.

3. We suppose the contrary, choose an element $g \in G$ such that $g^2 \neq e$ and construct a subset $A \in \tau^2(\mathcal{F}_G) \setminus \tau(\mathcal{F}_G)$. Assume that G is countable, $G = \{g_n \colon n < \omega\}$, $g_0 = e$ and put $G_n = \{g_i \colon i \leqslant n\}$. We put $x_0 = e$ and choose inductively a sequence $(x_n)_{n\in\omega}$ of elements of G such that, for every $n < \omega$,

$$(G_n \cup \{g, g^{-1}\})\{x_{n+1}, gx_{n+1}\} \cap (G_n \cup \{g, g^{-1}\})\{x_0, gx_0, \ldots, x_n, gx_n\} = \varnothing.$$

We consider the set $A = \{x_n, gx_n \colon n \in \omega\}$ and observe that

$$gA \cap A = \{gx_n \colon n \in \omega\},\ g^{-1}A \cap A = \{x_n \colon n \in \omega\}.$$

By the choice of $(x_n)_{n\in\omega}$, $gA \cap A \in \tau(\mathcal{F}_G)$, $g^{-1}A \cap A \in \tau(\mathcal{F}_G)$. If $f \in G \setminus \{g, g^{-1}, e\}$ then $fA \cap A$ is finite. Hence $A \in \tau^2(\mathcal{F}_G)$. Since $gA \cap A$ is infinite, $A \notin \tau(\mathcal{F}_G)$ so $A \in \tau^2(\mathcal{F}_G) \setminus \tau(\mathcal{F}_G)$.

If G is uncountable, we choose a countable subgroup G' of G containing g and repeat the construction of A inside G'.

4. Assuming the converse, we put $X = T_1 \cup T_2$. By Proposition 2(2), there exist $g, f \in G \setminus \{e\}$ such that $X \cap gX \cap fX \cap gfX \notin \mathcal{I}$. We observe that

$$X \cap gX \cap fX \cap fgX = \bigcup_{i,j,k,l \in \{1,2\}} (T_i \cap gT_j \cap fT_k \cap fgT_l).$$

We choose $i, j, k, l \in \{1, 2\}$ such that $T_i \cap gT_j \cap fT_k \cap fgT_l \notin \mathcal{I}$. Without loss of generality, $i = 1$. Since $T_1 \in \tau(\mathcal{I})$, we get $j = k = 2$. Since $T_2 \in \tau(\mathcal{I})$, we get, $g = f$. Since G has no elements of order 2, we have $fg \neq e$. Thus, $T_1 \cap fgT_1 \in \mathcal{I}$ and we get a contradiction.

Theorem 1 is proved.

***Remark* 3.** Let G be an infinite Boolean group. By Theorem 1(2), $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$. We take any infinite thin subset A and $x \in G \setminus \{e\}$. Then the union $A \cup xA$ is not thin because $(A \cup xA) \cap x(A \cup xA) \supseteq xA$ is infinite. Consequently, the family $\tau^*(\mathcal{F}_G)$ is not additive and $\tau^*(\mathcal{F}_G)$ is not an ideal.

In contrast, for every left-invariant ideal $\mathcal{F}$ in a torsion-free group G the family $\tau^{<\alpha}(\mathcal{F})$ is a left-invariant ideal for each limit ordinal α, see [4]. In particular, the family $\tau^*(\mathcal{F})$ is an ideal in $\mathcal{P}_G$.

Theorem 2. *Let G be an infinite group and $\mathcal{I}$ be a left-invariant ideal in $\mathcal{P}_G$. Then $\tau(\mathcal{I}) \subseteq \mathcal{S}(\mathcal{I})$, where $\mathcal{S}(\mathcal{I})$ is the ideal of all $\mathcal{I}$-small subsets of G.*

Proof. We suppose the contrary and fix $A \in \tau(\mathcal{I})$ such that $A \notin \mathcal{S}(\mathcal{I})$. Since A is not $\mathcal{I}$-small, by Proposition 1(3), there exists $F \in \mathcal{F}_G$ such that FA is $\mathcal{I}$-thick. Let $F = \{f_1, \ldots, f_n\}$. Since G is infinite, $G \setminus F^{-1}F \neq \varnothing$. We choose $h \in G \setminus F^{-1}F$ and put

$$A_{ij} = A \cap f_j^{-1} h f_i A, \quad i, j \in \{1, \ldots, n\}.$$

Taking into account that $A \in \tau(\mathcal{I})$ and $f_j^{-1} h f_i \neq e$, we conclude that $A_{ij} \in \mathcal{I}$. We put $B = \{x \in FA \colon hx \in FA\}$. By Proposition 1(4), B is $\mathcal{I}$-thick. Given any $x \in B$, we choose f_i, f_j and $a, b \in A$ such that $x = f_i a$, $hx = f_j b$ so $f_i a = h^{-1} f_j b$ and $b = f_j^{-1} h f_i a$. Hence, $a \in A_{ij}$ and $x \in FA_{ij}$. It follows that $B \subseteq \bigcup_{i,j=1}^{n} FA_{ij}$ so $B \in \mathcal{I}$, which is impossible because B is $\mathcal{I}$-thick.

Theorem 2 is proved.

***Corollary* 1.** *Let G be an infinite group, $\mathcal{I}$ be a left-invariant ideal in $\mathcal{P}_G$. Then the ideal $\mathcal{S}(\mathcal{I})$ is thin-complete.*

Proof. Applying Theorem 2 to $\mathcal{S}(\mathcal{I})$, we get $\mathcal{S}(\mathcal{I}) \subseteq \tau(\mathcal{S}(\mathcal{I})) \subseteq \mathcal{S}(\mathcal{S}(\mathcal{I}))$. To verify that $\mathcal{S}(\mathcal{S}(\mathcal{I})) \subseteq \mathcal{S}(\mathcal{I})$, we show that every $\mathcal{S}(\mathcal{I})$-large subset L is $\mathcal{I}$-large. We choose $F \in \mathcal{F}_G$ and $S \in \mathcal{S}(\mathcal{I})$ such that $G = FL \cup S$. By Proposition 1(1), $G \setminus S$ is $\mathcal{I}$-large. Hence, there exist $H \in \mathcal{F}_G$ and $I \in \mathcal{I}$ such that $G = H(G \setminus S) \cup I = HFL \cup I$, so L is $\mathcal{I}$-large.

Corollary 1 is proved.

For every group G, $\tau(\mathcal{I}_\varnothing)$ coincides with the family $[G]^{\leq 1}$ of all at most one-element subsets. If G is finite, then $\mathcal{S}(\mathcal{I}_\varnothing) = \{\varnothing\}$. Thus, Corollary 1 is not true for finite groups.

2. Relatively sparse subsets of groups. Let G be an infinite group and $\mathcal{F}$ be a subfamily in $\mathcal{P}_G$. We say that a subset $A \subseteq G$ is $\mathcal{F}$-*sparse* if, for any infinite subset S of G, there exists a finite subset $F \subset S$ such that

$$\bigcap_{x \in F} xA \in \mathcal{F}.$$

We denote by $\sigma(\mathcal{F})$ the family of all $\mathcal{F}$-sparse subsets of G. If $\mathcal{F}$ is left invariant and downward closed then so is $\sigma(\mathcal{F})$. Repeating the arguments from [2, p. 494, 495], the reader can verify that $\sigma(\mathcal{F})$ is a left invariant ideal provided that $\mathcal{F}$ is a left invariant ideal. Alternatively, this statement can be derived from Theorem 3, see Corollary 2.

Now we need some information on the algebraic structure of the Stone – Čech compactification βG of a discrete group G. We take βG to be the set of all ultrafilters on G identifying G with the set of all principal ultrafilters. The topology of can be described by stating that the sets $\{\overline{A} \colon A \subseteq G\}$ form a base for open sets in βG where $\overline{A} = \{p \in \beta G \colon A \in p\}$. The set $G^* = \beta G \setminus G$ of all free ultrafilters on G is closed in βG and the family $\{A^* \colon A \subseteq G\}$ is a base for open sets in G^* where $A^* = \{p \in G^* \colon A \in p\}$.

Using the universal property of the Stone-Čech compactifications, the multiplication on G can be extended to the semigroup operation on βG in such a way that all mappings $x \to gx$, $g \in G$ and $x \to xp$, $p \in \beta G$ from βG to βG are continuous. Given any $q, p \in \beta G$ and $A \subseteq G$, the product qp is defined by the rule:

$$A \in qp \Leftrightarrow \{x \in G \colon x^{-1}A \in q\} \in p.$$

For the structure of the compact right-topological semigroup βG and its combinatorial applications, see [5, 6].

For a family $\mathcal{F}$ of subsets of a group G, we put

$$\mathcal{F}^{\wedge} = \{q \in \beta G \colon G \setminus A \in q \text{ for any } A \in \mathcal{F}\}$$

and note that $^{\wedge}$ is a bijection between the family of all left-invariant ideals of $\mathcal{P}_G$ and the family of all closed left ideals of βG. For more information on this correspondence, see [2, 7].

Given an ultrafilter $p \in G^*$, we say that a subset A of G is $(\mathcal{F}, p)$-*sparse* if for any $P \in p$ there exists a finite subset $F \subset P$ such that $\bigcap_{x \in F} x^{-1}A \in \mathcal{F}$. We denote by $\sigma(\mathcal{F}, p)$ the family of all $(\mathcal{F}, p)$-sparse subsets of G. Clearly,

$$\sigma(\mathcal{F}) = \bigcap_{p \in G^*} \sigma(\mathcal{F}, p).$$

Theorem 3. *Let $\mathcal{F}$ be a left-invariant ideal in $\mathcal{P}_G$, $A \subseteq G$, $p \in G^*$. Then*

(1) *$A \in \sigma(\mathcal{F}, p)$ if and only if $A^* \cap p\mathcal{F}^{\wedge} = \varnothing$;*

(2) *$(\sigma(\mathcal{F}))^{\wedge} = cl(G^*\mathcal{F}^{\wedge})$.*

Proof. (1) $A \notin \sigma(\mathcal{F}, p) \Leftrightarrow \exists P \in p \; \forall F \subset P \; F \in \mathcal{F}_G \colon \bigcap_{x \in F} x^{-1}A \notin \mathcal{F} \;\Leftrightarrow$

$$\exists P \in p \quad \exists q \in \mathcal{F}^{\wedge} \quad \forall x \in P \colon \; x^{-1}A \in q \;\Leftrightarrow$$

$$\Leftrightarrow\; \exists q \in \mathcal{F}^{\wedge} \colon \; A \in pq \;\Leftrightarrow\; A^* \cap p\mathcal{F}^{\wedge} \neq \varnothing.$$

(2) $q \in (\sigma(\mathcal{F}))^{\wedge} \Leftrightarrow \forall Q \in q \colon G \setminus Q \in \sigma(\mathcal{F}) \Leftrightarrow \forall Q \in q \; \forall p \in G^* \colon G \setminus Q \in$ $\in \sigma(\mathcal{F}, p) \;\Leftrightarrow_{(1)} \forall Q \in q \quad \forall p \in G^* \colon \; (G \setminus Q)^* \cap p\mathcal{F}^{\wedge} = \varnothing \;\Leftrightarrow\; \forall Q \in q \colon (G \setminus Q)^* \cap$ $\cap\, G^*\mathcal{F}^{\wedge} = \varnothing \Leftrightarrow \forall Q \in q \colon (G \setminus Q)^* \cap cl(G^*\mathcal{F}^{\wedge}) = \varnothing \;\Leftrightarrow\; q \in cl(G^*\mathcal{F}^{\wedge})$.

***Corollary* 2.** *Let $\mathcal{F}$ be a left-invariant ideal in $\mathcal{P}_G$. Then*

(1) *$\sigma(\mathcal{F})$ is a left-invariant ideal in $\mathcal{P}_G$;*

(2) *if $\mathcal{F}^{\wedge}$ is a right ideal in βG then $(\sigma(\mathcal{F}))^{\wedge}$ is a right ideal.*

Proof. We note that a closure of an arbitrary left (right) ideal of βG is a left (right) ideal, see Theorems 2.15 and 2.17 in [5]. Then both statements follow from Theorem 3(2).

We say that a left-invariant ideal $\mathcal{F}$ in $\mathcal{P}_G$ is *sparse-complete* if $\sigma(\mathcal{F}) = \mathcal{F}$ (or equivalently, by Theorem 3(2), $\mathcal{F}^{\wedge} = cl(G^*\mathcal{F}^{\wedge})$) and denote by $\sigma^*(\mathcal{F})$ the intersection of all sparse-complete ideals containing $\mathcal{F}$. Clearly, the *sparse-completion* $\sigma^*(\mathcal{F})$ is the smallest sparse-complete ideal such that $\mathcal{F} \subseteq \sigma^*(\mathcal{F})$.

We define also a sequence $(\sigma^n(\mathcal{F}))_{n \in \omega}$ of ideals by recursion: $\sigma^0(\mathcal{F}) = \mathcal{F}$, $\sigma^{n+1}(\mathcal{F}) = \sigma(\sigma^n(\mathcal{F}))$ for $n \in \omega$.

Theorem 4. *Let G be an infinite group, $\mathcal{F}$ be a left invariant ideal in $\mathcal{P}_G$. Then*

(1) *$\sigma^*(\mathcal{F}) = \bigcup_{n \in \omega} \sigma^n(\mathcal{F})$;*

(2) *$\sigma^{n+1}(\mathcal{F}_G) \neq \sigma^n(\mathcal{F}_G)$ for every $n \in \omega$.*

Proof. 1. Clearly, $\bigcup_{n \in \omega} \sigma^n(\mathcal{F}) \subseteq \sigma^*(\mathcal{F})$. On the other hand,

$$\left(\sigma\left(\bigcup_{n \in \omega} \sigma^n(\mathcal{F})\right)\right)^{\wedge} = cl\left(G^*\left(\bigcup_{n \in \omega} \sigma^n(\mathcal{F})\right)^{\wedge}\right) =$$

$$= cl\left(G^* \bigcap_{n\in\omega} (\sigma^n(\mathcal{F}))^\wedge\right) \subseteq cl\left(\bigcap_{n\in\omega} G^*(\sigma^n(\mathcal{F}))^\wedge\right) \subseteq$$

$$\subseteq cl\left(\bigcap_{n\in\omega} (\sigma^{n+1}(\mathcal{F}))^\wedge\right) \subseteq cl\left(\bigcap_{n\in\omega} (\sigma^n(\mathcal{F}))^\wedge\right) =$$

$$= \bigcap_{n\in\omega} (\sigma^n(\mathcal{F}))^\wedge = \left(\bigcup_{n\in\omega} \sigma^n(\mathcal{F})\right)^\wedge,$$

so $\bigcup_{n\in\omega} \sigma^n(\mathcal{F})$ is sparse-complete and $\sigma^*(\mathcal{F}) \subseteq \bigcup_{n\in\omega} \sigma^n(\mathcal{F})$.

2. We suppose that G is countable, $G = \{g_n : n \in \omega\}$, $F_n = \{g_i : i \leqslant n\}$. For $n = 0$, we take an arbitrary countable thin subset T and note that $T \in \sigma(\mathcal{F}_G) \setminus \mathcal{F}_G$.

For $n > 0$, it suffices to choose an injective sequence $(x_m)_{x\in\omega}$ in G and a decreasing sequence $(X_m)_{m\in\omega}$ of subsets of G such that

$(1)_n$ $F_m x_m X_m \cap F_k x_k X_k = \varnothing$ for all $m < k < \omega$;

$(2)_n$ $g x_m X_m \cap x_m X_m = \varnothing$ for all $g \in F_m \setminus \{e\}$ and $m < \omega$;

$(3)_n$ $X_m \in \sigma^n(\mathcal{F}_G) \setminus \sigma^{n-1}(\mathcal{F}_G)$ for each $m < \omega$.

Indeed, we put $Q_n = \bigcup_{m\in\omega} x_m X_m$. Let $S = \{x_m^{-1} : m \in \omega\}$, F be a finite subset of S. Then $\bigcap_{z\in F} zQ_n$ contains some subset X_m and, by $(3)_n$, $Q_n \notin \sigma^n(\mathcal{F}_G)$. On the other hand, by $(1)_n$, $(2)_n$ and $(3)_n$, $Q_n \in \sigma^{n+1}(\mathcal{F}_G)$. Thus, $Q_n \in \sigma^{n+1}(\mathcal{F}_G) \setminus \sigma^n(\mathcal{F}_G)$.

We show only how to construct Q_1 and Q_2.

To satisfy $(1)_1$, $(2)_1$, $(3)_1$, we choose inductively two injective sequences $(x_m)_{m\in\omega}$, $(y_m)_{m\in\omega}$ in G (x_m after y_m) such that for every $N \in \omega$ and all $m < k \leqslant N$, $g \in F_m \setminus \{e\}$:

$$F_m x_m \{y_i : m < i \leqslant N\} \cap F_k x_k \{y_i : k < i \leqslant N\} = \varnothing,$$

$$g x_m \{y_i : m < i \leqslant N\} \cap x_m \{y_i : m < i \leqslant N\} = \varnothing,$$

$$F_m y_m \cap F_k y_k = \varnothing.$$

Then we put $X_m = \{y_i : m < i < \omega\}$, and note that $(x_m)_{m\in\omega}$, and $(X_m)_{m\in\omega}$ satisfy $(1)_1$, $(2)_1$, $(3)_1$.

To satisfy $(1)_2$, $(2)_2$, $(3)_2$, we choose inductively three injective sequences $(x_m)_{m\in\omega}$, $(y_m)_{m\in\omega}$, $(z_m)_{m\in\omega}$ in G (y_m after z_m, x_m after y_m) such that for every $N \in \omega$ and all $m < k \leqslant N$, $g \in F_m \setminus \{e\}$:

$$F_m x_m \{y_i z_j : m < i < j \leqslant N\} \cap F_k x_k \{y_i z_j : k < i < j \leqslant N\} = \varnothing,$$

$$g x_m \{y_i z_j : m < i < j \leqslant N\} \cap x_m \{y_i z_j : m < i < j \leqslant N\} = \varnothing,$$

$$F_m y_m \{z_i : m < i \leqslant N\} \cap F_k y_k \{z_i : k < i \leqslant N\} = \varnothing,$$

$$g y_m \{z_i : m < i \leqslant N\} \cap y_m \{z_i : m < i \leqslant N\} = \varnothing,$$

$$F_m z_m \cap F_k z_k = \varnothing.$$

Then we put $X_m = \{y_i z_j \colon m < i < j < \omega\}$, and note that $(x_m)_{m\in\omega}$ and $(X_m)_{m\in\omega}$ satisfy $(1)_2$, $(2)_2$, $(3)_2$.

If G is uncountable, we fix some countable subgroup of G and, for each $n \in \omega$, pick $Q \subseteq H$, $Q \in (\sigma^{n+1}(\mathcal{F}_H) \cap \mathcal{P}_H) \setminus (\sigma^n(\mathcal{F}_H) \cap \mathcal{P}_H)$. Clearly, $Q \in \sigma^{n+1}(\mathcal{F}_G) \setminus \sigma^n(\mathcal{F}_G)$.

Theorem 4 is proved.

Theorem 5. *For a left-invariant ideal $\mathcal{F}$ of subsets of an infinite group G, we have*

(1) $\tau(\mathcal{F}) \subseteq \sigma(\mathcal{F})$;

(2) *if G is torsion-free, then* $\sigma(\mathcal{F}) \subseteq \tau^*(\mathcal{F})$;

(3) *if G is Abelian and $\{g \in G \colon g^2 = e\}$ is finite, then* $\tau^2(\mathcal{F}_G) \nsubseteq \sigma(\mathcal{F}_G)$.

Proof. 1. The inclusion $\tau(\mathcal{F}) \subseteq \sigma(\mathcal{F})$ follows from the definitions.

2. Assume that the group G is torsion-free and let $A \subseteq G$ be $\mathcal{F}$-sparse. According to a characterization of $\tau^*(\mathcal{F})$ proved in [4] and mentioned in Remark 1, in order to prove that $A \in \tau^*(\mathcal{F})$ we need to show that for each sequence $(g_n)_{n\in\omega} \in (G \setminus \{e\})^\omega$ there is $n \in \omega$ such that

(4) $\bigcap\limits_{i_0,\ldots,i_n \in \{0,1\}} g_0^{i_0} \ldots g_n^{i_n} A \in \mathcal{F}.$

It follows from the torsion-free property of G that the set

$$C = \{g_0^{i_0} \ldots g_n^{i_n} \colon n \in \omega,\ i_0, \ldots, i_n \in \{0,1\}\}$$

is infinite. Since A is $\mathcal{F}$-sparse, there is a finite subset $F \subset C$ such that $\bigcap_{x\in F} xA \in \mathcal{F}$. For this set F we can find $n \in \omega$ such that

$$F \subset \{g_0^{i_0} \ldots g_n^{i_n} \colon i_0, \ldots, i_n \in \{0,1\}\}$$

and conclude that (4) holds.

3. First, we consider the case of countable group G. Suppose that we have constructed an infinite subset X of G such that

(5) $\forall a, b, c \in X,\ a \neq b \neq c \Rightarrow ab^{-1}c \notin X.$

We choose a sequence $(F_n)_{n\in\omega}$ of finite subsets of X such that each finite subset of X appears in $(F_n)_{n\in\omega}$ infinitely many times. We enumerate $G = \{g_n \colon n < \omega\}$, $g_0 = e$, and put $G_n = \{g_i \colon i \leqslant n\}$. We put $a_0 = e$ and choose inductively a sequence $(a_n)_{n\in\omega}$ in G such that, for all $n \in \omega$,

(6) $G_{n+1}a_{n+1}F_{n+1} \cap G_{n+1}(a_0F_0 \cup \ldots \cup a_nF_n) = \varnothing.$

We claim that the set $A = \bigcup_{n\in\omega} a_nF_n$ belongs to $\tau^2(\mathcal{F}_G) \setminus \sigma^*(\mathcal{F}_G)$.

Assuming that $A \in \sigma^*(\mathcal{F}_G)$, we can find a finite subset F^{-1} of X^{-1} such that $\bigcap_{g\in F^{-1}} gA = \varnothing$. Since the set F appears in $(F_n)_{n\in\omega}$ infinitely often, the intersection $\bigcap_{g\in F^{-1}} gA$ contains infinitely many members of the injective sequence $(a_n)_{n\in\omega}$, so we get a contradiction.

The inclusion $A \in \tau^2(\mathcal{F}_G)$ will follow from Proposition 2(2) as soon as we show that

$$|A \cap gA \cap fA \cap gfA| < \infty$$

for all $g, f \in G \setminus \{e\}$. Suppose the contrary and choose corresponding g, f. By (6), there exists $n \in \omega$ such that

$$a_nF_n \cap ga_nF_n \cap fa_nF_n \cap gfa_nF_n \neq \varnothing.$$

We pick $t \in F_n$ such that

$$gt \in F_n, \quad ft \in F_n, \quad gft \in F_n,$$

so t, tg, tf, $tfg \in X$. But $tf(t^{-1})tg = tfg$ and, by (5), $tfg \notin X$.

If G is uncountable, we take a countable subgroup G' and construct A inside G'.

To complete the proof, we construct X as a union $X = \bigcup_{n\in\omega} X_n$ of an increasing sequence of finite subsets $\{X_n : n \in \omega\}$, $|X_n| = n$. We put $X_0 = \varnothing$ and assume that, for some $n \in \omega$, we have chosen a subset X_n satisfying (5) with X_n instead of X, and $X_n \cap X_n^{-1} = \varnothing$. Since G has only finite number of elements of order 2, we can choose an element $x_{n+1} \in G$ such that

(7) $(x_{n+1}X_nX_n^{-1} \cup x_{n+1}^{-1}X_nX_n) \cap X_n = \varnothing$;

(8) $x_{n+1}^2 \neq e$;

(9) $x_{n+1} \notin X_n^{-1}$;

(10) $x_{n+1}X_n^{-1}x_{n+1} = \varnothing$.

We put $X_{n+1} = X_n \cup \{x_{n+1}\}$. By (8) and (9), $X_{n+1} \cap X_{n+1}^{-1} = \varnothing$. By (7) and (10), X_{n+1} satisfies (5).

Theorem 5 is proved.

***Corollary* 3.** *Let G be an infinite Abelian group with finite number of elements of order* 2. *Then $\tau^2(\mathcal{F}_G) \nsubseteq \tau(\mathcal{F}_G)$, $\tau(\sigma(\mathcal{F}_G)) \nsubseteq \sigma(\mathcal{F}_G)$, and $\sigma(\sigma(\mathcal{F}_G)) \nsubseteq \sigma(\mathcal{F}_G)$. In particular, the ideal $\sigma(\mathcal{F}_G)$ of sparse sets is not thin-complete.*

If the group G is torsion-free, then Theorem 5 guarantees that $\mathcal{F} \subseteq \tau(\mathcal{F}) \subseteq \sigma(\mathcal{F}) \subseteq \subseteq \tau^*(\mathcal{F})$, which implies the equivalence of the equalities $\mathcal{F} = \tau(\mathcal{F}) = \tau^*(\mathcal{F})$ (the thin-completeness) and $\mathcal{F} = \sigma(\mathcal{F})$ (the sparse-completeness). Thus we obtain the following surprising:

***Corollary* 4.** *Let $\mathcal{F}$ be a left-invariant ideal of subsets of an infinite group G. If G is torsion-free, then $\mathcal{F}$ is thin-complete if and only if $\mathcal{F}$ is sparse-complete. Consequently, $\tau^*(\mathcal{F}) = \sigma^*(\mathcal{F})$.*

We conclude this section discussing the intrinsic structure of $\sigma(\mathcal{F})$.

***Remark* 4.** Let G be an infinite group and $\mathcal{F}$ be a left-invariant ideal in $\mathcal{P}_G$. For a subset $A \subset G$, we consider the set

$$\Sigma_A = \left\{ F \in \mathcal{F}_G \colon \bigcap_{x\in F} xA \notin \mathcal{F} \right\},$$

partially ordered by the relation $\subset$. It follows from the definition that A is $\mathcal{F}$-sparse if and only if the Σ_A is well-founded in the sense that it contains no infinite chains. In this case we can assign to each set $F \in \Sigma_A$ the ordinal

$$\mathrm{rank}(F) = \sup\left\{\mathrm{rank}(E) + 1 \colon F \subset E \in \Sigma_A,\ |E \setminus F| = 1\right\},$$

where $\sup(\varnothing) = 0$. So, the maximal elements of Σ_A have rank 0, their immediate predecessors have rank 1 and so on. Let

$$\mathrm{rank}(\Sigma_A) = \sup\left\{\mathrm{rank}(F) + 1 \colon F \in \Sigma_A\right\} = \mathrm{rank}(\varnothing) + 1$$

be the rank of the family Σ_A.

For an ordinal α let

$$\sigma_\alpha(\mathcal{F}) = \{A \in \sigma(\mathcal{F})\colon \operatorname{rank}(\Sigma_A) \leqslant \alpha + 1\} \text{ and } \sigma_{<\alpha}(\mathcal{F}) = \bigcup_{\beta<\alpha} \sigma_\beta(\mathcal{F}).$$

Sets from the family $\sigma_\alpha(\mathcal{F})$ are called $(\alpha, \mathcal{F})$-*sparse*. Observe that a set $A \subset G$ is $(n, \mathcal{F})$-sparse for a natural number $n \in \omega$ if and only if for each infinite set $S \subset G$ there is a set $F \subset G$ of cardinality $|F| \le n+1$ such that $\bigcap_{x\in F} xA \in \mathcal{F}$. Now we see that $(n, \mathcal{F}_G)$-sparse sets coincide with $(n+1)$-sparse sets studied in [2]. By Lemma 1.2 of [2] the union $A \cup B$ of an $(n, \mathcal{F}_G)$-space set $A \subset G$ and an $(m, \mathcal{F}_G)$-sparse set $B \subset G$ is $(m+n, \mathcal{F}_G)$-sparse. Consequently, the family $\sigma_{<\omega}(\mathcal{F}_G)$ is an ideal in G.

***Question* 1.** For which ordinals α the family $\sigma_{<\alpha}(\mathcal{F})$ is an ideal in $\mathcal{P}_G$? Is it true for each limit (additively indecomposable*) ordinal α?

Repeating the argument of Proposition 3, we can prove that

$$\sigma(\mathcal{F}) = \bigcup_{\alpha<|G|^+} \sigma_\alpha(\mathcal{F}),$$

so $\sigma_{<|G|^+}(\mathcal{F}) = \sigma(\mathcal{F})$ is an ideal in $\mathcal{P}_G$ according to Corollary 2.

We note that a similar construction using the rank function of well-founded trees has been used in [4] for describing the intrinsic structure of the ideal $\tau^*(G)$.

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*An ordinal α is called *additively indecomposable* if for any ordinals $\beta, \gamma < \alpha$ we get $\beta + \gamma < \alpha$.