UDC 512.5

M. J. Nikmehr (K. N. Toosi Univ. Technology, Tehran, Iran)

STRONGLY SEMICOMMUTATIVE RINGS RELATIVE TO A MONOID
СИЛЬНО НАПІВКОМУТАТИВНІ КІЛЬЦЯ ВІДНОСНО МОНОЇДА

For a monoid M, we introduce strongly M-semicommutative rings, which are generalization of strongly semicommutative rings and investigate their properties. We show that if G is a finitely generated Abelian group, then G is torsion free if and only if there exists a ring R with $|R| \geq 2$ such that R is strongly G-semicommutative.

Для моноїда M ми вводимо сильно M-напівкомутативні кільця, що узагальнюють сильно напівкомутативні кільця, та вивчаємо їх властивості. Показано, що якщо G — скінченнопороджена абелева група, то G є вільною від скруту тоді і тільки тоді, коли існує кільце R з $|R| \geq 2$ таке, що R є сильно G-напівкомутативним.

1. Introduction. Throughout this article, R and M denote an associative ring with identity and a monoid, respectively. In [1] Cohn introduced the notion of reversible ring. A ring R is said to be *reversible*, whenever $a, b \in R$ satisfy $ab = 0$ then $ba = 0$. A ring R is called *symmetric*, whenever $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. A ring R is called *reduced*, whenever $a^2 = 0$ implies $a = 0$ for all $a \in R$. A ring R is called *semicommutative*, whenever $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. The following implication holds:

$$\text{reduced} \Longrightarrow \text{symmetric} \Longrightarrow \text{reversible} \Longrightarrow \text{semicommutative}.$$

In [13] Yang and Liu introduced the notion of strongly reversible. A ring R is called *strongly reversible*, whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. All reduced rings are strongly reversible but converse is not true. In [11] Singh and Juyal introduced the notion of strongly reversible. A ring R is called *strongly M-reversible*, whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$ where $\alpha, \beta \in R[M]$. In [5] Huh and Lee showed that polynomial rings over semicommutative rings need not be semicommutative. In [2] Gang and Ruijuan introduced the notion of strongly semicommutative. A ring R is called *strongly semicommutative*, whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $f(x)R[x]g(x) = 0$. All reduced rings are strongly semicommutative but converse is not true. Rege and Chhawchharia [10], introduced the notion of an Armendariz ring. A ring R is called *Armendariz*, whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots \ldots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. Some properties of Armendariz rings were given in [8, 9, 12]. In [7] Z. Liu studied a generalization of Armendariz rings, which is called M-Armendariz rings, where M is monoid. A ring R is called *M-Armendariz*, whenever $\alpha = a_1g_1 + a_2g_2 + \ldots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \ldots + b_mg_m \in R[M]$, with $g_i, h_j \in M$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$, for all i, j. A ring R is called *strongly M-semicommutative*, whenever $\alpha\beta = 0$ implies $\alpha R[M]\beta = 0$, where $\alpha, \beta \in R[M]$. Let $M = (N \cup \{0\}, +)$. Then a ring R is strongly M-semicommutative if and only if R is strongly semicommutative. Recall that a monoid M is called a unique product monoid (*u.p.-monoid*) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely in the form ab, where $a \in A$ and $b \in B$. We investigate a generalization of strongly semicommutative rings which we call strongly M-semicommutative rings. It is proved that a ring R is strongly M-semicommutative if and only if its polynomial ring $R[x]$

is strongly M-semicommutative if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly M-semicommutative. Also, we check the following questions:

(1) Does R being a strongly M-semicommutative imply $R(+)R$ being strongly M-semicommutative?

(2) R being a strongly M-semicommutative if and only if R is Abelian ring?

(3) R being strongly M-semicommutative if and only if R/I is strongly M-semicommutative?

2. Strongly M-semicommutative ring. We begin this section with the following definition which have the main role in the whole work.

Definition 2.1. *A ring R is called strongly M-semicommutative, whenever $\alpha\beta = 0$ implies $\alpha R[M]\beta = 0$, where $\alpha, \beta \in R[M]$.*

Lemma 2.1 [6]. *If R is a reduced ring, then*

$$T_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$$

is a semicommutative ring.

Lemma 2.2 [7]. *Let M be a monoid with $|M| \geq 2$. Then the following conditions are equivalent*:

(1) *R is M-Armendariz and reduced.*

(2) *$T_3(R)$ is M-Armendariz.*

Proposition 2.1. *Let M be a monoid with $|M| \geq 2$, and R is M-Armendariz and reduced. Then $T_3(R)$ is strongly M-semicommutative.*

Proof. Suppose that $\alpha = A_0 g_1 + \ldots + A_n g_n$, $\beta = B_0 h_1 + \ldots + B_m h_m \in T_3(R)[M]$, $\alpha\beta = 0$. Since $T_3(R)$ is M-Armendariz by Lemma 2.2, so $A_i B_j = 0$. Also $T_3(R)$ is semicommutative by Lemma 2.1, and hence $A_i T_3(R) B_j = 0$. Therefore $\alpha T_3(R)[M]\beta = 0$. This means that $T_3(R)$ is strongly M-semicommutative.

Before stating Proposition 2.2, we need the following lemmas.

Lemma 2.3 [11]. *Let M be u.p.-monoid and R be a reduced ring. Then R is strongly M-reversible.*

Lemma 2.4 [11]. *Let M be u.p.-monoid and R be a reduced ring. Then $R[M]$ is reduced.*

Proposition 2.2. *Let M be u.p.-monoid and R be a reduced ring. Then R is strongly M-semicommutative.*

Proof. Suppose $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j$ are in $R[M]$ with $a_i, b_j \in R$ and $g_i, h_j \in M$ for all i, j. Take $\alpha\beta = 0$. So $(\alpha R[M]\beta)^2 = (\alpha R[M]\beta)(\alpha R[M]\beta) = \alpha R[M](\beta\alpha)R[M]\beta = 0$, since R is strongly M-reversible by Lemma 2.3. Also by Lemma 2.4, we have $\alpha R[M]\beta = 0$. Hence R is strongly M-semicommutative ring.

Lemma 2.5. *Subrings and direct products of strongly M-semicommutative ring are strongly M-semicommutative.*

Proof. Let $I_\lambda (\lambda \in \Lambda)$ be ideals of R such that every $\dfrac{R}{I_\lambda}$ is strongly M-semicommutative and $\cap_{\lambda \in \Lambda} I_\lambda = 0$. Suppose that $\alpha = \sum_{i=0}^{m} a_i g_i$, $\beta = \sum_{j=0}^{n} b_j h_j \in R[M]$, satisfy $\alpha\beta = 0$. For any $\gamma = \sum_{k=0}^{l} c_k r_k \in R[M]$, we have that $\overline{\alpha}\,\overline{\gamma}\,\overline{\beta} = 0$ in $\left(\dfrac{R}{I_\lambda}\right)[M]$ for each $\lambda \in \Lambda$, since $\dfrac{R}{I_\lambda}$ is strongly M-semicommutative. So $\sum_{i+j+k=t} a_i c_k b_j \in I_\lambda$ for $t = 0, \ldots, m+n+l$ and any $\lambda \in \Lambda$, which implies that $\sum_{i+j+k=t} a_i c_k b_j = 0$ for $t = 0, \ldots, m+n+l$, since $\cap_{\lambda\in\Lambda} I_\lambda = 0$. Thus we obtain $\alpha R[M]\beta = 0$.

Proposition 2.3. *Let M be a cancelative monoid and N an ideal of M. If R is strongly N-semicommutative, then R is strongly M-semicommutative.*

Proof. Suppose that $\alpha = a_1g_1 + a_2g_2 + \ldots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \ldots + b_mh_m$ are in $R[M]$ such that $\alpha\beta = 0$. Take $g \in N$. Then $gg_1, gg_2, \ldots, gg_n$, $h_1g, h_2g, \ldots, h_mg \in N$ and $gg_i \neq gg_j$ and $h_ig \neq h_jg$ for all $i \neq j$. So $\alpha_1\beta_1 = \left(\sum_{i=1}^n a_igg_i\right)\left(\sum_{j=1}^m b_jh_jg\right) = 0$. Since R is strongly N-semicommutative, so $\alpha_1 R[N]\beta_1 = 0$. Thus $\alpha R[M]\beta = 0$. Therefore R is strongly M-semicommutative.

Lemma 2.6. *Let M be a cyclic group of order $n \geq 2$ and R a ring with unity. Then R is not strongly M-semicommutative.*

Proof. Suppose that $M = e,\ g,\ g^2, \ldots,\ g^n - 1$. Let $\alpha = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} e + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} g + \ldots$ $\ldots + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} g^{n-1}$ and $\beta = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} e + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} g \in R[M]$.

Then $\alpha\beta = 0$. But $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} R[M] \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \neq 0$, so $\alpha R[M]\beta \neq 0$. Thus R is not strongly M-semicommutative.

Lemma 2.7. *M be a monoid and N a submonoid of M. If R is strongly M-semicommutative ring, then R is strongly N-semicommutative.*

Lemma 2.8. *Let M and N be u.p.-monoids. Then so is the monoid $M \times N$.*

Proof. See [7] (Lemma 1.13).

Let $T(G)$ be set of elements of finite order in an Abelian group G. Then $T(G)$ is fully invariant subgroup of G. G is said to be torsion-free if $T(G) = \{e\}$.

Theorem 2.1. *Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent*:

(1) *G is torsion-free.*

(2) *There exists a ring R with $|R| \geq 2$ such that R is strongly G-semicommutative.*

Proof. (2) $\Longrightarrow$ (1). If $g \in T(G)$ and $g \neq e$, then $N = \langle g\rangle$ is cyclic group of finite order. If a ring $R \neq 0$ is strongly M-semicommutative. Then by Lemma 2.7 R is strongly N-semicommutative, a contradiction with Lemma 2.6. Thus every ring $R \neq 0$ is not strongly M-semicommutative.

(1) $\Longrightarrow$ (2). Let G be a finitely generated Abelian group with $T(G) = \{e\}$. Then $G = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$ a finite direct product of group $\mathbb{Z}$. By Lemma 2.8 G is u.p.-monoid. Let R be a commutative reduced ring. Then by Proposition 2.2, R is strongly G-semicommutative.

It is natural to conjecture that R is a strongly semicommutative ring if for any nonzero proper ideal I of R, R/I and I are strongly semicommutative, where I is considered as a strongly semicommutative ring without identity. Note that strongly semicommutative rings are Abelian, and so every n by n upper (or lower) triangular matrix ring, for $n \geq 2$, over any ring with identity can not be strongly semicommutative.

***Example* 2.1** (see [13], Example 3.7)**.** Let S be a division ring and

$$R = \left\{ \begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in S \right\}.$$

Take an ideal $I = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is strongly M-semicommutative nonzero proper ideal of R. Take

$$\alpha = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_j & 0 \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} g_i, \qquad \beta = \sum_{j=0}^{m} \begin{pmatrix} u_j & v_j & 0 \\ 0 & u_j & w_j \\ 0 & 0 & u_j \end{pmatrix} h_j$$

are in $R/I[M]$ satisfying $\alpha\beta = 0$. Then we have that

$$\begin{pmatrix} \sum_{i=0}^{n} a_i g_i & \sum_{i=0}^{n} b_i g_i & 0 \\ 0 & \sum_{i=0}^{n} a_i g_i & \sum_{i=0}^{n} c_i g_i \\ 0 & 0 & \sum_{i=0}^{n} a_i g_i \end{pmatrix} \begin{pmatrix} \sum_{j=0}^{m} u_j h_j & \sum_{j=0}^{m} v_j h_j & 0 \\ 0 & \sum_{j=0}^{m} u_j h_j & \sum_{j=0}^{m} w_j h_j \\ 0 & 0 & \sum_{j=0}^{m} u_j h_j \end{pmatrix} = 0$$

which implies $\sum_{i=0}^{n} a_i g_i \sum_{j=0}^{m} u_j h_j = 0$, and hence $\sum_{i=0}^{n} a_i g_i = 0$ or $\sum_{j=0}^{m} u_j h_j = 0$, since S is division ring, and it is easy to prove that $\alpha R[M]\beta = 0$. There by we get that for any strongly M-semicommutative nonzero proper ideal I of R, R/I is strongly M-semicommutative.

However we take a stronger condition I is reduced then we may have an affirmative answer as in the following.

Proposition 2.4. *For a ring R suppose that R/I is strongly M-semicommutative ring for some ideal I of R. If I is reduced then R is strongly M-semicommutative.*

Proof. Let $\alpha\beta = 0$ with $\alpha, \beta \in R[M]$. Then we have $\alpha R[M]\beta \subseteq I[M]$ and $\beta I[M]\alpha = 0$ since $\beta I[M]\alpha \subseteq I[M]$, $(\beta I[M]\alpha)^2 = 0$ and $I[M]$ is reduced. According

$$((\alpha R[M]\beta)I[M])^2 = \alpha R[M]\beta I[M]\alpha R[M]\beta I[M] = \alpha R[M](\beta I[M]\alpha)R[M]\beta I[M] = 0$$

and so $\alpha R[M]\beta I[M] = 0$, and hence $(\alpha R[M]\beta)^2 \subseteq \alpha R[M]\beta I[M] = 0$ implies $(\alpha R[M]\beta)^2 = 0$. But $\alpha R[M]\beta \subseteq I[M]$ and so $\alpha R[M]\beta = 0$, therefore R is strongly M-semicommutative.

As a kind of converse of Proposition 2.4, we obtain the following situation.

Proposition 2.5. *Let R be a strongly M-semicommutative ring and I be an ideal of R. If I is an annihilator in R, then R/I is a strongly M-semicommutative ring.*

Proof. Set $I = r_R(S)$ for some $S \subseteq R$ and write $\overline{t} = t + I \in \dfrac{R}{I}$. Let $\overline{\alpha}\overline{\beta} = 0$, so $S[M]\alpha R[M]\beta = 0$, since R is strongly M-semicommutative by hypothesis and we have $r_R(S)[M] = r_{R[M]}(S[M])$. Thus $\alpha R[M]\beta \in r_{R[M]}(S[M])$ implies $\overline{\alpha}\left(\dfrac{R}{I}\right)[M]\overline{\beta} = 0$.

Lemma 2.9. *For an Abelian ring R, R is strongly M-semicommutative if and only if eR and $(1-e)R$ are strongly M-semicommutative for every idempotent e of R if and only if eR and $(1-e)R$ are strongly M-semicommutative for some idempotent e of R.*

Proof. Suppose that $\alpha\beta = 0$, since eR and $(1-e)R$ are strongly M-semicommutative, thus $e\alpha eR[M]e\beta e = 0$ and $(1-e)\alpha(1-e)R[M](1-e)\beta(1-e) = 0$. So

$$\alpha R[M]\beta = e\alpha R[M]\beta + (1-e)\alpha R[M]\beta =$$

$$= e\alpha e R[M] e\beta e + (1-e)\alpha(1-e)R[M](1-e)\beta(1-e) = 0,$$

and therefore R is strongly M-semicommutative.

For semicommutative rings relative to monoids, we have following results.

Proposition 2.6. *Let M and N be a u.p.-monoid. If R is a reduced ring, then $R[M]$ is strongly N-semicommutative.*

Proof. By Lemma 2.4 $R[M]$ is reduced, since N is a u.p.-monoid and $R[M]$ is reduced, therefore by Proposition 2.2, $R[M]$ is strongly N-semicommutative.

Proposition 2.7. *Let M and N be a u.p.-monoid. If R is a reduced, then R is strongly $M \times N$-semicommutative.*

Proof. Suppose that $\sum_{i=1}^{s} a_i(m_i, n_i)$ is in $R[M \times N]$. Without loss of generality, we assume that $\{n_1, n_2, \ldots, n_s\} = \{n_1, n_2, \ldots, n_t\}$ with $n_i \neq n_j$ when $1 \leq i \neq j \leq t$. For any $1 \leq p \leq t$, denote $A_p = \{i \mid 1 \leq i \leq s,\ n_i = n_p\}$. Then $\sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p \in R[M][N]$. Note that $m_i \neq m_{i'}$ for any $i, i' \in A_p$ with $i \neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \rightarrow R[M][N]$ defined by

$$\sum_{i=1}^{s} a_i(m_i, n_i) \longrightarrow \sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p.$$

Suppose that $\left(\sum_{i=1}^{s} a_i(m_i, n_i)\right)\left(\sum_{j=1}^{s'} b_j({m'}_j, {n'}_j)\right) = 0$ in $R[M \times N]$. Then from the above isomorphism, it follows that

$$\left(\sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p\right)\left(\sum_{q=1}^{t'}\left(\sum_{j\in B_q} b_j m'_j\right) n'_q\right) = 0$$

in $R[M][N]$. Therefore by Proposition 2.6 we have

$$\left(\sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p\right) R[M][N] \left(\sum_{q=1}^{t'}\left(\sum_{j\in B_q} b_j m'_j\right) n'_q\right) = 0,$$

so R is strongly $M \times N$-semicommutative.

Let M_i, $i \in I$, be monoids. Denote $\coprod_{i\in I} M_i = \{(g_i)_{i\in I} \mid$ there exist only finite i's such that $g_i \neq$ $\neq e_i$, the identity of $M_i\}$. Then $\coprod_{i\in I} M_i$ is a monoid with the operation $(g_i)_{i\in I}({g'}_i)_{i\in I} = (g_i{g'}_i)_{i\in I}$.

***Corollary* 2.1.** *Let M_i, $i \in I$ be u.p.-monoids and R be a reduced ring. If R is strongly M_i-semicommutative for some $i_0 \in I$, then R is strongly $\coprod_{i\in I} M_i$-semicommutative.*

Proof. Let $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R\left[\coprod_{i\in I} M_i\right]$ such that $\alpha\beta = 0$. Then α, $\beta \in R[M_1 \times M_2 \times \ldots \times M_n]$ for some finite subset $\{M_1, M_2, \ldots, M_n\} \subseteq \{M_i \mid i \in I\}$. Thus $\alpha, \beta \in R[M_{i_0} \times M_1 \times \ldots \times M_n\}$. The ring R, by Proposition 2.7 and by induction, is strongly $M_{i_0} \times M_1 \times \ldots \times M_n$-semicommutative, so $\alpha R[M_{i_0} \times M_1 \times \ldots \times M_n]\beta = 0$. Hence R is strongly $\coprod_{i\in I} M_i$-semicommutative.

Let R be an algebra over a commutative ring S. The Dorroh extension of R by S is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + + s_2 r_1, s_1 s_2)$, where $r_i \in R$ and $s_i \in S$. Let R be a commutative ring, M be an R-module, and σ be an endomorphism of R. Rege and Chhawchharia [10] (Definition 1.3), give $R \oplus M$ a (possibly noncommutative) ring structure with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1) m_2 + r_2 m_1)$, where $r_i \in R$ and $m_i \in M$. We shall call this extension the skewtrivial extension of R by M and σ.

Proposition 2.8. (1) *Let R be an algebra over a commutative ring S, and D be the Dorroh extension of R by S. If R is strongly M-semicommutative and S is a domain, then D is strongly M-semicommutative.*

(2) *Let R be a commutative domain, and σ be an injective endomorphism of R. Then the skewtrivial extension of R by R and σ is strongly M-semicommutative.*

Proof. (1) Let $\alpha = (\alpha_1, \alpha_2) = \sum (r_i, s_i) g_i$, $\beta = (\beta_1, \beta_2) = \sum (s_j, n_j) h_j \in D[M]$ with $(\alpha_1, \alpha_2)(\beta_1, \beta_2) = 0$. Then $(\alpha_1\beta_1 + \alpha_2\beta_1 + \beta_2\alpha_1, \alpha_2\beta_2) = 0$, so we have $\alpha_1\beta_1 + \alpha_2\beta_1 + \beta_2\alpha_1 = 0$ and $\alpha_2\beta_2 = 0$. Since S is a domain, $\alpha_2 = 0$ or $\beta_2 = 0$. In the following computations we use freely the condition that R is strongly M-semicommutative. Say $\alpha_2 = 0$, then $0 = \alpha_1\beta_1 + \beta_2\alpha_1 = \alpha_1(\beta_1 + \beta_2)$ and since R is strongly M-semicommutative, we have $\alpha_1(\gamma_1 + \gamma_2)(\beta_1 + \beta_2) = 0$ such that $\gamma_1 + \gamma_2 \in \in R[M]$ and so $(\alpha_1\gamma_1\beta_1 + \alpha_2\gamma_1\beta_1 + \gamma_2\alpha_1\beta_1 + \alpha_2\gamma_2\beta_1 + \beta_2\alpha_1\gamma_1 + \beta_2\alpha_2\gamma_1 + \beta_2\gamma_2\alpha_1, \alpha_2\gamma_2\beta_2) = 0$. Also $\beta_2 = 0$, then $0 = \alpha_1\beta_1 + \alpha_2\beta_1 = (\alpha_1 + \alpha_2)\beta_1$ and so we have $(\alpha_1 + \alpha_2)(\gamma_1 + \gamma_2)\beta_1 = 0$ such that $\gamma_1 + \gamma_2 \in R[M]$ and so $(\alpha_1\gamma_1\beta_1 + \alpha_2\gamma_1\beta_1 + \gamma_2\alpha_1\beta_1 + \alpha_2 h_2\beta_1 + \beta_2\alpha_1\gamma_1 + \beta_2\alpha_2\gamma_1 + \beta_2\gamma_2\alpha_1, \alpha_2\gamma_2\beta_2) = = 0$. Therefore we obtain $(\alpha_1, \alpha_2)(\gamma_1, \gamma_2)(\beta_1, \beta_2) = 0$ for any $\gamma = (\gamma_1, \gamma_2) \in D[M]$, so in any case, proving that D is strongly M-semicommutative.

(2) Let N be the skewtrivial extension of R by R and σ. Set $(\alpha_1, \alpha_2)(\beta_1, \beta_2) = 0$ for $(\alpha_i, \beta_i) \in N$ with $i = 1, 2, 3$. Then $\alpha_1\beta_1 = 0$ and $\sigma(\alpha_1)\beta_2 + \beta_1\alpha_2 = 0$, so $\alpha_1 = 0$ and so $\beta_1 = 0$, since R is a domain. Say $\alpha_1 = 0$, then $0 = \sigma(\alpha_1)\beta_2 + \beta_1\alpha_2 = g_1\alpha_2$, therefore $\beta_1\gamma_1\alpha_2 = 0$ for any $y_1 \in N[M]$, since R is strongly semicommutative, and so $0 = (\alpha_1\gamma_1\beta_1, \beta_1\gamma_1\alpha_2) = (\alpha_1\gamma_1\beta_1, \sigma(\alpha_1)\sigma(\gamma_1)\beta_2 + + \sigma(\alpha_1)\beta_1\gamma_2 + \beta_1\gamma_1\alpha_2 = (\alpha_1, \alpha_2)(\gamma_1, \gamma_2)(\beta_1, \beta_2)$ for any $\gamma = (\gamma_1, \gamma_2) \in N[M]$. Say $\beta_1 = 0$, then $\sigma(\alpha_1)\beta_2 = 0$ and it follows that $\sigma(\alpha_1) = 0,$ or $\beta_2 = 0$, then $\alpha_1 = 0$ since σ is injective and R is a domain. Hence we have $(\alpha_1, \alpha_2)(\gamma_1, \gamma_2)(\beta_1, \beta_2) = 0$ in any case.

Now we will study some conditions under which polynomial rings may be strongly M-semicommutative. The Laurent polynomial ring with an indeterminate x over a ring R consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integer; we denote it $R[x; x^{-1}]$.

Proposition 2.9. (1) *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then R is strongly M-semicommutative if and only if so is $\Delta^{-1}R$.*

(2) *For a ring R, $R[x]$ is strongly M-semicommutative if and only if so is $R[x; x^{-1}]$.*

Proof. (1) Let $\alpha\beta = 0$ with $\alpha = \sum_{i=0}^{n} (u^{-1}a_i) g_i$, $\beta = \sum_{j=0}^{m} (v^{-1}b_j) h_j$, $u, v \in \Delta$ and $a, b \in R$. Since Δ is contained in the center of R, we have $0 = \alpha\beta = \sum_{i=0}^{n} (u^{-1}a_i) g_i \sum_{j=0}^{m} (v^{-1}b_j) h_j = = \sum_{s=i+j} (a_i b_j)(g_i h_j)(uv)^{-1}$, so

$$\sum_{i=0}^{n} a_i g_i \sum_{j=1}^{m} b_j h_j = 0.$$

But R is strongly M-semicommutative by the condition, and hence for any $\sum_{k=0}^{l} c_k p_k \in R[M]$ we have that

$$\sum_{i=0}^{n} a_i g_i \sum_{k=0}^{l} c_k p_k \sum_{j=0}^{m} b_j h_j = \sum_{i+j+k=t} (a_i c_k b_j)(g_i p_k h_j) = 0$$

for $t = 0, 1, \ldots, m+n+l$. Hence

$$\alpha\gamma\beta = \sum_{i=0}^{n} (u^{-1} a_i) g_i \sum_{k=0}^{l} (\omega^{-1} c_k) p_k \sum_{j=0}^{m} (v^{-1} b_j) h_j = \sum_{t=i+j+k} (a_i c_k b_j)(g_i p_k h_j)(u\omega v)^{-1} = 0$$

for any $\gamma = \sum_{k=0}^{l} (\omega^{-1} c_k) p_k \in \Delta^{-1} R[M]$. Hence $\Delta^{-1} R$ is strongly M-semicommutative.

(2) Let $\Delta = 1, x, x^2, \ldots$. Then clearly Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1} R[x]$, it follows that $R[x; x^{-1}]$ is strongly M-semicommutative by the result (1).

Given a ring R we denote the center of R by $Z(R)$, i.e.,

$$Z(R) = \{ s \in R \mid sr = rs \ \text{ for all } \ r \in R \}.$$

Proposition 2.10. *Let R be a ring and suppose that $Z(R)$ contains an infinite subring every nonzero element of which is regular in R. Then R is strongly M-semicommutative ring if and only if $R[x]$ is strongly M-semicommutative ring if and only if $R[x; x^{-1}]$ is strongly M-semicommutative ring.*

Proof. It suffices to prove that $R[x]$ is strongly M-semicommutative ring when so is R, by Lemma 2.5 and Proposition 2.9 (2). Since $Z(R)$ contains an infinite subring every nonzero element of which is regular in R by hypothesis, it follows that $R[x]$ is a subdirect product of infinite number of copies of R. Thus $R[x]$ is strongly M-semicommutative by Lemma 2.5 because R is strongly M-semicommutative ring by the assumption.

We study following proposition the connections between Armendariz rings and strongly M-semicommutative rings. Recall that reduced rings, M is u.p.-monoid are both M-Armendariz and strongly M-semicommutative rings Abelian. So it is natural to observe the relationships between them.

Proposition 2.11. *Let $R[M]$ be a Armendariz ring. Then the following statements are equivalent*:

(1) *R is a strongly M-semicommutative ring.*

(2) *$R[x]$ is a strongly M-semicommutative ring.*

(3) *$R[x, x^{-1}]$ is a strongly M-semicommutative ring.*

Proof. (1) $\Rightarrow$ (2). It is easy to see that there exists an isomorphism of $R[x][M] \longrightarrow R[M][x]$ via $\sum_i \left(\sum_p a_{ip} x^p\right) g_i \longrightarrow \sum_p \left(\sum_i a_{ip} g_i\right) x^p$. Let

$$\alpha = \sum_p \left(\sum_i a_{ip} g_i \right) x^p, \qquad \beta = \sum_q \left(\sum_j b_{jq} h_j \right) x^q$$

be polynomial in $R[M][x]$, such that $\alpha\beta = 0$, where $\alpha_i = \sum_p a_{ip} g_i$ and $\beta_j = \sum_q b_{jq} h_j \in R[M]$. Since $R[M]$ is Armendariz, so $R[M][x]$ is a Armendariz ring, therefore $\alpha_i \beta_j = 0$ for all i, j. Also R is strongly M-semicommutative by the hypothesis, therefore $\alpha_i \gamma_k \beta_j = 0$ for all i, j, k. Thus $\alpha R[M][x] \beta = 0$.

(2) $\Rightarrow$ (3). By the Proposition 2.9 (2) is trivial.

(3) $\Rightarrow$ (1). It is clear.

Proposition 2.12. *Let R be an M-Armendariz ring. If R is a semicommutative ring, then R is strongly M-semicommutative.*

Proof. Suppose that $\alpha = \sum_{i=0}^{m} a_i g_i, \beta = \sum_{j=0}^{n} b_j h_j \in R[M]$ satisfy $\alpha\beta = 0$. Since R is M-Armendariz, and hence $a_i b_j = 0$ for all i, j, also R is semicommutative, therefore $a_i c b_j = 0$ for any element c in R, for all i, j. Now it is easy to check that $\alpha\gamma\beta = 0$ for any $\gamma = \sum_{k=0}^{s} c_k l_k \in R[M]$.

Since reversible rings are semicommutative, the following corollary is clear.

Corollary 2.2. *Let R be an M-Armendariz ring. If R is a reversible ring, then R is a strongly M-Armendariz.*

Let R be a commutative ring and M an R-module. The R-module $R \oplus M$ acquires a ring structure where the product is defined by $(a, m)(b, n) = (ab, an + bm)$. We shall use the notation $R(+)M$ for this ring. If M is not zero, this ring is not reduced, since M can be identified with the ideal $0 \oplus M$ which has square zero. (It seems appropriate to call this ring as "R Nagata M".)

Let R be a ring and A an ideal of R. The factor ring $\overline{R} = R/A$ has the natural structure of a left R-, right R-bimodule. Denote $\overline{a} = a + A \in \overline{R}$ for each $a \in R$. We use this structure to define a ring structure on $R \oplus (R/A)$ as follows:

$$(r, \overline{a})(r', \overline{a'}) = (rr', \overline{ra' + ar'}).$$

We denote this ring by $R(+)R/A$. Its properties are similar to those of $R(+)M$.

Proposition 2.13. *Let R be a domain, A be an ideal of R. Suppose R/A is strongly M-semicommutative. Then $R(+)R/A$ is strongly M-semicommutative.*

Proof. Let α, β be elements of $\{R(+)R/A\}[M]$, where

$$\alpha = \sum_{i=0}^{m} (a_i, \overline{u_i}) g_i = (\alpha_0, \overline{\alpha_1})$$

and

$$\beta = \sum_{j=0}^{n} (b_j, \overline{v_j}) h_j = (\beta_0, \overline{\beta_1}).$$

If $\alpha\beta = 0$, we have $(\alpha_0, \overline{\alpha_1})(\beta_0, \overline{\beta_1}) = 0$. Thus we have the following equations:

$$\alpha_0 \beta_0 = 0, \tag{2.1}$$

$$\overline{\alpha_0 \beta_1 + \alpha_1 \beta_0} = 0. \tag{2.2}$$

Let $\alpha_0 = 0$. Then (2.2) becomes $\overline{\alpha_1 \beta_0} = 0$ over R/A. Since R/A is strongly M-semicommutative, it follows that $\overline{\alpha_1}\left(\dfrac{R}{A}\right)[M]\overline{\beta_0} = 0$. Also for any $\gamma_0 \in R[M]$ implies that $\overline{\alpha_1 \gamma_0 \beta_0} = 0$. We conclude that $0 = \left(\alpha_0\gamma_0\beta_0, \overline{\alpha_0\gamma_0\beta_1} + \overline{\alpha_0\gamma_1\beta_0} + \overline{\alpha_1\gamma_0\beta_0}\right) = (\alpha_0, \overline{\alpha_1})(\gamma_0, \overline{\gamma_1})(\beta_0, \overline{\beta_1})$. This case $\beta_0 = 0$ is similar.

Corollary 2.3. *Let R be a domain, A be an ideal of R. Suppose R/A is strongly semicommutative. Then $R(+)R/A$ is strongly semicommutative.*

It follows from Proposition 2.13 that if R is a domain then $R(+)R$ is strongly semicommutative. This result can be extended to reduced rings. The following properties of these rings will be used:

(1) If a, b are elements of a reduced ring, then $ab = 0$ if and only if $ba = 0$.

(2) Reduced rings are strongly semicommutative.

(3) If R is reduced, then so is the ring $R[x]$. We shall also identify $\{R(+)R\}[x]$ with the ring $R[x](+)R[x]$ in a natural manner. Therefore if R is a reduced ring, then the ring $R(+)R$ is strongly semicommutative.

Proposition 2.14. *Let M be u.p.-monoid and R be a reduced ring. Then the ring $R(+)R$ is strongly M-semicommutative.*

Proof. Let $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1)$ be elements of $\{R(+)R\}[M]$, we claim that $\alpha\{R(+)R\}[M]\beta = 0$. Write $\alpha = \sum_{i=0}^{m}(a_i, u_i)g_i = (\alpha_0, \alpha_1)$ and $\beta = \sum_{j=0}^{n}(b_j, v_j)h_j = (\beta_0, \beta_1)$, with corresponding representations for α_k, β_k (for $k = 0, 1$). Now we have

$$\alpha_0\beta_0 = 0, \tag{2.3}$$

$$\alpha_0\beta_1 + \alpha_1\beta_0 = 0. \tag{2.4}$$

By Lemma 2.4 $R[M]$ is reduced, (2.3) implies

$$\beta_0\alpha_0 = 0. \tag{2.5}$$

Multiplying equation (2.4) by β_0 on the left and using (2.5) we get $\beta_0\alpha_1\beta_0 = 0$. This implies that $(\alpha_1\beta_0)^2 = 0$ and so (since $R[M]$ is reduced)

$$\alpha_1\beta_0 = 0. \tag{2.6}$$

This implies (on account of (2.4))

$$\alpha_0\beta_1 = 0. \tag{2.7}$$

Now (2.3), (2.6) and (2.7) yield (since R is strongly M-semicommutative)

$$\alpha_0R[M]\beta_0 = 0, \qquad \alpha_1R[M]\beta_0 = 0, \qquad \text{and} \qquad \alpha_0R[M]\beta_1 = 0.$$

Therefore $(\alpha_0, \alpha_1)(\gamma_0, \gamma_1)(\beta_0, \beta_1) = (\alpha_0\gamma_0\beta_0, \alpha_0\gamma_0\beta_1 + \alpha_0\gamma_1\beta_0 + \alpha_1\gamma_0\beta_0) = 0$ for each (γ_0, γ_1) of $\{R(+)R\}[M]$.

The following theorem generalization of Proposition 2.14 has a similar proof.

Theorem 2.2. *Let M be u.p.-monoid, R be a reduced ring and A an ideal of R such that R/A is reduced. Then $R(+)R/A$ is strongly M-semicommutative.*

***Remark* 2.1.** Recall that a ring R is strongly regular [3] if for each element a in R, there exists an element b in R such that $a = a^2b$. A ring is strongly regular, if and only if it is (von Neumann) regular and reduced. If R is a strongly regular ring, then for each ideal A of R, R/A is strongly regular and reduced. On applying Theorem 2.2 we get the following result: If R is a strongly regular ring, then for each ideal A of R, then ring $R(+)R/A$ is strongly M-semicommutative.

The ring R is called Abelian if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$.

Recall that a ring R is a called right principally projective ring (or simples right p.p.-ring) if the right annihilator of an element of R is generated by an idempotent.

Lemma 2.10. *Let M be an monoid and R be strongly M-semicommutative. Then R is an Abelian ring. The converse holds if R is a right p.p.-ring.*

Proof. If e is an idempotent in R, then $e(1-e)=0$. Since R is strongly M-semicommutative, we have $e\alpha(1-e)=0$ for any $\alpha\in R[M]$ and so $e\alpha=e\alpha e$. On the other hand, $(1-e)e=0$ implies that $(1-e)\alpha e=0$, so we have $\alpha e=e\alpha e$. Therefore, $\alpha e=e\alpha$. For converse suppose now R is an Abelian and right p.p.-ring. Let $\alpha,\beta\in R[M]$ with $\alpha\beta=0$. Then $\alpha\in\mathrm{Ann}(\beta)=eR[M]$ for some $e^2=e\in R$ and so $\beta\alpha=0$ and $\alpha=e\alpha$. Since R is Abelian, we have $\alpha\gamma\beta=e\alpha\gamma\beta=\alpha\gamma\beta e=0$ for any $\gamma\in R[M]$, so, $\alpha R[M]\beta=0$. Therefore R is strongly M-semicommutative.

Before stating Example 2.2, we need the following lemmas.

Lemma 2.11 ([4], Lemma 1)**.** *Given a ring R we have the following assertion: R is an Abelian ring if and only if R is a reduced ring if and only if R is a semicommutative ring, when R is a right p.p.-ring.*

Lemma 2.12 ([4], Lemma 2)**.** *Let S be an Abelian ring and define*

$$\left\{\left.\begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n}\\ 0 & a & a_{23} & \dots & a_{2n}\\ 0 & 0 & a & \dots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a\end{pmatrix}\right|\, a,\ a_{ij}\in S\right\}=R_n$$

with n a positive integer ≥ 2. Then every idempotent in R_n is of the form

$$\begin{pmatrix} f & 0 & 0 & \dots & 0\\ 0 & f & 0 & \dots & 0\\ 0 & 0 & f & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & f\end{pmatrix}$$

with $f^2=f\in S$ and so R_n is Abelian.

Example **2.2.** Let S be Abelian ring and

$$R=\left\{\left.\begin{pmatrix} a & a_{12} & \dots & a_{1n}\\ 0 & a & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & a\end{pmatrix}\right|\, a,\ a_{ij}\in S\right\}.$$

Then R is Abelian by Lemma 2.12. Let M be a monoid with $|M|\geq 2$. Take $e,g\in M$ such that $e\neq g$. Consider

$$\alpha=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}e+\begin{pmatrix} 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}g\in R[M],$$

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} g \in R[M].$$

Then $\alpha\beta = 0$, but

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0,$$

so R is not strongly M-semicommutative. Assuming that R is a right p.p.-ring, then R is reduced by Lemma 2.11, a contradiction by the element

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in R. Thus, R is not a right p.p.-ring. In fact there can not be an idempotent $e \in R$ such that

$$\mathrm{Ann}_R \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = eR.$$

Proposition 2.15. *The direct limit of a direct system of strongly M-semicommutative rings is also strongly M-semicommutative.*

Proof. Let $A = \{R_i, \alpha_{ij}\}$ be a direct system of strongly M-semicommutative rings R_i for $i \in I$ and ring homomorphism $\alpha_{ij} \colon R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Let $R = \lim R_i$ be the direct limit of D with $l_i \colon R_i \to R$ and $l_j\alpha_{ij} = l_i$, we will prove that R is strongly M-semicommutative ring. Take $x, y \in R$, then $x = l_i(x_i)$, $y = l_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k$, $j \leq k$ define $x + y = l_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j))$ and $xy = l_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j))$, where $\alpha_{ik}(x_i)$, $\alpha_{jk}(y_j)$ are in R_k. Then R forms a rings with $0 = l_i(0)$ and $1 = l_i(1)$. Now suppose $\alpha\beta = 0$ for $\alpha = \sum_{s=1}^{m} a_s g_s$, $\beta = \sum_{t=1}^{n} b_t h_t$ in $R[M] - \{0\}$. There exist $i_s, j_t, k \in I$ such that $a_s = l_{i_s}(a_{i_s})$, $b_t = l_{j_t}(b_{j_t})$, $i_s \leq k$, $j_t \leq k$. So $a_s b_t = l_k(\alpha_{i_s k}(a_{i_s})\alpha_{j_t k}(b_{j_t}))$. Thus $\alpha\beta = \left(\sum_{s=1}^{m} l_k(\alpha_{i_s k}(a_{i_s}))g_s\right)\left(\sum_{t=0}^{n} l_k(\alpha_{j_t k}(b_{j_t})h_t\right) = 0$. But R_k is strongly M-semicommutative ring and so $l_k(\alpha_{i_s k}(a_{i_s})R_k[M]\alpha_{j_t k}(b_{j_t})) = 0$. Thus $\alpha R[M]\beta = 0$, and hence R is strongly M-semicommutative ring.

1. *Cohn P. M.* Reversible rings // Bull. London Math. Soc. – 1999. – **31**. – P. 641 – 648.
2. *Gang Y., Ruijuan D.* Rings over which polynomial rings are semicommutative // Vietnam J. Math. – 2009. – **37**, № 4. – P. 527 – 535.

3. *Hirano Y., Tominaga H.* Regular rings, V-rings and their generalization // Hiroshima Math. J. – 1979. – **9**. – P. 137 – 149.
4. *Huh G., Kim K., Lee Y.* P.p.-rings and generalized p.p.-ring // J. Pure and Appl. Algebra. – 2002. – **167**. – P. 37 – 52.
5. *Huh C., Lee Y., Smoktunowicz A.* Armendariz rings and semicommutative rings // Communs Algebra. – 2002. – **30**, № 2. – P. 751 – 761.
6. *Kim N. K., Lee Y.* Extensions of reversible rings // J. Pure and Appl. Algebra. – 2003. – **185**. – P. 207 – 223.
7. *Liu Z.* Armendariz rings relative to monoid // Communs Algebra. – 2005. – **33**. – P. 649 – 661.
8. *Nikmehr M. J.* The structure of ideals over a monoid with applications // World Appl. Sci. J. – 2012. – **20**, № 12. – P. 1636 – 1641.
9. *Nikmehr M. J., Fatahi F., Amraei H.* Nil-Armendariz rings with applications to a monoid // World Appl. Sci. J. – 2011. – **13**, № 12. – P. 2509 – 2514.
10. *Rege M. B., Chhawcharia S.* Armendariz rings // Proc. Jap. Acad. Ser. A. Math. Sci. – 1997. – **73**. – P. 14 – 17.
11. *Singh A. B., Juyal P., Khan M. R.* Strongly reversible ring to monoid // Int. J. Pure and Appl. Math. – 2010. – **63**, № 1. – P. 1 – 7.
12. *Tavallaee H. T., Nikmehr M. J., Pazoki M.* Weak α-skew Armendariz ideals // Ukr. Math. J. – 2012. – **64**, № 3. – P. 456 – 469.
13. *Yang G., Liu Z. K.* On strongly reversible rings // Taiwan. J. Math. – 2008. – **12**, № 1. – P. 129 – 136.

Received 08.11.12