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ON A PROBLEM OF DETERMINING THE PARAMETER OF A PARABOLIC EQUATION

ПРО ЗАДАЧУ ВИЗНАЧЕННЯ ПАРАМЕТРА ПАРАБОЛІЧНОГО РІВНЯННЯ

The boundary-value problem of determining the parameter p of a parabolic equation

$$v'(t) + Av(t) = f(t) + p(0 \leq t \leq 1), \quad v(0) = \varphi, \quad v(1) = \psi$$

in arbitrary Banach space E with the strongly positive operator A is considered. The exact estimates in Hölder norms for the solution of this problem are established. In applications, exact estimates for the solution of the boundary-value problems for parabolic equations are obtained.

Розглянуто крайову задачу визначення параметра p параболічного рівняння

$$v'(t) + Av(t) = f(t) + p(0 \leq t \leq 1), \quad v(0) = \varphi, \quad v(1) = \psi$$

у довільному банаховому просторі E із сильно додатним оператором A. Встановлено точні за нормами Гельдера оцінки для розв'язку цієї задачі. У застосуваннях одержано точні оцінки для розв'язків крайових задач для параболічних рівнянь.

1. Introduction. Methods of solutions of the nonlocal boundary-value problems for evolution equations with a parameter have been studied extensively by many researchers (see, e.g., [1 – 21] and the references given therein).

We consider the following local boundary-value problem for the differential equation

$$v'(t) + Av(t) = f(t) + p(0 \leq t \leq 1), \quad v(0) = \varphi, \quad v(1) = \psi \tag{1.1}$$

in an arbitrary Banach space with linear (unbounded) operator A and an unknown parameter p.

In the paper [1] the solvability of the problem (1.1) in the space $C(E)$ of the continuous E-valued functions $\varphi(t)$ defined on $[0,1]$, equipped with the norm

$$\|\varphi\|_{C(E)} = \max_{0 \leq t \leq 1} \|\varphi(t)\|_E$$

was studied under the necessary and sufficient conditions for the operator A. The solution depends continuously on the initial and boundary data. Namely:

Theorem 1.1. *Assume that $-A$ is the generator of the analytic semigroup $\exp\{-tA\}(t \geq 0)$ and all points $2\pi ik$, $k \in Z$, $k \neq 0$ are not belongs to the spectrum $\sigma(A)$. Let $v(0) \in E$, $v(1) \in D(A)$ and $f(t) \in C^\beta(E)$, $0 < \beta \leq 1$. Then for the solution $(v(t), p)$ of problem* (1.1) *in $C(E) \times E$ the estimates*

$$\|p\|_E \leq M\left[\|v(0)\|_E + \|v(1)\|_E + \|Av(1)\|_E + \frac{1}{\beta}\|f\|_{C^\beta(E)}\right],$$

$$\|v\|_{C(E)} \leq M[\|v(0)\|_E + \|v(1)\|_E + \|f\|_{C(E)}]$$

hold, where M does not depend on $\beta, v(0), v(1)$ and $f(t)$. Here $C^\beta(E)$ is the space obtained by completion of the space of all smooth E-valued functions $\varphi(t)$ on $[0,1]$ in

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the norm

$$\|\varphi\|_{C^{\beta}(E)} = \max_{0\le t\le 1}\|\varphi(t)\|_E + \sup_{0\le t<t+\tau\le 1}\frac{\|\varphi(t+\tau)-\varphi(t)\|_E}{\tau^{\beta}}.$$

We say $(v(t),p)$ is the solution of the problem (1.1) in $C_0^{\beta,\gamma}(E)\times E_1$ if the following conditions are satisfied:

i) $v'(t),\ Av(t)\in C_0^{\beta,\gamma}(E),\ p\in E_1\subset E,$

ii) $(v(t),p)$ satisfies the equation and boundary conditions (1.1).

Here $C_0^{\beta,\gamma}(E)$, $(0\le\gamma\le\beta,\ 0<\beta<1)$ is the Hölder space with weight obtained by completion of the space of all smooth E-valued functions $\varphi(t)$ on $[0,1]$ in the norm

$$\|\varphi\|_{C_0^{\beta,\gamma}(E)} = \max_{0\le t\le 1}\|\varphi(t)\|_E + \sup_{0\le t<t+\tau\le 1}\frac{(t+\tau)^{\gamma}\|\varphi(t+\tau)-\varphi(t)\|_E}{\tau^{\beta}}.$$

In the present paper the exact estimates in Hölder norms for the solution of problem (1.1) are proved. In applications, exact estimates for the solution of the boundary-value problems for parabolic equations are obtained.

2. $C_0^{\beta,\gamma}(E)$-estimates for the solution of problem (1.1). We study the problem (1.1) in the spaces $C_0^{\beta,\gamma}(E)$. To these spaces there correspond the spaces of traces $E_1^{\beta,\gamma}$, which consist of the elements $w\in E$ for which the following norm is finite:

$$|w|_1^{\beta,\gamma} = \max_{0\le z\le 1}\|A\exp\{-zA\}w\|_E +$$

$$+\sup_{0\le z<z+\tau\le 1}\frac{(z+\tau)^{\gamma}\|A\left(\exp\{-\left(z+\tau\right)A\}-\exp\{-zA\}\right)w\|_E}{\tau^{\beta}}.$$

Assume that $-A$ is the generator of the analytic semigroup $\exp\{-tA\}(t\ge 0)$ with exponentially decreasing norm, when $t\to+\infty$, i.e., the following estimates hold:

$$\|\exp\{-tA\}\|_{E\to E}\le Me^{-\delta t}, t\|A\exp\{-tA\}\|_{E\to E}\le M,\quad t>0,\quad M>0,\quad \delta>0. \tag{2.1}$$

From (2.1) it follows that

$$\|T\|_{E\to E}\le M(\delta). \tag{2.2}$$

Here $T=(I-\exp\{-A\})^{-1}$. We have that

$$\begin{gathered} v(t)=\exp\{-tA\}v(0)+\int_0^t\exp\{-(t-s)A\}f(s)ds+(I-\exp\{-tA\})A^{-1}p, \\ p=T\{Av(1)-A\exp\{-A\}v(0)-\int_0^1 A\exp\{-(1-s)A\}f(s)ds\} \end{gathered} \tag{2.3}$$

for the solution of problem (1.1) in the space $C_0^{\beta,\gamma}(E)$ (see, for example, [1, 22]).

Theorem 2.1. *Let* $v(0)-A^{-1}f(0),\ v(1)-A^{-1}f(1)\in E_1^{\beta,\gamma}$ *and* $f(t)\in C_0^{\beta,\gamma}(E)$, $0\le\gamma\le\beta$, $0<\beta<1$. *Then for the solution* $(v(t),\,p)$ *of problem* (1.1) *in* $C_0^{\beta,\gamma}(E)\times\times E_1^{\beta,\gamma}$ *the estimates*

$$\|v'\|_{C_0^{\beta,\gamma}(E)}+\|Av-p\|_{C_0^{\beta,\gamma}(E)}+\max_{0\le t\le 1}\left|v(t)-A^{-1}f(t)\right|_1^{\beta,\gamma}\le$$

$$\le M\left[\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|v(1)-A^{-1}f(1)\right|_1^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}\right], \tag{2.4}$$

$$\left|A^{-1}p\right|_1^{\beta,\gamma}\le M\left[\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\right.$$

$$\left.+\left|v(1)-A^{-1}f(1)\right|_1^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}\right] \tag{2.5}$$

hold, where M *does not depend on* γ, β, $v(0)$, $v(1)$ *and* $f(t)$.

Proof. Using formula (2.3), we can write

$$Av(t)-f(t)=\exp\{-tA\}\,(Av(0)-f(0))+\exp\{-tA\}\,(f(0)-f(t))+$$

$$+\int_0^t A\exp\{-(t-s)A\}\,(f(s)-f(t))\,ds+(I-\exp\{-tA\})\,p=$$

$$=\exp\{-tA\}\,(Av(0)-f(0))+(I-\exp\{-tA\})\,p+J(t), \tag{2.6}$$

$$p=T\{Av(1)-f(1)-\exp\{-A\}\,(Av(0)-f(0))-$$

$$-\int_0^1 A\exp\{-(1-s)A\}\,(f(s)-f(1))\,ds+\exp\{-A\}\,(f(1)-f(0))\}=$$

$$=T\{Av(1)-f(1)-\exp\{-A\}\,(Av(0)-f(0))-J(1)\}, \tag{2.7}$$

where

$$J(t)=\exp\{-tA\}\,(f(0)-f(t))+\int_0^t A\exp\{-(t-s)A\}\,(f(s)-f(t))\,ds.$$

By [22], Theorems 5.1 and 5.2 in Chapter 1,

$$\|J\|_{C_0^{\beta,\gamma}(E)}\le M\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}, \tag{2.8}$$

$$\left|A^{-1}J(t)\right|_1^{\beta,\gamma}\le M\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}. \tag{2.9}$$

From the definition of the space $E_1^{\beta,\gamma}$ and the estimate (2.1) it follows that

$$\left|\exp\{-tA\}\left(v(0)-A^{-1}f(0)\right)+(I-\exp\{-tA\})\,A^{-1}p\right|_1^{\beta,\gamma}\le$$

$$\leq \|\exp\{-tA\}\|_{E\to E}\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\|I-\exp\{-tA\}\|_{E\to E}\left|A^{-1}p\right|_1^{\beta,\gamma}\leq$$

$$\leq M\left[\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|A^{-1}p\right|_1^{\beta,\gamma}\right] \tag{2.10}$$

for all $t, t\in[0,1]$. By (2.2), (2.9), (2.10) and the triangle inequality

$$\max_{0\leq t\leq 1}\left|v(t)-A^{-1}f(t)\right|_1^{\beta,\gamma}\leq$$

$$\leq M\left[\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|A^{-1}p\right|_1^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}\right], \tag{2.11}$$

$$\left|A^{-1}p\right|_1^{\beta,\gamma}\leq\|T\|_{E\to E}\left[\left|v(1)-A^{-1}f(1)\right|_1^{\beta,\gamma}+\right.$$

$$\left.+\|\exp\{-A\}\|_{E\to E}\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|A^{-1}J(1)\right|_1^{\beta,\gamma}\right]\leq$$

$$\leq M\left[\left|v(1)-A^{-1}f(1)\right|_1^{\beta,\gamma}+\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|A^{-1}J(1)\right|_1^{\beta,\gamma}\right]. \tag{2.12}$$

Estimate for $\max\limits_{0\leq t\leq 1}\left|v(t)-A^{-1}f(t)\right|_1^{\beta,\gamma}$ and estimate (2.5) are proved. Using formula (2.6) and equation (1.1), we can write

$$v'(t)=\exp\{-tA\}\left(Av(0)-f(0)-p\right)+J(t). \tag{2.13}$$

Applying the triangle inequality and the definition of the space $C_0^{\beta,\gamma}(E)$, we obtain

$$\|v'\|_{C_0^{\beta,\gamma}(E)}\leq\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\left|A^{-1}p\right|_1^{\beta,\gamma}+\|J\|_{C_0^{\beta,\gamma}(E)}.$$

From this estimate and (2.12), (2.12) it follows that

$$\|v'\|_{C_0^{\beta,\gamma}(E)}\leq M\left[\left|v(0)-A^{-1}f(0)\right|_1^{\beta,\gamma}+\right.$$

$$\left.+\left|v(1)-A^{-1}f(1)\right|_1^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\|f\|_{C_0^{\beta,\gamma}(E)}\right].$$

The estimate for $Av(t)-p$ in the norm $C_0^{\beta,\gamma}(E)$ follows from this estimate and the triangle inequality. Theorem 2.1 is proved.

***Remark* 2.1.** The spaces $C_0^{\beta,\gamma}(E)$ in which exact estimates has been established, depend on parameters β and γ. However, the constants in these inequalities depend only on β. Hence, we can be choose the parameter γ freely, which increases the number of spaces.

With the help of A we introduce the fractional space $E_\alpha(E,A)$, $0<\alpha<1$, consisting of all $v\in E$ for which the following norms are finite:

$$\|v\|_\alpha=\sup_{\lambda>0}\left\|\lambda^{1-\alpha}A\exp\{-\lambda A\}v\right\|_E+\|v\|_E\,.$$

3. $C_0^{\beta,\gamma}(E_{\alpha-\beta})$-estimates for the solution of problem (1.1). We study the problem (1.1) in the spaces $C_0^{\beta,\gamma}(E_{\alpha-\beta})$. To these spaces there correspond the spaces of traces

$E^{\beta,\gamma}_{1+\alpha-\beta}$, which consist of the elements $w \in E$ for which the following norm is finite:

$$|w|^{\beta,\gamma}_{1+\alpha-\beta} = \max_{0\leq z\leq 1} \|A\exp\{-zA\}w\|_{\alpha-\beta} +$$

$$+ \sup_{0\leq z<z+\tau\leq 1} \frac{(z+\tau)^{\gamma}\|A\left(\exp\{-\left(z+\tau\right)A\} - \exp\{-zA\}\right)w\|_{\alpha-\beta}}{\tau^{\beta}}.$$

Theorem 3.1. *Let* $v(0) - A^{-1}f(0),\ v(1) - A^{-1}f(1) \in E^{\beta,\gamma}_{1+\alpha-\beta}$ *and* $f(t) \in \in C^{\beta,\gamma}_0(E_{\alpha-\beta})$, $0 \leq \gamma \leq \beta \leq \alpha$, $0 < \alpha < 1$. *Then for the solution* $(v(t), p)$ *of problem* (1.1) *in* $C^{\beta,\gamma}_0(E_{\alpha-\beta}) \times E^{\beta,\gamma}_{1+\alpha-\beta}$ *the estimates*

$$\|v'\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})} + \|Av - p\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})} + \max_{0\leq t\leq 1}\left|v(t) - A^{-1}f(t)\right|^{\beta,\gamma}_1 \leq$$

$$\leq M\left[\left|v(0) - A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta} +\right.$$

$$\left. + \left|v(1) - A^{-1}f(1)\right|^{\beta,\gamma}_{1+\alpha-\beta} + \alpha^{-1}(1-\alpha)^{-1}\|f\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})}\right], \tag{3.1}$$

$$\left|A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta} \leq M\left[\left|v(0) - A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta} +\right.$$

$$\left. + \left|v(1) - A^{-1}f(1)\right|^{\beta,\gamma}_{1+\alpha-\beta} + \alpha^{-1}(1-\alpha)^{-1}\|f\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})}\right] \tag{3.2}$$

hold, where M *does not depend on* γ, β, α, $v(0)$, $v(1)$ *and* $f(t)$.

Proof. By [22], Theorem 5.3 in Chapter 1,

$$\|J\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})} \leq M\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})}, \tag{3.3}$$

$$\left|A^{-1}J(t)\right|^{\beta,\gamma}_{1+\alpha-\beta} \leq M\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C^{\beta,\gamma}_0(E_{\alpha-\beta})}. \tag{3.4}$$

From the definition of the space $E^{\beta,\gamma}_{1+\alpha-\beta}$ and the estimate (2.1) it follows that

$$\left|\exp\{-tA\}\left(v(0) - A^{-1}f(0)\right) + \left(I - \exp\{-tA\}\right)A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta} \leq$$

$$\leq \|\exp\{-tA\}\|_{E\to E}\left|v(0) - A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta} +$$

$$+\| - \exp\{-tA\}\|_{E\to E}\left|A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta} \leq$$

$$\leq M\left[\left|v(0) - A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta} + \left|A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta}\right] \tag{3.5}$$

for all $t, t \in [0,1]$. By (2.2), (3.4), (3.5) and the triangle inequality

$$\max_{0\leq t\leq 1}\left|v(t) - A^{-1}f(t)\right|^{\beta,\gamma}_{1+\alpha-\beta} \leq$$

$$\leq M\left[\left|v(0)-A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\left|A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta}+\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\right], \tag{3.6}$$

$$\left|A^{-1}p\right|^{\beta,\gamma}_{1+\alpha-\beta}\leq\|T\|_{E\to E}\left[\left|v(1)-A^{-1}f(1)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\right.$$

$$\left.+\|\exp\{-A\}\|_{E\to E}\left|v(0)-A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\left|A^{-1}J(1)\right|^{\beta,\gamma}_{1+\alpha-\beta}\right]\leq$$

$$\leq M\left[\left|v(1)-A^{-1}f(1)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\left|v(0)-A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\left|A^{-1}J(1)\right|^{\beta,\gamma}_{1+\alpha-\beta}\right]. \tag{3.7}$$

Estimate for $\max\limits_{0\leq t\leq 1}\left|v(t)-A^{-1}f(t)\right|^{\beta,\gamma}_{1+\alpha-\beta}$ and estimate (3.2) are proved. Applying (2.13), the triangle inequality and the definition of the space $C_0^{\beta,\gamma}(E_{\alpha-\beta})$, we obtain

$$\|v'\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\leq\left|v(0)-A^{-1}f(0)\right|^{\beta,\gamma}_{1}+\left|A^{-1}p\right|^{\beta,\gamma}_{1}+\|J\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}.$$

From this estimate and (3.7), (3.7) it follows that

$$\|v'\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\leq M\left[\left|v(0)-A^{-1}f(0)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\right.$$

$$\left.+\left|v(1)-A^{-1}f(1)\right|^{\beta,\gamma}_{1+\alpha-\beta}+\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\right].$$

The estimate for $Av(t)-p$ in the norm $C_0^{\beta,\gamma}(E_{\alpha-\beta})$ follows from this estimate and the triangle inequality.

Theorem 3.1 is proved.

Note that applying the definition of the space $E^{\beta,\gamma}_{1+\alpha-\beta}$, we can obtain

$$|w|^{\beta,\gamma}_{1+\alpha-\beta}\leq M\|Aw\|_{E_{\alpha-\gamma}},\quad Aw\in E_{\alpha-\gamma}.$$

We have not been able to establish the opposite inequality necessary for the equivalence of norms. Nevertheless, we have the following result.

Theorem 3.2. *Let* $v(0)-A^{-1}f(0),\ v(1)-A^{-1}f(1)\in E_{\alpha-\gamma}$ *and* $f(t)\in C_0^{\beta,\gamma}(E_{\alpha-\beta})$, $0\leq\gamma\leq\beta\leq\alpha$, $0<\alpha<1$. *Then for the solution* $(v(t),p)$ *of problem* (1.1) *in* $C_0^{\beta,\gamma}(E_{\alpha-\beta})\times E_{\alpha-\gamma}$ *the estimates*

$$\|v'\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}+\|Av-p\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}+\max_{0\leq t\leq 1}\|Av(t)-f(t)\|_{\alpha-\gamma}\leq$$

$$\leq M\left[\|Av(0)-f(0)\|_{\alpha-\gamma}+\|Av(1)-f(1)\|_{\alpha-\gamma}+\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\right], \tag{3.8}$$

$$\|p\|_{\alpha-\gamma}\leq M\left[\|Av(0)-f(0)\|_{\alpha-\gamma}+\right.$$

$$\left.+\|Av(1)-f(1)\|_{\alpha-\gamma}+\alpha^{-1}(1-\alpha)^{-1}\|f\|_{C_0^{\beta,\gamma}(E_{\alpha-\beta})}\right] \tag{3.9}$$

hold, where M *does not depend on* γ, β, α, $v(0)$, $v(1)$ *and* $f(t)$.

***Remark* 3.1.** The spaces $C_0^{\beta,\gamma}(E_{\alpha-\beta})$ in which exact estimates has been established, depend on parameters α, β and γ. However, the constants in these inequalities depend only on α. Hence, we can be choose parameters β, γ freely, which increases the number of spaces. In particular, Theorems 3.1 and 3.2 imply the well-posedness theorem in $C(E_\alpha)$.

***Remark* 3.2.** Theorems 2.1 and 3.1, 3.2 hold for the following boundary-value problems:

$$v'(t) + Av(t) = f(t) + p(0 \le t \le 1), \quad v(0) = \varphi, v(\lambda) = \psi, \quad 0 < \lambda \le 1,$$

$$v'(t) - Av(t) = f(t) + p(0 \le t \le 1), \quad v(1) = \varphi, v(\lambda) = \psi, \quad 0 \le \lambda < 1$$

in an arbitrary Banach space with positive operator A and an unknown parameter p.

Applications. First, the boundary-value problem on the range $\{0 \le t \le 1, x \in \mathrm{R}^n\}$ for the $2m$-order multidimensional parabolic equation is considered:

$$\frac{\partial v(t,x)}{\partial t} + \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} v(t,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \sigma v(t,x) = f(t,x) + p(x), \; 0 < t < 1,$$

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} v(0,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \sigma v(0,x) = f(0,x), \quad x \in \mathrm{R}^n, \tag{4.1}$$

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} v(1,x)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \sigma v(1,x) = f(1,x), \quad x \in \mathrm{R}^n,$$

$$|r| = r_1 + \ldots + r_n,$$

where $a_r(x)$ and $f(t,x)$ are given as sufficiently smooth functions. Here, σ is a sufficiently large positive constant.

It is assumed that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x) \left(i\xi_1\right)^{r_1} \ldots \left(i\xi_n\right)^{r_n}, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathrm{R}^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{4.2}$$

acting on functions defined on the space R^n, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$.

The problem (4.1) has a unique smooth solution. This allows us to reduce the problem (4.1) to the problem (1.1) in a Banach space $E = C^\mu(R^n)$ of all continuous bounded functions defined on $\mathbb{R}^n$ satisfying a Hölder condition with the indicator $\mu \in (0,1)$ with a strongly positive operator $A = B^x + \sigma I$ defined by (4.2) (see [25] and [26]).

Theorem 4.1. *For the solution of the boundary problem* (4.1) *the following estimates are satisfied:*

$$\|v\|_{C_0^{1+\beta,\gamma}(C^\mu(R^n))} \leq \frac{M(\mu)}{\beta(1-\beta)}\|f\|_{C_0^{\beta,\gamma}(C^\mu(R^n))}, \quad 0 \leq \gamma \leq \beta < 1, \quad 0 < \mu < 1,$$

$$\|v\|_{C_0^{1+\beta,\gamma}\left(C^{2m(\alpha-\beta)}(R^n)\right)} \leq M(\alpha,\beta)\|f\|_{C_0^{\beta,\gamma}\left(C^{2m(\alpha-\beta)}(R^n)\right)},$$

$$\|p\|_{C^{2m(\alpha-\gamma)}(R^n)} \leq M(\alpha,\beta,\gamma)\|f\|_{C_0^{\beta,\gamma}\left(C^{2m(\alpha-\beta)}(R^n)\right)},$$

$$0 \leq \gamma \leq \beta, \quad 0 < 2m(\alpha-\beta) < 1,$$

where $M(\mu)$, $M(\alpha,\beta)$ *and* $M(\alpha,\beta,\gamma)$ *does not depend on* $f(t,x)$.

The proof of Theorem 4.1 is based on the abstract Theorems 2.1, 3.2 and on the following theorem on the structure of the fractional spaces $E_\alpha(A, C^\mu(R^n))$.

Theorem 4.2. $E_\alpha(A, C^\mu(R^n)) = C^{2m\alpha+\mu}(R^n)$ *for all* $0 < \alpha < \dfrac{1}{2m}$, $0 < \mu < 1$ [22].

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$, $0 < x_k < 1$, $1 \leq k \leq n$, with boundary $S, \overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$ we consider the mixed boundary-value problem for the multidimensional parabolic equation

$$\begin{aligned}
&\frac{\partial v(t,x)}{\partial t} - \sum_{r=1}^{n} \alpha_r(x)\frac{\partial^2 v(t,x)}{\partial x_r^2} + \sigma v(t,x) = f(t,x) + p(x),\\
&x = (x_1,\ldots,x_n) \in \Omega, \quad 0 < t < 1,\\
&-\sum_{r=1}^{n} \alpha_r(x)\frac{\partial^2 v(0,x)}{\partial x_r^2} + \sigma v(0,x) = f(0,x), \quad x \in \overline{\Omega},\\
&-\sum_{r=1}^{n} \alpha_r(x)\frac{\partial^2 v(1,x)}{\partial x_r^2} + \sigma v(1,x) = f(1,x), \quad x \in \overline{\Omega},\\
&v(t,x) = 0, \quad x \in S,
\end{aligned} \tag{4.3}$$

where $\alpha_r(x)$ $(x \in \Omega)$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $\alpha_r(x) \geq a > 0$. Here, σ is a sufficiently large positive constant.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$, $\beta = (\beta_1,\ldots,\beta_n)$, $0 < x_k < 1$, $k = 1,\ldots,n$, of all continuous functions satisfying a Hölder condition with the indicator $\beta = (\beta_1,\ldots,\beta_n)$, $\beta_k \in (0,1)$, $1 \leq k \leq n$ and with weight $x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}$, $0 \leq x_k < x_k + h_k \leq 1$, $1 \leq k \leq n$, which equipped with the norm

$$\|f\|_{C_{01}^{\beta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + \sup_{\substack{0 \leq x_k < x_k+h_k \leq 1,\\ 1 \leq k \leq n}} \big|f(x_1,\ldots,x_n) -$$

$$-f(x_1+h_1,\ldots,x_n+h_n)\big| \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k}(1-x_k-h_k)^{\beta_k},$$

where $C(\overline{\Omega})$-is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |f(x)|\,.$$

It is known that the differential expression [24]

$$Av = -\sum_{r=1}^{n} \alpha_r(t,x)\frac{\partial^2 v(t,x)}{\partial x^2} + \delta v(t,x)$$

defines a positive operator A acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S.

Therefore, we can replace the mixed problem (4.3) by the abstract boundary problem (1.1). Using the results of Theorem 2.1, we can obtain that the following theorem.

Theorem 4.3. *For the solution of the mixed boundary-value problem* (4.3) *the following estimate is valid:*

$$\|v\|_{C_0^{1+\beta,\gamma}\left(C_{01}^{\mu}(\overline{\Omega})\right)} \le \frac{M(\mu)}{\beta(1-\beta)}\|f\|_{C_0^{\beta,\gamma}\left(C_{01}^{\mu}(\overline{\Omega})\right)},$$

$$0 \le \gamma \le \beta < 1, \quad \mu = \{\mu_1,\ldots,\mu_n\}, \quad 0 < \mu_k < 1, \quad 1 \le k \le n,$$

where $M(\mu)$ is independent of β, γ and $f(t,x)$.

Third, we consider the mixed boundary-value problem for parabolic equation

$$\begin{aligned}
&\frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial^2 v(t,x)}{\partial x^2} + \sigma v(t,x) = f(t,x) + p(x),\ 0<t<1,\ 0<x<1,\\
&-a(x)\frac{\partial^2 v(0,x)}{\partial x^2} + \sigma v(0,x) = f(0,x), \quad 0 \le x \le 1,\\
&-a(x)\frac{\partial^2 v(1,x)}{\partial x^2} + \sigma v(1,x) = f(1,x), \quad 0 \le x \le 1,\\
&u(t,0) = u(t,1),\ \ u_x(t,0) = u_x(t,1), \quad 0 \le t \le 1,
\end{aligned} \tag{4.4}$$

where $a(t,x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(t,x) \ge a > 0$. Here, σ is a sufficiently large positive constant.

We introduce the Banach spaces $C^{\beta}[0,1]$, $0 < \beta < 1$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\|\varphi\|_{C^{\beta}[0,1]} = \|\varphi\|_{C[0,1]} + \sup_{0\le x<x+\tau\le 1}\frac{|\varphi(x+\tau)-\varphi(x)|}{\tau^{\beta}},$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on [0,1] with the usual norm

$$\|\varphi\|_{C[0,1]} = \max_{0\le x\le 1}|\varphi(x)|.$$

It is known that the differential expression [23]

$$Av = -a(x)v''(x) + \delta v(x)$$

define a positive operator A acting in $C^{\beta}[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the conditions $v(0) = v(1)$, $v_x(0) = v_x(1)$.

Therefore, we can replace the mixed problem (4.4) by the abstract boundary value problem (4.4). Using the results of Theorems 2.1, 3.2, we can obtain that

Theorem 4.4. *For the solution of the mixed problem* (4.4) *the following estimates are valid*:

$$\|v\|_{C_0^{1+\beta,\gamma}(C^{\mu}[0,1])} \leq \frac{M(\mu)}{\beta(1-\beta)}\|f\|_{C_0^{\beta,\gamma}(C^{\mu}[0,1])}, \quad 0 \leq \gamma \leq \beta < 1, \quad 0 < \mu < 1,$$

$$\|v\|_{C_0^{1+\beta,\gamma}\left(C^{2(\alpha-\beta)}[0,1]\right)} \leq M(\alpha,\beta)\|f\|_{C_0^{\beta,\gamma}\left(C^{2(\alpha-\beta)}[0,1]\right)},$$

$$\|p\|_{C^{2(\alpha-\gamma)}[0,1]} \leq M(\alpha,\beta,\gamma)\|f\|_{C_0^{\beta,\gamma}\left(C^{2(\alpha-\beta)}[0,1]\right)},$$

$$0 \leq \gamma \leq \beta, \quad 0 < 2m(\alpha-\beta) < 1,$$

where $M(\mu)$, $M(\alpha,\beta)$ *and* $M(\alpha,\beta,\gamma)$ *does not depend on* $f(t,x)$.

The proof of Theorem 4.4 is based on the abstract Theorems 2.1, 3.2 and on the following theorem on the structure of the fractional spaces $E_{\alpha}(A, C[0,1])$.

Theorem 4.5 [23]. $E_{\alpha}(A, C[0,1]) = C^{2\alpha}[0,1]$ *for all* $0 < \alpha < \dfrac{1}{2}$.

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1. *Eidelman Yu. S.* Boundary value problems for differential equations with parameters: PhD Thesis (in Russian). – Voronezh, 1984.
2. *Eidelman Yu. S.* The boundary value problem for differential equations with a parameter // Differents. Uravneniya. – 1978. – **14**. – P. 1335–1337.
3. *Eidelman Yu. S.* Two-point boundary value problem for differential equations with a parameter // Dop. Akad. Nauk Ukr. RSR. Ser. A. – 1983, № 4. – S. 15 – 18.
4. *Prilepko A. I.* Inverse problems of potential theory // Mat. Zametki. – 1973. – **14**. – S. 755 – 767.
5. *Iskenderov A. D., Tagiev R. G.* The inverse problem of determining the right-hand sides of evolution equations in Banach space // Nauchn. Trudy Azerbaidzhan. Gos. Univ. – 1979. – **1**. – S. 51 – 56.
6. *Rundell W.* Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data // Applicable Anal. – 1980. – **10**. – P. 231 – 242.
7. *Prilepko A. I., Vasin I. A.* Some time-dependent inverse problems of hydrodynamics with final observation // Dokl. Akad. Nauk SSSR. – 1990. – **314**. – S. 1075 – 1078.
8. *Prilepko A. I., Kostin A. B.* On certain inverse problems for parabolic equations with final and integral observation // Mat. Sb. – 1992. – **183**, № 4. – 49 – 68.
9. *Prilepko A. I. and Tikhonov I. V.* Uniqueness of the solution of an inverse problem for an evolution equation and applications to the transfer equation // Mat. Zametki. – 1992. – **51**, № 2. – S. 77 – 87.
10. *D. G. Orlovski* On a problem of determining the parameter of an evolution equation // Differents. Uravneniya. – 1990. – **26**. – S. 1614 – 1621.
11. *Choulli M. and Yamamoto M.* Generic well-posedness of an inverse parabolic problem- the Hölder-space approach // Inverse Problems. – 1996. – **12**. – P. 195 – 205.
12. *Choulli M. and Yamamoto M.* Generic well-posedness of a linear inverse parabolic problem with diffusion parameters // J. Inv. Ill-Posed Problems. – 1999. – **7**, № 3. – P. 241 – 254.
13. *Eidelman Yu. S.* Conditions for the solvability of inverse problems for evolution equations // Dokl. Akad. Nauk Ukr. SSR Ser. A. – 1990. – № 7. – S. 28 – 31.

14. *Clement Ph., Guerre-Delabrire S.* On the regularity of abstract Cauchy problems and boundary value problems // Atti Accad. Naz. Lincei. Cl. sci. Fis., mate. natur. Rend. – 1998. – Sez. 9. – **9**, № 4. – P. 245 – 266.
15. *Agarwal R., Bohner M., Shakhmurov V. B.* Maximal regular boundary value problems in Banach-valued weighted spaces // Boundary-Value Problems. – 2005. – **1**. – P. 9 – 42.
16. *Gorbachuk V. I., Gorbachuk M. L.* Boundary value problems for differential-operator equations (in Russian). – Kiev: Naukova Dumka, 1984.
17. *Lunardi A.* Analytic semigroups and optimal regularity in parabolic problems // Oper. Theory: Adv. and Appl. – Basel etc.: Birkhäuser, 1995.
18. *Ashyralyev A.* Well-posedness of the boundary value problem for parabolic equations in difference analogues of spaces of smooth functions // Math. Probl. Eng. – 16 p., 2007.
19. *Ashyralyev A. O., Sobolevskii P. E.* The linear operator interpolation theory and the stability of difference-schemes // Dokl. Akad. Nauk SSSR. – 1984. – **275**, № 6. – S. 1289 – 1291.
20. *Ashyralyev A., Karatay I., Sobolevskii P. E.* Well-posedness of the nonlocal boundary value problem for parabolic difference equations // Discrete Dynam. Nature and Soc. – 2004. – **2004**, № 2. – P. 273 – 286.
21. *Ashyralyev A., Hanalyev A., Sobolevskii P. E.* Coercive solvability of nonlocal boundary value problem for parabolic equations // Abstract and Applied Analysis. – 2001. – **6**, № 1. – P. 53 – 61.
22. *Ashyralyev A., Sobolevskii P. E.* Well-posedness of parabolic difference equations // Oper. Theory: Adv. and Appl. – Basel etc.: Birkhäuser, 1994.
23. *Ashyralyev A.* Fractional spaces generated by the positivite differential and difference operator in a Banach space // Proc. Conf. "Math. Methods and Engineering", Kenan Taş et al. – Netherlands: Springer, 2007. – P. 13 – 22.
24. *Sobolevskii P. E.* The coercive solvability of difference equations // Dokl. Acad. Nauk SSSR. – 1971. – **201**, № 5. – S. 1063 – 1066.
25. *Smirnitskii Yu. A., Sobolevskii P. E.* Positivity of multidimensional difference operators in the C-norm // Uspechi. Mat. Nauk. – 1981. – **36**, № 4. – S. 202 – 203.
26. *Smirnitskii Yu. A.* Fractional powers of elliptic difference operators: PhD Thesis (in Russian). – Voronezh, 1983).

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