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CHARACTERIZATION OF $\mathbb{A}_{16}$ BY NON-COMMUTING GRAPH

ХАРАКТЕРИЗАЦІЯ $\mathbb{A}_{16}$ НЕПЕРЕСТАВНИМ ГРАФОМ

Let G be a finite non-Abelian group. We define a graph Γ_G, called the non-commuting graph of G, with vertex set $G - Z(G)$ such that two vertices x and y are adjacent if and only if $xy \neq yx$. A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture, the AAM's Conjecture as follows: If S is a finite non-Abelian simple group and G is a group such that $\Gamma_S \cong \Gamma_G$, then $S \cong G$. It is still unknown if this conjecture holds for all simple finite groups with connected prime graph except $\mathbb{A}_{10}$, $L_4(8)$, $L_4(4)$ and $U_4(4)$. In this paper we prove that if $\mathbb{A}_{16}$ denotes the alternating group of degree 16, then for any finite group G, the graph isomorphism $\Gamma_{\mathbb{A}_{16}} \cong \Gamma_G$ implies $\mathbb{A}_{16} \cong G$.

Нехай G — скінченна неабелівська група. Граф Γ_G, який називається непереставним графом групи G, визначено за допомогою множини вершин $G - Z(G)$ таких, що дві вершини x та y є суміжними тоді і тільки тоді, коли $xy \neq yx$. А. Абдоллахі, С. Акбарі та Г. Р. Маймані висунули наступну гіпотезу — ААМ гіпотезу: якщо S є скінченною неабелевою простою групою і G є групою такою, що $\Gamma_S \cong \Gamma_G$, то $S \cong G$. Досі залишається невідомим, чи справджується ця гіпотеза для всіх простих скінченних груп зі зв'язними простими графами, окрім $\mathbb{A}_{10}$, $L_4(8)$, $L_4(4)$ та $U_4(4)$. У статті доведено, що якщо $\mathbb{A}_{16}$ позначає знакозмінну групу степеня 16, то для будь-якої скінченної групи G з ізоморфізму графів $\Gamma_{\mathbb{A}_{16}} \cong \Gamma_G$ випливає $\mathbb{A}_{16} \cong G$.

1. Introduction. The study of relation between groups and graphs is one of the main research topic in group theory. There are several ways to associate a graph to a group G. The graph we will consider in this paper is denoted by Γ_G and is called the non-commuting graph of G. The vertex set of Γ_G is $V(\Gamma_G) = G - Z(G)$ where $Z(G)$ is the center of G and two distinct vertices x and y are joined whenever $xy \neq yx$. It is clear that if G is abelian, then Γ_G is the null graph. Hence in what follows we will assume that G is a non-Abelian group. Another graph associated to a finite group G is the prime graph $GK(G)$ introduced by Gruenberg – Kegel. The vertex set of $GK(G)$ is $\pi(G)$, the set of all the prime divisors of the order of G. Two distinct primes p and q are adjacent if and only if G contains an element of order pq.

For a graph X, we denote the set of vertices and edges of X by $V(X)$ and $E(X)$ respectively. Two graphs X and Y are isomorphic and we denote it by $X \cong Y$, if there exists a bijective map $\phi\colon V(X) \longrightarrow V(Y)$ such that if x and y are adjacent in X, then $\phi(x)$ and $\phi(y)$ are adjacent in Y and vice-versa. For a group G, we denote by $k(G)$ the number of conjugacy classes of G and $N(G) = \{n \in \mathbb{N} | G$ has a conjugacy class C such that $|C| = n\}$. Also $\mathrm{Cl}_G(g)$ denotes the conjugacy class containing $g \in G$.

In [1] relation between some graph theoretical properties of Γ_G and the group theory properties of the group G are studied. In particular the following two conjectures are raised.

Conjecture 1. *Let G be a finite non-Abelian group. If there is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Conjecture 2. *Let S be a finite non-Abelian simple group. If G is a group such that $\Gamma_G \cong \Gamma_S$, then $G \cong S$.*

There are many articles dealing with the characterization of simple groups by its non-commuting graph. In [3], M. R. Darafsheh proved Conjecture 1 for any simple group G. Also if $GK(G)$ is a non-connected graph, Conjecture 2 is verified for many simple groups. In [6], A. Iranmanesh and J. Jafarzadeh verified Conjecture 2 when G and S are both simple groups. In [10], L. Wang and W. Shi verified Conjecture 2 for $S \cong L_2(q)$. But if $GK(G)$ is a connected graph the structure theorem for the group G does not work in the general case and the problem of characterizing the group G via its non-commuting graph becomes difficult. In the case that $GK(G)$ is a connected graph, the following partial results have been obtained so far. In [11], L. Wang and W. Shi verified Conjecture 2 For $S \cong \mathbb{A}_{10}$. In [12], L. Zhang and W. Shi proved that Conjecture 2 is true for $L_4(8)$. In [4], Conjecture 2 is verified for the groups $L_4(4)$ and $U_4(4)$. The groups $\mathbb{A}_{10}$, $L_4(8)$, $L_4(4)$ and $U_4(4)$ have connected prime graphs. Our aim in this paper is to verify the above Conjecture for the alternating group of degree 16, $\mathbb{A}_{16}$, that has connected prime graph. In fact, we will prove the following theorem.

Theorem 1. *Let G be a finite group such that $\Gamma_G \cong \Gamma_{\mathbb{A}_{16}}$, then $G \cong \mathbb{A}_{16}$.*

2. Preliminaries. In this section we list some basic and known results which will be used in proving Theorem 1.

Lemma 1 [7, p. 98]. *If $|G| = pqr$, where p, q and r are distinct primes, then G is not simple.*

Lemma 2 (Lemma 3.27 of [1]). *If G is a finite group, then $2|E(\Gamma_G)| = |G|(|G| - k(G))$.*

Theorem 2 (P. Hall [8, p. 108]). *If G is a solvable group of order mn where $(m, n) = 1$, then G contains a subgroup of order m. Moreover any two subgroup of order m are conjugate.*

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ which are pairwise nonadjacent in $GK(G)$. Also we denote by $t(2, G)$ the maximal number of vertices containing 2 but pairwise nonadjacent in $GK(G)$. $t(G)$ is called the independence number of $GK(G)$ and $t(2, G)$ is called the 2-independence number of the graph $GK(G)$.

Theorem 3 [9]. *Let G be a finite group satisfying the two conditions*:

(a) *there exist three primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$, i.e., $t(G) \geq 3$;*

(b) *there exist an odd prime in $\pi(G)$ nonadjacent to 2 in $GK(G)$, i.e., $t(2, G) \geq 2$.*

Then there is a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut}(S)$ for the maximal normal solvable subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$ and one of the following statements holds:

(1) *$S \cong \mathbb{A}_7$ or $PSL(2, q)$ for some odd q, and $t(S) = t(2, S) = 3$;*

(2) *for every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

3. Characterization of $\mathbb{A}_{16}$ by non-commuting graph. We know that $|\mathbb{A}_{16}| = 10461394944000 = 2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and by [5] it has 123 conjugacy classes. For proving Theorem 1, we need the size of conjugacy classes, centralizer orders, order of elements and the number of conjugacy classes in $\mathbb{A}_{16}$. We omit the details of these items and refer to [5] for some of them in the following.

Lemma 3. *Let G be a finite group such that $\Gamma_G \cong \Gamma_{\mathbb{A}_{16}}$, then*

(1) $|G| = |\mathbb{A}_{16}|$,

(2) $k(G) = k(\mathbb{A}_{16})$,

(3) *if* $\phi\colon \Gamma_G \longrightarrow \Gamma_{\mathbb{A}_{16}}$ *is a graph isomorphism, then* $|C_G(g)| = |C_{\mathbb{A}_{16}}(\phi(g))|$ *and* $|\operatorname{Cl}_G(g)| = |\operatorname{Cl}_{\mathbb{A}_{16}}(\phi(g))|$ *for all* $g \in G - Z(G)$. *In particular* $N(G) = N(\mathbb{A}_{16})$.

Proof. By [3] we have $|G| = |\mathbb{A}_{16}|$ and proof of (1) is immediate. Also from Lemma 2 we know that $2|E(\Gamma_G)| = |G|(|G| - k(G))$ and by $\Gamma_G \cong \Gamma_{\mathbb{A}_{16}}$ and (1) we obtain (2). It is clear that $\deg(g) = |G| - |C_G(g)|$ for every $g \in G - Z(G)$ where $\deg(g)$ denotes the degree of g in the graph Γ_G. From the graph isomorphism, we have $\deg(g) = \deg(\phi(g))$ and so $|G| - |C_G(g)| = |\mathbb{A}_{16}| - |C_{\mathbb{A}_{16}}(\phi(g))|$. Hence from (1) we have $|C_G(g)| = |C_{\mathbb{A}_{16}}(\phi(g))|$ and $|\operatorname{Cl}_G(g)| = |\operatorname{Cl}_{\mathbb{A}_{16}}(\phi(g))|$ where $\operatorname{Cl}_G(g)$ denotes the conjugacy class containing g. Finally from the equality of sizes of conjugacy classes, we obtain $N(G) = N(\mathbb{A}_{16})$.

Lemma 3 is proved.

In $GK(\mathbb{A}_{16})$, from the set of orders of elements, we know that 2, 3, 5 and 7 are adjacent to each other, 11 is adjacent to 2, 3 and 5 and finally 13 is adjacent to 3. In the following we draw $GK(\mathbb{A}_{16})$ (Fig. 1).

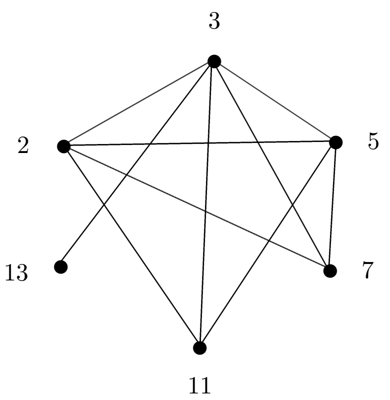

Fig. 1

In the next lemma, we show that if the non-commuting graphs of G and $\mathbb{A}_{16}$ are isomorphic, then their prime graphs are equal.

Lemma 4. *Let* G *be a finite group such that* $\Gamma_G \cong \Gamma_{\mathbb{A}_{16}}$, *then* $GK(G) = GK(\mathbb{A}_{16})$.

Proof. By Lemma 3, G and $\mathbb{A}_{16}$ have the same set of centralizer orders and $\pi(G) = \pi(\mathbb{A}_{16}) = \{2, 3, 5, 7, 11, 13\}$. By [5] we know that $\mathbb{A}_{16}$ has a centralizer order $|C_{\mathbb{A}_{16}}(a)| = 2^4 \cdot 3^3$ for $a \in \mathbb{A}_{16}$. So G has a centralizer order $|C_G(g)| = 2^4 \cdot 3^3$ where $\phi(g) = a$ in the graph isomorphism $\phi\colon \Gamma_G \to \Gamma_{\mathbb{A}_{16}}$. If $o(g) = 2^\alpha$, $1 \le \alpha \le 4$, then g commutes with an element of order 3 and so G has an element of order 6. If $o(g) = 2^\alpha \cdot 3^\beta$, $1 \le \alpha \le 4$ and $1 \le \beta \le 3$, then G has an element of order 6. If $o(g) = 3^\beta$, $1 \le \beta \le 3$, then g commutes with an element of order 2 and so G has an element of order 6. Thus in any case G has an element of order 6 and 2 is adjacent to 3 in $GK(G)$. By [5] and using similar argument we obtain G has elements g_1, g_2, g_3, g_4, g_5, g_6, g_7 and g_8 with $|C_G(g_1)| = 3\cdot 5$, $|C_G(g_2)| = 2\cdot 7$, $|C_G(g_3)| = 3\cdot 13$, $C_G(g_4) = 3\cdot 11$, $|C_G(g_5)| = 5 \cdot 11$, $|C_G(g_6)| = 2^2 \cdot 11$, $|C_G(g_7)| = 2^3 \cdot 5$ and $|C_G(g_8)| = 3^2 \cdot 7\cdot$ So $2, 3$ and 5 are adjacent to each other, 7 is adjacent to 2 and 3, 11 is adjacent to $2, 3$, 5 and 13 is adjacent to 3. Next we show that 5 is adjacent to 7. From centralizer orders of $\mathbb{A}_{16}$

and by Lemma 3, we deduce that G has a centralizer order $|C_G(g)| = 3 \cdot 5 \cdot 7$. Now by Lemma 1 and Theorem 2, G has a subgroup K with $|K| = 5 \cdot 7 = 35$. But every group of order 35 is Abelian (and cyclic). So G has an element h with $o(h) = 35$. Thus 5 is adjacent to 7 in $GK(G)$.

Next we prove that 2 is not adjacent to 13. If 2 is adjacent to 13, then G has an element g with $o(g) = 26$. It implies that $26 \big| |C_G(g)|$. From Lemma 3 and centralizer orders of $\mathbb{A}_{16}$ taken from [5] we obtain $|C_G(g)| = 2^9 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Now by $o(g) = 26$ and $|C_G(g)|$, G has elements of order 2, 3, 5, 7, and 11 such that both 2 and 13 divide centralizer order of these elements. So G has at least five conjugacy classes of elements with different orders where both 2 and 13 divide the centralizer order of each element. But by Lemma 3 and [5], G has only one conjugacy class of element with centralizer order divisible by 26, a contradiction. Similarly 13 is not adjacent to $5, 7$, and 11. We will prove that 7 is not adjacent to 11 in $GK(G)$. If this happens, then G has an element h with $o(h) = 77$. So $77 \big| |C_G(h)|$ and from Lemma 3 and centralizer orders of $\mathbb{A}_{16}$, we have $|C_G(h)| = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$, $2^9 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ or $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. In these cases G has elements of order 2, 3, 5, 7 and 11 such that 77 divides centralizer order of these elements. But G has only three centralizer orders divisible by 77. This is a contradiction and 7 is not adjacent to 11 in $GK(G)$. Therefore $GK(G) = GK(\mathbb{A}_{16})$ and the proof is completed.

Lemma 4 is proved.

Now by Theorem 3, there is a non-Abelian simple group S such that $S \leq G/K \leq \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G. Furthermore, $t(S) \geq \geq t(G) - 1$ and one of the following statements holds:

(1) $S \cong \mathbb{A}_7$ or $PSL(2, q)$ for some odd q, and $t(S) = t(2, S) = 3$,

(2) for every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.

In the next lemma we will prove that condition (1) of Theorem 3 does not hold. So condition (2) holds.

Lemma 5. *The non-Abelian simple group S in Theorem 5 has a Sylow 13-subgroup of order 13.*

Proof. We show that conclusion (1) of Theorem 5 does not hold. If $S \cong \mathbb{A}_7$, then $|S| = 7!/2$ and $|\mathrm{Aut}(S)| = 7!$. So $7!/2 \leq |G/K| \leq 7!$ and since $|G| = 16!/2$ we obtain $4151347200 \leq |K| \leq 8302694400$. So $|K| = 4151347200$ or 8302694400 and in any case $11 \cdot 13 \big| |K|$. Since K is solvable, by Theorem 2 K has a subgroup K_1 of order $11 \cdot 13$. By Sylow's theorems, every group of order $11 \cdot 13$ is cyclic. Hence G has an element of order 143 that implies 11 is adjacent to 13 in $GK(G)$ which is a contradiction by Fig. 1. Therefore $S \ncong \mathbb{A}_7$.

If $S \cong PSL(2, q)$ for some odd q, then by $S \leq G/K$, $|PSL(2, q)| = \dfrac{1}{(n, q-1)} q(q^2 - - 1)$ and comparing q in $|G|$ and $|PSL(2, q)|$, the following cases are raised: $S \cong \cong PSL(2, 3^n)$, $2 \leq n \leq 6$, $PSL(2, 5)$, $PSL(2, 5^2)$, $PSL(2, 5^3)$, $PSL(2, 7)$, $PSL(2, 7^2)$, $PSL(2, 11)$ or $PSL(2, 13)$.

If $S \cong PSL(2, 3^2)$, then $|S| = 2^4 \cdot 3^2 \cdot 5$ and $|\mathrm{Aut}(S)| = 2^5 \cdot 3^2 \cdot 5$. So $2^4 \cdot 3^2 \cdot 5 \leq \leq |G/K| \leq 2^5 \cdot 3^2 \cdot 5$. From $|G|$ we deduce that $2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \leq |K| \leq \leq 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$. Since $|K| \big| |G|$ we obtain $|K| = 2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ or $|K| = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$. In any case by Theorem 2, K has a subgroup of order

$11 \cdot 13$. Every group of order $11 \cdot 13$ is Abelian (and cyclic) which implies that 11 is adjacent to 13 in $GK(G)$ and this contradicts Lemma 4. If $S \cong PSL(2, 3^3)$, then by similar argument we obtain 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. If $S \cong PSL(2, 3^4)$, then $|S| = 265680 = 2^4 \cdot 3^4 \cdot 5 \cdot 41$. By Theorem 3 we know that $|S|\big||G|$. So $41\big|G|$ and this contradicts $41 \nmid |G|$. If $S \cong PSL(2, 3^5)$, then $61\big||S|$ and this contradicts $61 \nmid |G|$. If $S \cong PSL(2, 3^6)$, then $73\big||S|$ and this contradicts $73 \nmid |G|$. If $S \cong PSL(2, 5)$, then similar to the case $S \cong PSL(2, 3^2)$, we deduce that G has a subgroup of order $11 \cdot 13$. Thus 11 is adjacent to 13 in $GK(G)$ and this contradicts $GK(G) = GK(\mathbb{A}_{16})$. If $S \cong PSL(2, 5^2)$, then $|S| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^5 \cdot 3 \cdot 5^2 \cdot 13$. So $2^9 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11 \leq |K| \leq 2^{11} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11$ and possibilities for $|K|$ are $|K| = 2^9 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11$, $2^{10} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11$, $2^9 \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 11$ or $2^{11} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11$. In any case by Theorem 2, K has a subgroup K_1 with $|K_1| = 11 \cdot 7^2$. If P is an 11-Sylow subgroup of K_1, then by Sylow's theorms P is normal in K_1. If $t \in K_1$ with $o(t) = 7$, then $P\langle t\rangle$ is a subgroup of K_1 of order 77. Every group of order 77 is Abelian (and cyclic) implying that 7 is adjacent to 11 in $GK(G)$ and this contradicts $GK(G) = GK(\mathbb{A}_{16})$. If $S \cong PSL(2, 5^3)$, then $31\big||S|$ and this contradicts $31 \nmid |G|$. If $S \cong PSL(2, 7)$ or $PSL(2, 7^2)$, then similar to the case $S \cong PSL(2, 3^2)$ we have 11 is adjacent to 13 in $GK(G)$ which is a contradiction. If $S \cong PSL(2, 11)$, then similar to the case $S \cong PSL(2, 3^3)$, we obtain 7 is adjacent to 13 in $GK(G)$, a contradiction. If $S \cong PSL(2, 13)$, then similarly we show that 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. Therefore conclusion (1) of Theorem 3 does not hold. So conclusion (2) of Theorem 3 holds and S has a 13-Sylow subgroup of order 13.

Lemma 5 is proved.

3.1. Proof of the main theorem. Now by [2] we consider each of the finite non-Abelian simple groups as a candidate for S.

(1) $S \cong \mathbb{A}_n$ for $n \geq 5$. By Lemma 5 we know that $13\big||S|$, so $n \geq 13$. Therefore $S \cong \mathbb{A}_{13}$, $\mathbb{A}_{14}$, $\mathbb{A}_{15}$ or $\mathbb{A}_{16}$. If $S \cong \mathbb{A}_{13}$, then by Theorem 3 we obtain $13!/2 \leq G/K \leq 13!$. Thus $3360 \leq |K| \leq 6720$ and $|K| = 3360 = 2^5 \cdot 3 \cdot 5 \cdot 7$ or $6720 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot$ On the other hand K is normal in G, so in any case K contains four distinct conjugacy classes of elements with orders 2, 3, 5 and 7. But the four smallest orders for conjugacy classes of G are 1120, 5460, 104832, 320320. Therefore K can not contain four conjugacy classes which is a contradiction. The cases $S \cong \mathbb{A}_{14}$ or $\mathbb{A}_{15}$ similarly lead to contradiction. If $S \cong \mathbb{A}_{16}$, then from $S \leq G/K$ we obtain $|S| = |G|$ and $|K| = 1$. So $S = G$ and in this case $S = G \cong \mathbb{A}_{16}$. Thus if $S \cong \mathbb{A}_n$ for $n \geq 5$, then $S \cong \mathbb{A}_{16}$ and in this case $S = G \cong \mathbb{A}_{16}$.

(2) If S is isomorphic to one of the sporadic simple groups, then by Lemma 5 we have $13\big||S|$. Also the possible prime divisors of $|S|$ are 2, 3, 5, 7, 11 or 13, hence we obtain $S \cong Suz$ or Fi_{22}. If $S \cong Suz$, then $|S| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. So $3^7\big||S|$ implying that $3^7\big||G|$ and this is a contradiction. If $S \cong Fi_{22}$, then $|S| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. So $2^{17}\big||S|$ implying that $2^{17}\big||G|$, a contradiction. Therefore S is not one of the sporadic simple groups.

(3) S is one of the classical groups $PSL(n, q)$ for $n \in \mathbb{N}$ and prime power q. Since $|PSL(n, q)| = \dfrac{1}{(n, q-1)} q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1)$ and $S \leq G/K$ and $|G| = 2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, q must be a power of 2, 3, 5, 7, 11 or 13. In Lemma 5 we showed that S is not one of the classical groups $PSL(2, q)$ for odd q. So S may

be isomorphic to one of the following cases: $PSL(2,2^n)$, $2 \le n \le 14$, $PSL(3,2^n)$, $1 \le n \le 4$, $PSL(3,3)$, $PSL(3,3^2)$, $PSL(3,5)$, $PSL(4,2)$, $PSL(4,2^2)$ or $PSL(5,2)$. If $S \cong PSL(2,2^n)$, $2 \le n \le 14$ and $n \ne 6,12$, $PSL(3,2)$, $PSL(3,2^2)$, $PSL(3,2^3)$, $PSL(3,5)$, $PSL(4,2)$, $PSL(4,4)$ or $PSL(5,2)$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong PSL(2,2^6)$, $PSL(3,3)$ or $PSL(3,3^2)$ then similar to the Lemma 5, the case $S \cong PSL(2,3^2)$, we deduce that 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. If $S \cong PSL(2,2^{12})$ or $PSL(3,2^4)$, then $17\big||S|$ and this contradicts $17 \nmid |G|$. So S is not one of the classical groups $PSL(n,q)$.

(4) S is one of the classical groups $PSU(n,q^2)$ for $n \in \mathbb{N}$ and prime power q. Similar to the case (3) we obtain that $S \cong PSU(2,2^{2n})$, $2 \le n \le 7$, $PSU(2,3^2)$, $PSU(2,3^4)$, $PSU(2,3^6)$, $PSU(2,5^2)$, $PSU(2,7^2)$, $PSU(3,2^2)$, $PSU(3,2^4)$ or $PSU(3,3^2)$. If $S \cong PSU(2,2^4)$, $PSU(2,2^8)$, $PSU(2,2^{10})$, $PSU(2,2^{14})$, $PSU(2,3^2)$, $PSU(2,3^4)$ or $PSU(2,7^2)$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong$ $\cong PSU(2,2^6)$, then $|S| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$. So $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \le |G/K| \le 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ and by $|G|$ we have $2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \le$ $\le |K| \le 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. Then the possibilities for $|K|$ are $2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$, $2^8 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$, $2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11$, $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ or $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. By Theorem 2, in any case K has a subgroup K_1 of order 77. But every group of order 77 is Abelian (and cyclic) which means that 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. Similar to this argument S can not be isomorphic to either $PSU(2,5^2)$ or $PSU(3,2^2)$. If $S \cong PSU(2,2^{12})$ or $PSU(3,2^4)$, then $17\big||S|$ and this contradicts $17 \nmid |G|$. If $S \cong PSU(2,3^6)$, then $73\big||S|$ and this contradicts $73 \nmid |G|$. Therefore S is not one of the classical groups $PSU(n,q^2)$.

(5) S is one of the classical groups $PSP(2l,q)$ or $P\Omega(2l+1,q)$. At first we assume that $S \cong PSP(2l,q)$. From $PSP(2,q) \cong PSL(2,q)$ and proof of Lemma 5 and case (3), we obtain $S \ncong PSP(2,q)$. Using the fact that $|S|\big||G|$ and $|PSP(2l,q)| =$ $= \dfrac{1}{(2,q-1)} q^{l^2} \prod_{i=1}^{l} (q^{2i}-1)$ we obtain $S \cong PSP(4,2)$, $PSP(4,2^2)$, $PSP(4,2^3)$, $PSP(4,3)$ or $PSP(6,2)$. If $S \cong PSP(4,2)$, $PSP(4,\ 2^2)$, $PSP(4,3)$ or $PSP(6,2)$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong PSP(4,2^3)$, then $|S| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot$ $\cdot 13$ and $|\operatorname{Aut}(S)| = 2^{13} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 13$ and so $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13 \le |G/K| \le 2^{13} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 13$. Thus the possibilities for $|K|$ are $2 \cdot 3 \cdot 5^2 \cdot 11$, $2^2 \cdot 3 \cdot 5^2 \cdot 11$, $2 \cdot 3 \cdot 5^3 \cdot 11$, $2 \cdot 3^2 \cdot 5^2 \cdot 11$, $2^3 \cdot 3 \cdot 5^2 \cdot 11$ or $2^2 \cdot 3^2 \cdot 5^2 \cdot 11$. In any case K contains four distinct conjugacy classes of elements of orders 2, 3, 5 and 11. The first four smallest conjugacy classes of G by Lemma 3 and [5] have orders 1120, 5460, 104832 and 320320. From $|K| \le 9900$ we deduce that K can not contain four distinct conjugacy classes and this is a contradiction. So S is not one of the symplectic groups $PSP(2l,q)$. If S is one of the orthogonal groups $P\Omega(2l+1,q)$, then using $PSL(2,q) \cong P\Omega(3,q)$ and proof of Lemma 5 and case (3) we get that $S \ncong P\Omega(3,q)$. So S may be isomorphic to one of the following cases: $P\Omega(5,2)$, $P\Omega(5,2^2)$, $P\Omega(5,2^3)$, $P\Omega(5,3)$ or $P\Omega(7,2)$. If $S \cong P\Omega(5,2)$, $P\Omega(5,2^2)$, $P\Omega(5,3)$ or $P\Omega(7,2)$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong P\Omega(5,2^3)$, then we obtain $|S| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^{13} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 13$. By similar argument to the case $S \cong PSP(4,2^3)$, we get a contradiction. So S is not one of the orthogonal groups $P\Omega(2l+1,q)$. Therefore S is not one of the classical groups $PSP(2l,q)$ or $P\Omega(2l+1,q)$.

(6) S is one of the classical groups $P\Omega^{\varepsilon}(2l,q)$ for $\varepsilon \in \{+1,-1\}$. Then using the order of G and $|P\Omega^{\varepsilon}(2l,q)|$, q can be a power of 2, 3, 5 or 7. So S may be one of the groups $P\Omega^{\varepsilon}(4,2^n)$, $1 \le n \le 7$, $P\Omega^{\varepsilon}(4,3)$, $P\Omega^{\varepsilon}(4,3^2)$, $P\Omega^{\varepsilon}(4,3^3)$, $P\Omega^{\varepsilon}(4,5)$, $P\Omega^{\varepsilon}(4,7)$, $P\Omega^{\varepsilon}(6,2)$, $P\Omega^{\varepsilon}(6,2^2)$, $P\Omega^{\varepsilon}(6,3)$ or $P\Omega^{\varepsilon}(8,\ 2)$. If $S \cong P\Omega^{\varepsilon}(4,2)$, $P\Omega^{\varepsilon}(4,2^2)$, $P\Omega^{\varepsilon}(4,2^5)$, $P\Omega^{\varepsilon}(4,2^7)$, $P\Omega^{\varepsilon}(4,3)$, $P\Omega^{\varepsilon}(4,3^2)$, $P\Omega^{\varepsilon}(4,7)$, $P\Omega^{\varepsilon}(6,2)$ or $P\Omega^{\varepsilon}(8,2)$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong P\Omega^{\varepsilon}(4,8)$, $P\Omega^{\varepsilon}(4,5)$, or $P\Omega^{\varepsilon}(6,3)$, then we consider two cases. If $\varepsilon = +1$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $\varepsilon = -1$, then similar to the case (4) for $S \cong PSU(2,2^6)$, we obtain that 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. If $S \cong P\Omega^{\varepsilon}(4,2^4)$ or $P\Omega^{\varepsilon}(6,2^2)$, then $17 \big| |S|$ and this contradicts $17 \nmid |G|$. If $S \cong P\Omega^{\varepsilon}(4,2^6)$, then we consider two cases. If $\varepsilon = +1$, then $13^2 \big| |S|$ and this contradicts $13^2 \nmid |G|$. If $\varepsilon = -1$, then $17 \big| |S|$ and this contradicts $17 \nmid |G|$. So S is not one of the classical groups $P\Omega^{\varepsilon}(2l,q)$.

(7) S is one of the exceptional Chevalley groups $F_4(q)$, $G_2(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$. If $S \cong G_2(q)$, then using the orders of $|G_2(q)|$ and $|G|$ and the fact that $S \le G/K$, q may be one of the $2, 3$ or 2^2. If $S \cong G_2(2)$, then $13 \nmid |S|$ which is a contradiction. If $S \cong G_2(2^2)$ or $G_2(3)$, then similar to the case (4) for $PSU(2,2^6)$, we obtain that 7 is adjacent to 11 in $GK(G)$ which is a contradiction. If $S \cong F_4(q)$, $E_6(q)$, $E_7(q)$ or $E_8(q)$, then $|S|$ has a factor of form q^{24}, q^{36}, q^{63} or q^{120}. But from $S \le G/K$ we know that the maximum factor in $|S|$ can be q^{14} (for $q = 2$) and this is a contradiction. So S is not one of the exceptional Chevalley groups.

(8) S is one of the twisted Chevalley groups or Tits group. Then $S \cong^2 D_4(q)$, ${}^2F_4(2^{2n+1})$, ${}^2E_6(q)$, ${}^2G_2(3^{2n+1})$, ${}^2G_2(3)'$, ${}^2B_2(2^{2n+1})$ or T, the Tits group. If $S \cong$ $\cong {}^3D_4(q)$, then using $S \le G/K$, $|G|$ and $|{}^3D_4(q)|$, the only possibility for q is 2. If $S \cong^3 D_4(2)$, then $|S| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$. So from $S \le G/K \le \operatorname{Aut}(S)$ and $|G|$ we have $|K| = 2^2 \cdot 3 \cdot 5^3 \cdot 11$, $2^3 \cdot 3 \cdot 5^3 \cdot 11$ or $2^2 \cdot 3^2 \cdot 5^3 \cdot 11$. K is normal in G and in any case has four conjugacy classes of elements of order $2, 3, 5$ and 11. But the four smallest conjugacy classes of G (by [5] and Lemma 3) have orders 1120, 5460, 104832 and 320320. From $|K|$ we deduce that K can not contain four distinct conjugacy classes which is a contradiction. If $S \cong {}^2F_4(2^{2n+1})$ for $n > 0$, then by $|{}^2F_4(2^{2n+1})|$ we have $2^{36} \big| |S|$ and this contradicts $2^{36} \nmid |G|$. If $S \cong^2 G_2(3^{2n+1})$ for $n > 0$, then $3^9 \big| |S|$ and this contradicts $3^9 \nmid |G|$. If $S \cong^2 E_6(q)$ for $n > 0$, then $|S|$ has a factor of the form q^{36}. But the largest factor in $|S|$ can be q^{14} (for $q = 2$) and this is a contradiction. If $S \cong^2 B_2(2^{2n+1})$ for $n > 0$, then by $|S|$ we obtain that n may be $1, 2$ or 3. If $n = 1$, then $S \cong^2 B_2(2^3)$. So $|S| = 2^6 \cdot 5 \cdot 7 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. Similar to the case (4) for $PSU(2,2^6)$ we get a contradiction with 7 adjacent to 11 in $GK(G)$. If $n = 2$ or 3, then $S \cong^2 B_2(2^5)$ or ${}^2B_2(2^7)$. In any case $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong^2 G_2(3)'$, then $13 \nmid |S|$ and this contradicts Lemma 5. If $S \cong T$, the Tits group, then $|S| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ and $|\operatorname{Aut}(S)| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 13$. Similar to the case (4) for $PSU(2,2^6)$, we deduce that 7 is adjacent to 11 in $GK(G)$ and this is a contradiction. So S is not one of the twisted Chevalley groups or Tits group.

Conclusion of the proof of Theorem 1. In cases (1) – (8) we considered S to be a non-Abelian finite simple group and showed that the only possibility is $S \cong \mathbb{A}_{16}$. So $|S| = |G| = 2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and from $S \le G/K \le \operatorname{Aut}(S)$ for a maximal normal solvable subgroup K of G, we obtain $S = G = \mathbb{A}_{16}$ and $K = \{1\}$. Thus $S = G = \mathbb{A}_{16}$ and the proof is completed.

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