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CHARACTERIZATION M_{11} AND $L_3(3)$ BY THEIR COMMUTING GRAPHS

ХАРАКТЕРИЗАЦІЯ M_{11} ТА $L_3(3)$ ЇХ КОМУТУЮЧИМИ ГРАФАМИ

For the groups M_{11} and $L_3(3)$, we show that their commuting graphs are unique.

Показано, що комутуючі графи груп M_{11} та $L_3(3)$ єдині.

1. Introduction. Throughout this article G is a finite group. One can associate a graph to G in many different ways (see, for example, [2, 9–11]). One of the graphs is the commuting graph associated to a finite group. For a finite group G, the *commuting graph* of G has $G - \{1\}$ as its vertex set and two distinct vertices x and y are joined by an edge if $[x, y] = 1$(x and y commute). In [10, 11] properties of the commuting graph for finite simple groups were used to prove the Margulis – Platonov conjecture for arithmetic groups.

Let X and Y be two graphs with vertex sets $V(X)$ and $V(Y)$, respectively. Then X is called *isomorphic* to Y if there is a bijection $f : V(X) \mapsto V(Y)$ such that any two vertices u and v of $V(X)$ are adjacent in X if and only if $f(u)$ and $f(v)$ are adjacent in Y. The bijection f is called a *graph isomorphism*.

In this short note we show that the commuting graphs of the groups M_{11} and $L_3(3)$ are unique. We note that our proofs do not require the classification of the finite simple groups.

Theorem 1.1. *Let G be a finite group isomorphic to M_{11} or $L_3(3)$ and H be a finite group such that $X(G) \cong X(H)$. Then $G \cong H$.*

Our strategy for identifying the groups M_{11} and $L_3(3)$ is to determine the structures of the centralizers of involutions. By assumptions and notations in Theorem 1.1 we will show that H is simple and for each involution $h \in H$ we have $C_H(h) \cong GL_2(3)$. Then the main result will follow from the following theorem.

Theorem 1.2 ([3], XII, Theorem 5.2)**.** *Let G be a finite simple group and $t \in G$ be an involution in the center of a Sylow 2-subgroup of G. If $C_G(t) \cong GL_2(3)$, then either $G \cong M_{11}$ or $G \cong L_3(3)$.*

For a finite group G, $O(G)$ is the largest normal subgroup of G of odd order and $M(G)$ is the schur multiplier of G. We have used the Atlas [4] notations for simple groups. The other notations follow [1] and [8].

For a vertex $x \in V(X)$, $d(x)$ is the number of vertices adjointed with x. In this paper all graphs are simple and without loop. We have used [5] for other notations in graph theory.

2. Preliminaries. In this section we recall some known theorems in finite groups.

Theorem 2.1 [6]**.** *Let G be a finite group which contains a self-centralizing subgroup of order three. Then one of the following statements is true*:

i) *G contains a nilpotent normal subgroup N such that G/N is isomorphic to either Z_3 or S_3*;

ii) *G contains a normal* 2*-subgroup N and $G/N \cong A_5$;*

iii) $G \cong L_3(2) \cong L_2(7)$.

Theorem 2.2 (Glauberman Z*-theorem [7])**.** *Let G be a finite group and $1 \neq t \in G$ be an involution. If $G \neq O(G)C_G(t)$, then t is conjugate in G to an involution in $C_G(t) \setminus \langle t \rangle$.*

Lemma 2.1. *Let G be a finite group. If a Sylow* 2*-subgroup of G is cyclic or is isomorphic to $Z_2 \times D_8$ or Q_8, then G is not a non-Abelian simple group.*

Proof. Let G be a finite group and $T \in \mathrm{Syl}_2(G)$. Let T be isomorphic to Q_8 or T be cyclic. Then there is a unique involution $t \in T$. So by Theorem 2.2, we get that G is not a non-Abelian simple group. Now let T be isomorphic to $Z_2 \times D_8$. Let $\langle t \rangle = T'$ and $Z(T) = \langle s, t \rangle$. Then s is not conjugate to t in G and so $\mathrm{Aut}(T)$ is a 2-group. Since $T \in \mathrm{Syl}_2(G)$, we have $N_G(T) = T$ and then by [8] (Theorem 7.1.1) we get that no two involutions in $Z(T)$ are conjugate in G. Let N be a subgroup of T isomorphic to D_8, then $t \in Z(N)$ and N is a maximal subgroup of T. If t is conjugate to an involution of $N \setminus \{t\}$ in G, then s is not conjugate to an involution of N in G. Then by Thompson transfer lemma ([3], XII.8.2) we get that G is not simple. If s is conjugate to an involution of N in G, then t is not conjugate to an involution of $T \setminus \{t\}$ in G. Then by Theorem 2.2 we get that G is not simple. If neither s nor t is conjugate to an involution in $N \setminus \{t\}$, then by Theorem 2.2 or by Thompson transfer lemma we get that G is not simple and the lemma is proved.

3. Proof of Theorem 1.1. In this section we shall prove Theorem 1.1. We recall that for a finite group G, $X(G)$ is the commuting graph of G.

Notations. In this section G is a finite group isomorphic to M_{11} or $L_3(3)$ and H is a finite group such that $X(G) \cong X(H)$. Let $\phi : X(G) \to X(H)$ be an isomorphism and $g \in V(X(G))$ be an involution. Set $h = \phi(g)$.

Lemma 3.1. *For each involution $x \in H$ we have that $C_H(x)$ is isomorphic to either $GL_2(3)$ or $Z_2 \times S_4$.*

Proof. Since $X(G) \cong X(H)$, we have $|G| = |H|$ and $X(C_G(g)) \cong X(C_H(h))$. By [4, p. 13, 18] we have that $C_G(g) \cong GL_2(3)$. As $X(C_G(g)) \cong X(C_H(h))$ and there is a vertex of degree 4 in $X(C_G(g))$, there is a vertex of degree 4 in $X(C_H(h))$ say r. Since $d(r) = 4$, we get that $|C_H(r, h)| = 6$ and so $C_H(r, h) = \langle r \rangle \times \langle h \rangle$. Hence r is of order 3 and h is of order 2. Therefore a Sylow 3-subgroup S of $C_H(h)$ is of order three and $C_{C_H(h)}(S) = S \times \langle h \rangle$. It gives us that a Sylow 3-subgroup $\overline{S}$ of $C_H(h)/\langle h \rangle$ is of order three and $\overline{S}$ is self-centralizing. Now by Theorem 2.1 and as $|C_H(h)| = |C_H(g)| = 2^4 \cdot 3$ we get that $C_H(h)/\langle h \rangle \cong S_4$. Therefore $C_H(h) \cong GL_2(3)$ or $C_H(h)$ is isomorphic to $Z_2 \times S_4$ and the lemma is proved.

***Proof of Theorem* 1.1.** By Lemma 3.1, $C_H(h)$ is isomorphic to $GL_2(3)$ or $Z_2 \times S_4$. Assume that H is a simple group. Then by Lemma 2.1 we get that $C_H(h) \cong Z_2 \times S_4$ does not happen and therefore $C_H(h) \cong GL_2(3)$. This and Theorem 1.2 give us that $H \cong M_{11}$ or $L_3(3)$. Since $|H| = |G|$, we get that $H \cong G$ and theorem is proved. Hence it is enough to show that H is simple. We assume that H is not simple and N is a minimal normal subgroup of H. Let $\langle h, f \rangle$ be an elementary Abelian 2-group of order 4 in H. Then by coprime action we have

$$O(H) = \left\langle C_H(x) \cap O(H); x \in \langle h, f \rangle^\sharp \right\rangle.$$

Since for each involution $x \in H$, either $C_H(x) \cong GL_2(3)$ or $C_H(x) \cong Z_2 \times S_4$, we deduce that $O(H) = 1$. This gives us that 2 divides the order of N. First assume that $C_H(x) \cong GL_2(3)$ for each involution $x \in H$. Then by Lemma 2.1 and as $C_H(h) \cap N$ is normal in $C_H(h)$, we get that $C_H(h) \leq N$. Now by Theorem 1.2, $N \cong M_{11}$ or $L_3(3)$. By [4, p. 13, 18], $\mathrm{Out}(M_{11}) = 1$ and $|\mathrm{Out}(L_3(2))| = 2$. As $|N|_2 = |H|_2$, we get that $N = H$ and the theorem is proved in this case.

Now assume that $C_H(x) \cong Z_2 \times S_4$ for each involution $x \in H$. Then by Lemma 2.1 and as $C_H(h) \cap N$ is normal in $C_H(h)$, we get that a Sylow 2-subgroup of N is elementary Abelian of order 8, N is simple and for each involution $t \in N$ we have that $C_N(t)$ is elementary Abelian of order 8. Now by ([8], Theorem 16.6.1) we get that $N \cong J_1$ or a group of Ree type. In a group of Ree type the centralizer of an involution is isomorphic to $Z_2 \times L_2(q)$ for $q \geq 3$. Therefore N is not a group of Ree type. Assume that $N \cong J_1$, then by [4, p. 36] we get that the centralizer of each involution in N is isomorphic to $Z_2 \times A_5$, therefore N is not isomorphic to J_1 and hence this case does not happen. Now the theorem is proved.

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