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GRÜSS-TYPE AND OSTROWSKI-TYPE INEQUALITIES IN APPROXIMATION THEORY

НЕРІВНОСТІ ТИПУ ГРЮССА ТА ОСТРОВСЬКОГО В ТЕОРІЇ НАБЛИЖЕНЬ

We discuss the Grüss inequalities on spaces of continuous functions defined on a compact metric space. Using the least concave majorant of the modulus of continuity, we obtain a Grüss inequality for the functional $L(f) = H(f;x)$, where $H\colon C[a,b] \to C[a,b]$ is a positive linear operator and $x \in [a,b]$ is fixed. We apply this inequality in the case of known operators, for example, the Bernstein, Hermite – Fejér operator the interpolation operator, convolution-type operators. Moreover, we derive Grüss-type inequalities using Cauchy's mean value theorem, thus generalizing results of Čebyšev and Ostrowski. A Grüss inequality on a compact metric space for more than two functions is given, and an analogous Ostrowski-type inequality is obtained. The latter in turn leads to one further version of Grüss' inequality. In an appendix, we prove a new result concerning the absolute first-order moments of the classical Hermite – Féjer operator.

Розглянуто нерівності Грюсса на просторах неперервних функцій, які визначено на компактному метричному просторі. З використанням найменшої опуклої мажоранти модуля неперервності одержано нерівність Грюсса для функціонала $L(f) = H(f;x)$, де $H\colon C[a,b] \to C[a,b]$ — додатний лінійний оператор, а $x \in [a,b]$ зафіксовано. Цю нерівність застосовано до випадку відомих операторів, наприклад оператора Бернштейна, інтерполяційного оператора Ерміта – Фейєра, операторів типу конволюції. Крім того, виведено нерівності типу Грюсса на основі теореми Коші про середнє, що узагальнює результати Чебишова та Островського. Представлено нерівність Грюсса на компактному метричному просторі для більш ніж двох функцій та отримано аналогічну нерівність типу Островського, яка, в свою чергу, приводить до ще однієї версії нерівності Грюсса. У додатку доведено новий результат щодо абсолютних моментів першого порядку класичного оператора Ерміта – Фейєра.

1. Introduction. The original form of Grüss' inequality estimates the difference between the integral of a product of two functions and the product of integrals of the two functions and was published by G. Grüss in 1935 [11]:

Theorem A. *Let f and g be two functions defined and integrable on $[a,b]$. If $m \le f(x) \le M$ and $p \le g(x) \le P$ for all $x \in [a,b]$, then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \right| \le \frac{1}{4}(M-m)(P-p).$$

The constant $1/4$ is the best possible.

Grüss' inequality attracted considerable interest after its publication. Here we mention only papers by E. Landau [14], J. Karamata [12], and a particularly useful one by A. M. Ostrowski [21]. We also note that a whole chapter in a book by D. S. Mitrinović et al. [19] is devoted to the inequality we discuss here.

Our present work is to a large extent motivated by a theorem which can be found in the paper [2] by D. Andrica and C. Badea. Here we cite a special form of it.

Theorem B. *Let $I = [a, b]$ be a compact interval of the real axis, $B(I)$ be the space of real-valued and bounded functions defined on I and $L\colon B(I) \to \mathbb{R}$ be a positive linear functional satisfying $L(e_0) = 1$ where $e_0\colon I \ni x \mapsto 1$. Assuming that for $f, g \in B(I)$ one has $m \le f(x) \le M$, $p \le g(x) \le P$ for all $x \in I$, the following holds*:

$$|L(fg) - L(f)L(g)| \le \frac{1}{4}(M - m)(P - p).$$

Another celebrated classical inequality was proved by A. M. Ostrowski [20] in 1938 which we cite below in the form given by Anastassiou in 1995 (see [3]).

Theorem C. *Let f be in $C^1[a, b]$, $x \in [a, b]$. Then*

$$|f(x) - \mu(f)| \le \varphi(x)\|f'\|_\infty,$$

where $\mu(f) := \dfrac{1}{b-a}\displaystyle\int_a^b f(t)dt$, $\varphi(x) := \dfrac{(x-a)^2 + (b-x)^2}{2(b-a)}$.

There is a relationship between the classical inequalities of Grüss and Ostrowski observed by S. S. Dragomir and S. Wang [7] in 1997 and further studied by X.-L. Cheng [5] in 2001. The two first-named authors proved that Grüss' classical inequality basically implies the following result (which we cite in its improved form given by Cheng in his Theorem 1.5).

Theorem D. *Let $f \in C^1[a, b]$ satisfy $m \le f'(x) \le M$ for $x \in [a, b]$. Then*

$$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(t)dt - \frac{f(b) - f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(M-m).$$

***Corollary* E.** *Under the assumptions of Theorem D we also have*

(i) $$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(t)dt\right| \le \left|\frac{f(b) - f(a)}{b-a}\right| \left| x - \frac{a+b}{2}\right| + \frac{1}{8}(b-a)(M-m);$$

(ii) *if $f(b) = f(a)$, then*

$$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(t)dt\right| \le \frac{1}{8}(b-a)(M-m);$$

(iii) *if we choose $m = \inf\limits_{x\in[a,b]} f'(x)$, $M = \sup\limits_{x\in[a,b]} f'(x)$, then*

$$\left| f(x) - \frac{1}{b-a}\int\limits_a^b f(t)dt - \frac{f(b) - f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)\|f'\|.$$

Note that for $f(b) = f(a)$ the left-hand side in (iii) is Ostrowski's classical expression. The right-hand side is in terms of $\|f'\|$; however, it is not pointwise. Note that $\dfrac{(x-a)^2 + (b-x)^2}{2(b-a)} = \dfrac{1}{4}(b-a)$ for $x = \dfrac{b+a}{2}$. The right-hand side in the theorem is of Grüss-type, i.e., it contains $M - m$, a difference of upper and lower bounds. It is thus justified to call an inequality, as given in the theorem, an Ostrowski – Grüss-type inequality (although, historically speaking, Grüss – Ostrowski-type inequality might be the more adequate term).

In [1] the first two authors gave a generalization of Ostrowski's inequality for arbitrary $f \in C[a,b]$ and certain linear operators. In order to formulate the result given here we need the following definition.

Definition 1. *Let $f \in C[a,b]$. If for $t \in [0,\infty)$ the quantity*

$$\omega(f;t) = \sup\{|f(x) - f(y)|\,,\ |x-y| \le t\}$$

is the usual modulus of continuity, its least concave majorant is given by

$$\tilde{\omega}(f;t) = \sup\left\{\frac{(t-x)\omega(f;y) + (y-t)\omega(f;x)}{y-x};\ 0 \le x \le t \le y \le b-a\right\}.$$

Let $I = [a,b]$ be a compact interval of the real axis and $f \in C(I)$. In [24] the following result for the least concave majorant is proved:

$$K\left(\frac{t}{2}, f; C[a,b], C^1[a,b]\right) := \inf_{g\in C^1(I)}\left(\|f-g\|_\infty + \frac{t}{2}\|g'\|_\infty\right) = \frac{1}{2}\tilde{\omega}(f;t), \quad t \ge 0.$$

Theorem F. *Let $L\colon C[a,b] \to C[a,b]$ be non-zero, linear and bounded, and such that $L\colon C^1[a,b] \to C^1[a,b]$ with $\|(Lg)'\| \le c_L\|g'\|$ for all $g \in C^1[a,b]$. Then for all $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$|Lf(x) - \mu(Lf)| \le \|L\|\tilde{\omega}\left(f; \frac{c_L}{\|L\|}\varphi(x)\right).$$

If $L = I_d$ is the identity on $C[a,b]$, then $\|L\| = c_L = 1$, and in this case we get

$$|f(x) - \mu(f)| \le \tilde{\omega}(f;\varphi(x)), \quad f \in C[a,b]. \tag{1}$$

***Remark* G.** If $f \in C^1[a,b]$, then the inequality (1) can be written as

$$|f(x) - \mu(f)| \le \tilde{\omega}(f;\varphi(x)) \le \varphi(x)\|f'\|_\infty.$$

This is Ostrowski's classical inequality in Anastassiou's form (see above). If $f \in \in \mathrm{Lip}_M\alpha$, $0 < \alpha \le 1$, then $|f(x) - \mu(f)| \le \tilde{\omega}(f;\varphi(x)) \le M(\varphi(x))^\alpha$. For $\alpha = 1$ we obtain Dragomir's inequality [6].

It is the aim of this paper to look again at Grüss' inequality from a somewhat different point of view, and to eventually relate it again to Ostrowski's inequality. In doing so we will be guided by the contribution of Andrica and Badea. That is: how non-multiplicative is a linear functional in the worst case? This is quite an intriguing question from the point of view of approximation theory.

2. A pre-Grüss inequality on a compact metric space. In 2004 A. Mc. D. Mercer and P. R. Mercer [17] gave the following pre-Grüss inequality for a positive linear functional $L\colon B(I) \to \mathbb{R}$, with $L(1) = 1$:

$$|L(fg) - L(f)L(g)| \le \frac{1}{2}\min\{(M-m)L(|g-G|), (P-p)L(|f-F|)\}, \tag{2}$$

where $m \le f(x) \le M$, $p \le g(x) \le P$ for all $x \in I$, $F := Lf$ and $G := Lg$.

In this section we will prove a pre-Grüss inequality on a compact metric space. Let $L\colon C(X) \to \mathbb{R}$ be a linear bounded functional, $L(1) = 1$, where $C(X)$ is a compact metric space with metric d. Then there are positive linear functionals L_+, L_-, $|L|$ such that $L = L_+ - L_-$ and $|L| = L_+ + L_-$. If L is a positive functional we have $|L| = L_+ = L$.

Since $M - m = \omega(f; d(X))$, $P - p = \omega(g; d(X))$, where $m = \inf f(x)$, $M = \sup f(x)$, $p = \inf g(x)$, $P = \sup g(x)$, we can prove, using the idea of A. Mercer and P. Mercer's proof, the following inequality:

Theorem 1. *Let $L\colon C(X) \to \mathbb{R}$ be a linear, bounded functional, $L(1) = 1$, defined on the compact metric space $C(X)$. Then the inequality*

$$|L(fg)-L(f)L(g)| \leq \frac{1}{2}\min\left\{\omega(f;d(X))|L|(|g-G|), \omega(g;d(X))|L|(|f-F|)\right\} \quad (3)$$

holds.

***Remark* 1.** The inequality is sharp in the sense that a non-positive functional A with $A(1) = 1$ exists such that equality occurs.

***Example* 1.** Let us consider the following non-positive functional

$$A\colon C[0,1] \to \mathbb{R}, \ A(f) = 2f(0) - f(1).$$

For this functional we have $A(1) = 1$, $A_+(f) = 2f(0)$, $A_-(f) = f(1)$, $|A|(f) = 2f(0) + f(1)$ and $A(fg) - A(f)A(g) = 2(f(1) - f(0))(g(0) - g(1))$. If we choose $f(t) = g(t) = t$, then $F = G = -1$ and

$$|A(fg) - A(f)A(g)| = 2 = \frac{1}{2}\min\left\{\omega(f;1)|A|(g+1), \omega(g,1)|A|(f+1)\right\}.$$

***Corollary* 1.** *If $L\colon C(X) \to \mathbb{R}$ is a positive linear (and thus bounded) functional with $L(1) = 1$, then for all $f, g \in C(X)$ we have*

$$|L(fg)-L(f)L(g)| \leq \frac{1}{2}\min\left\{\omega(f;d(X))L(|g-G|), \omega(g;d(X))L(|f-F|)\right\}, \quad (4)$$

$$|L(fg) - L(f)L(g)| \leq \frac{1}{4}\omega(f;d(X))\omega(g;d(X)). \quad (5)$$

Proof. Since L is a positive functional it follows $|L| = L$; so the first inequality is proved.

In [17] A. Mercer and P. Mercer show that the inequalities

$$L(|g-G|) \leq \frac{1}{2}(P-p) \text{ and } L(|f-F|) \leq \frac{1}{2}(M-m) \quad (6)$$

hold. The inequality (5) can be obtained by using in (4) the inequalities (6).

Corollary 1 is proved.

In [8], B. Gavrea and I. Gavrea raised the following problem.

Problem. *Let L be a linear positive functional defined on $C[0,1]$ with $L(1) = 1$ and f, g be two continuous functions. Do positive numbers $\delta_1 = \delta_1(f) < 1$ and $\delta_2 = \delta_2(f) < 1$ exist such that*

$$|L(fg) - L(f)L(g)| \leq \frac{1}{4}\tilde{\omega}(f;\delta_1)\tilde{\omega}(f;\delta_2)?$$

We will show that the answer to Gavreas' question is negative. Let us consider

$$L(f) = B_1\left(f; \frac{1}{2}\right) = \frac{1}{2}\left(f(0) + f(1)\right), \quad f \in C[0,1],$$

where B_1 is the first Bernstein operator on $C[0,1]$.

If we choose $f(t) = g(t) = t$ we have, with $e_i(t) := t^i$,

$$|L(fg) - L(f)L(g)| = |L(e_2) - L(e_1)^2| = \left|B_1\left(e_2; \frac{1}{2}\right) - B_1\left(e_1; \frac{1}{2}\right)^2\right| = \frac{1}{4}.$$

Moreover, for $0 \le t \le 1$, $\omega_1(e_1; t) = \tilde{\omega}(e_1; t) = t$, implying

$$\frac{1}{4}\tilde{\omega}(f;t)\tilde{\omega}(g;t) = \frac{1}{4}t^2 < \frac{1}{4} \text{ for } 0 \le t < 1.$$

Hence the conjecture of the two Gavreas is not true.

One more question is if the upper bound in (5) has a corresponding lower bound, i.e., if there is a constant $c > 0$ such that for all $f, g \in C(X)$ we also have

$$c\omega(f; d(X))\omega(g; d(X)) \le |L(f \cdot g) - L(f)L(g)|. \tag{7}$$

The following example shows that this is not the case.

***Example* 2.** Suppose $A\colon C(X) \to \mathbb{R}$ is a positive linear functional satisfying $A(1) = 1$. Write $D(f,g) := A(f \cdot g) - A(f)A(g)$.

***Case* 1:** supp $A = \{x\}$ for $x \in X$. Then $A = \delta_x$, the point evaluation functional at x. Hence $D(f,g) = 0$ for all $f, g \in C(X)$, and for appropriate choices of X, f and g the left-hand side of (7) is non-zero.

***Case* 2:** supp $A = \{x, y\}$, meaning that $A = \alpha \cdot \delta_x + \beta \cdot \delta_y$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Hence

$$D(f,g) = \alpha \cdot \beta(f(y) - f(x))(g(y) - g(x)) = 0$$

if and only if f or/and g is/are constant on supp A. Again for suitable choices of X, f and g the left-hand side of (7) is non-zero.

***Case* 3:** $|\text{supp } A| \ge 3$. Then there is an $h \in C(X)$ taking at least 3 distinct values on supp A. Let $a := A(h)$, $b := A(h^2)$, $c := A(h^3)$.

For all $t \in \mathbb{R}$ we have $(h - t \cdot 1)^2 \ge 0$, implying $A(h^2) - 2tA(h) + t^2 \ge 0$. Taking $t = A(h)$ shows that $A(h^2) \ge A^2(h)$. If $A(h^2) = A^2(h)$, then there is a $t_0 \in \mathbb{R}$ such that $A(h^2) - 2t_0A(h) + t_0^2 = 0$, i.e., $A((h - t_0 1)^2) = 0$. This implies that $h - t_0 1$ is constant on supp A, which is a contradiction. Thus $A(h^2) - A^2(h) = b - a^2 > 0$. Let $f := h - a$, $g := h^2 + \dfrac{ab - c}{b - a^2}h$. Then $A(f) = 0$, $A(f \cdot g) = 0$, and so $D(f,g) = 0$. Clearly f is non-constant on supp A. Assuming that $g = d$ is constant on A, means $h^2 + \dfrac{ab - c}{b - a^2}h = d$, or $h^2 + \dfrac{ab - c}{b - a^2}h - d = 0$ on supp A. But this means that h attains at most two values on supp A, again a contradiction. Thus f and g are both non-constant on supp A and again the right-hand side of (7) is non-zero.

3. Grüss-type inequalities for positive linear operators. Let $H_n\colon C[a,b] \to C[a,b]$ be positive linear operators which reproduce constant functions. For $x \in [a,b]$ we consider $L = \varepsilon_x \circ H_n$, so $L(f) = H_n(f; x)$. Denote by

$$D(f,g) := H_n(fg;x) - H_n(f;x)H_n(g;x).$$

The following result suggests how non-multiplicative the functional $L(f) = H_n(f;x)$ is for a given $x \in [a,b]$.

Theorem 2. *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right)\tilde{\omega}\left(g; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right)$$

holds.

Proof. Using the Cauchy – Schwarz inequality for positive linear functionals we can write

$$|H_n(f;x)| \le H_n(|f|;x) \le \sqrt{H_n(f^2;x)H_n(1;x)} = \sqrt{H_n(f^2;x)},$$

so

$$D(f,f) = H_n(f^2;x) - H_n(f;x)^2 \ge 0.$$

Then D is a positive semidefinite bilinear form on $C[a,b]$. For $f, g \in C[a,b]$, using Cauchy – Schwarz for D, it follows that

$$|D(f,g)| \le \sqrt{D(f,f)D(g,g)} \le \|f\|_\infty \|g\|_\infty. \tag{8}$$

Since $H_n\colon C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, $H_n(f;x)$, with $x \in [a,b]$ fixed, is a positive linear functional and can be represented as $H_n(f;x) = \int_a^b f(t)d\mu(t)$, where μ is a probability measure on $[a,b]$, i.e., $\int_a^b d\mu(t) = 1$.

We have

$$H_n(f^2;x) - H_n(f;x)^2 = \int_a^b f^2(t)d\mu(t) - \left(\int_a^b f(s)d\mu(s)\right)^2 =$$

$$= \int_a^b \left(f(t) - \int_a^b f(s)d\mu(s)\right)^2 d\mu(t) = \int_a^b \left(\int_a^b (f(t)-f(s))d\mu(s)\right)^2 d\mu(t) \le$$

$$\le \int_a^b \left(\int_a^b (f(t)-f(s))^2 d\mu(s)\right) d\mu(t) \le \|f'\|_\infty^2 \int_a^b \left(\int_a^b (t-s)^2 d\mu(s)\right) d\mu(t) =$$

$$= \|f'\|_\infty^2 \int_a^b \left(t^2 - 2t\int_a^b s d\mu(s) + \int_a^b s^2 d\mu(s)\right) d\mu(t) =$$

$$= \|f'\|_\infty^2 \left[\int_a^b t^2 d\mu(t) - 2\int_a^b s d\mu(s) \int_a^b t d\mu(t) + \int_a^b s^2 d\mu(s)\right] =$$

$$= 2\|f'\|_\infty^2 \left[H_n(e_2;x) - H_n(e_1;x)^2\right] \le 2\|f'\|_\infty^2 H_n\left((e_1-x)^2;x\right), \quad f \in C^1[a,b].$$

Therefore

$$D(f,f) = H_n(f^2;x) - H_n(f;x)^2 \leq 2\|f'\|_\infty^2 H_n\left((e_1-x)^2;x\right). \tag{9}$$

Using relation (9) for differentiable functions $r, s \in C^1[a,b]$, we obtain the following estimate:

$$|D(r,s)| \leq \sqrt{D(r,r)D(s,s)} \leq 2\|r'\|_\infty \|s'\|_\infty H_n\left((e_1-x)^2;x\right). \tag{10}$$

Moreover, if $f \in C[a,b]$, $s \in C^1[a,b]$, then

$$|D(f,s)| \leq \sqrt{D(f,f)D(s,s)} \leq \|f\|_\infty \sqrt{2}\|s'\|_\infty \sqrt{H_n\left((e_1-x)^2;x\right)}. \tag{11}$$

Likewise, for $r \in C^1[a,b]$, $g \in C[a,b]$, we have

$$|D(r,g)| \leq \|g\|_\infty \sqrt{2}\|r'\|_\infty \sqrt{H_n\left((e_1-x)^2;x\right)}. \tag{12}$$

Now let $f, g \in C[a,b]$ be fixed, $r, s \in C^1[a,b]$ arbitrary. Then

$$|D(f,g)| = |D(f-r+r, g-s+s)| \leq$$

$$\leq |D(f-r,g-s)| + |D(f-r,s)| + |D(r,g-s)| + |D(r,s)| \leq$$

$$\leq \|f-r\|\|g-s\| + \sqrt{2}\|f-r\|\|s'\|\sqrt{H_n\left((e_1-x)^2;x\right)} +$$

$$+\sqrt{2}\|g-s\|\|r'\|\sqrt{H_n\left((e_1-x)^2;x\right)} + 2\|r'\|\|s'\|H_n\left((e_1-x)^2;x\right) =$$

$$= \left\{\|f-r\| + \|r'\|\sqrt{2H_n\left((e_1-x)^2;x\right)}\right\}\left\{\|g-s\| + \|s'\|\sqrt{2H_n\left((e_1-x)^2;x\right)}\right\}.$$

Passing to the infimum over r and $s \in C^1[a,b]$, respectively, shows

$$|D(f,g)| \leq K\left(\sqrt{2H_n\left((e_1-x)^2;x\right)}, f; C^0, C^1\right) \times$$

$$\times K\left(\sqrt{2H_n\left((e_1-x)^2;x\right)}, g; C^0, C^1\right) =$$

$$= \frac{1}{2}\tilde{\omega}\left(f; \sqrt{8H_n\left((e_1-x)^2;x\right)}\right)\frac{1}{2}\tilde{\omega}\left(g; \sqrt{8H_n\left((e_1-x)^2;x\right)}\right) =$$

$$= \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right)\tilde{\omega}\left(g; 2\sqrt{2H_n\left((e_1-x)^2;x\right)}\right),$$

which concludes the proof.

***Remark* 2.** If we choose $H_n = B_n$, the Bernstein operator, then this gives

$$|B_n(fg;x) - B_n(f;x)B_n(g;x)| \leq$$

$$\leq \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{2B_n\left((e_1-x)^2;x\right)}\right)\tilde{\omega}\left(g; 2\sqrt{2B_n\left((e_1-x)^2;x\right)}\right) =$$

$$= \frac{1}{4}\tilde{\omega}\left(f; 2\sqrt{\frac{2x(1-x)}{n}}\right)\tilde{\omega}\left(g; 2\sqrt{\frac{2x(1-x)}{n}}\right) \leq$$

$$\leq \tilde{\omega}\left(f;\frac{1}{\sqrt{2n}}\right)\tilde{\omega}\left(g;\frac{1}{\sqrt{2n}}\right), \quad f,g\in C[0,1].$$

***Remark* 3.** The above result can be remarkably generalized if we replace $([a,b],|\cdot|)$ by a compact metric space (X,d), $H_n((e_1-x)^2;x)$ by $H_n(d^2(\cdot,x);x)$, and $K(\cdot,f;C[a,b],C^1[a,b])$ by $K(\cdot,f;C(X),\mathrm{Lip}1)$.

4. Grüss-type inequality for the classical Hermite – Fejér interpolation operator. The classical Hermite – Fejér interpolation operator is a positive linear operator and can be written as

$$L_n(f;x)=\sum_{k=1}^{n} f(x_k)(1-xx_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2, \tag{13}$$

where $f\in C[-1,1]$ and $x_k=\cos\dfrac{2k-1}{2n}\pi$, $1\leq k\leq n$, are the zeros of $T_n(x)=$ $=\cos(n\arccos)$, the n-th Chebyshev polynomial of the first kind.

For this operator we have $L_n\left((e_1-x)^2;x\right)=\dfrac{1}{n}T_n^2(x)$.

***Remark* 4.** If we choose in Theorem 2 $H_n=L_n$, the classical Hermite – Fejér interpolation operator, then this gives

$$|L_n(fg;x)-L_n(f;x)L_n(g;x)|\leq\frac{1}{4}\tilde{\omega}\left(f;\frac{2\sqrt{2}}{\sqrt{n}}|T_n(x)|\right)\tilde{\omega}\left(g;\frac{2\sqrt{2}}{\sqrt{n}}|T_n(x)|\right). \tag{14}$$

This is disappointing in view of the fact that L_n approximates much faster than B_n. Indeed, in [9] the following pointwise inequality was proved:

$$|L_n(f;x)-f(x)|\leq 5\omega_1\left(f;\frac{|T_n(x)|}{n}\left\{\sqrt{1-x^2}\ln n+1\right\}\right).$$

In this section we will give a different approach adapted to the Hermite – Fejér case. Denote by

$$D(f,g):=L_n(fg;x)-L_n(f;x)L_n(g;x).$$

Theorem 3. *If $f,g\in C[-1,1]$ and $x\in[-1,1]$ is fixed, then the following inequality is verified*:

$$|D(f,g)|\leq\frac{1}{2}\min\left\{\|f\|_\infty\tilde{\omega}\left(g;\frac{40\ln n}{n}\right),\|g\|_\infty\tilde{\omega}\left(f;\frac{40\ln n}{n}\right)\right\}. \tag{15}$$

Proof. For $f\in C[-1,1]$, $s\in C^1[-1,1]$ proceed as follows:

$$|D(f,s)|=|L_n(f\cdot s;x)-L_n(f;x)L_n(s;x)|=|L_n(f(s-L_n(s;x));x)|=$$

$$=|L_n^t(f(t)(s(t)-s(x)+s(x)-L_n(s;x));x)|\leq$$

$$\leq\|f\|_\infty L_n^t(|s(t)-s(x)|+|s(x)-L_n(s;x)|;x)\leq$$

$$\leq\|f\|_\infty L_n(\|s'\||e_1-x|+\|s'\|L_n(|e_1-x|;x);x)=$$

$$=2\|f\|_\infty\|s'\|_\infty L_n(|e_1-x|;x).$$

Now, for $f, g \in C[-1,1]$ fixed and $s \in C^1[-1,1]$ arbitrary we get

$$|D(f,g)| = |D(f, g-s+s)| \le |D(f, g-s)| + |D(f,s)| \le$$

$$\le \|f\|_\infty \|g-s\|_\infty + 2\|f\|_\infty \|s'\|_\infty L_n(|e_1 - x|; x) =$$

$$= \|f\|_\infty \left\{ \|g-s\|_\infty + 2L_n(|e_1-x|;x)\|s'\|_\infty \right\}.$$

Passing to the infimum over $s \in C^1$ yields

$$|D(f,g)| \le \|f\|_\infty K(2L_n(|e_1-x|;x), g; C^0, C^1) =$$

$$= \|f\|_\infty \frac{1}{2}\tilde{\omega}(g, 4L_n(|e_1-x|;x)).$$

By symmetry the same holds with f and g interchanged. Hence

$$|D(f,g)| \le \frac{1}{2}\min\{\|f\|_\infty \tilde{\omega}(g, 4L_n(|e_1-x|;x)); \|g\|_\infty \tilde{\omega}(f, 4L_n(|e_1-x|;x))\}.$$

But in [9] it was proved that (see the appendix for a detailed proof)

$$L_n(|e_1-x|;x) \le \frac{4}{n}|T_n(x)|(\sqrt{1-x^2}\ln n + 1) \le 10\frac{\ln n}{n}, \quad n \ge 2, \qquad (16)$$

and so

$$|D(f,g)| \le \frac{1}{2}\min\left\{\|f\|_\infty \tilde{\omega}\left(g; \frac{40\ln n}{n}\right), \|g\|_\infty \tilde{\omega}\left(f; \frac{40\ln n}{n}\right)\right\}.$$

***Remark* 5.** If one of the functions f or g is in Lip1, we have $|D(f,g)| = \mathcal{O}\left(\frac{\ln n}{n}\right)$, $n \to \infty$. The relation (14) implies in this case only $|D(f,g)| = o\left(\frac{1}{\sqrt{n}}\right)$. Also, the relation (14) implies $|D(f,g)| = o\left(\frac{1}{n}\right)$ for $f, g \in \operatorname{Lip} 1$. This cannot be concluded from (15).

5. A Grüss inequality for convolution-type operators.

Definition 2. *For every function $f \in C(I)$, $I = [-1,1]$, and any natural number n, the operator $G_{m(n)}$ is defined by*

$$G_{m(n)}(f,t) := \pi^{-1}\int_{-\pi}^{\pi} f\left(\cos(\arccos t + v)\right) K_{m(n)}(v)dv,$$

where the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ with the following properties:

(i) $K_{m(n)}$ *is positive and even*;

(ii) $\int_{-\pi}^{\pi} K_{m(n)}(v)dv = \pi$, *i.e.*, $G_{m(n)}(1,t) = 1$ *for* $t \in I$.

For each $f \in C(I)$ the integral $G_{m(n)}(f,\cdot)$ from Definition 2 is an algebraic polynomial of degree $m(n)$. Moreover, in view of (i) and (ii) one has

$$K_{m(n)}(v) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cos kv, \quad v \in [-\pi, \pi].$$

Lemma 1 [15]. *For $x \in I$ the inequality*

$$G_{m(n)}\left((e_1-x)^2,x\right)=x^2\left\{\frac{3}{2}-2\rho_{1,m(n)}+\frac{1}{2}\rho_{2,m(n)}\right\}+(1-x^2)\left\{\frac{1}{2}-\frac{1}{2}\rho_{2,m(n)}\right\}$$

holds. Here e_1 denotes the first monomial given by $e_1(t)=t$ for $|t|\le 1$.

If $K_{m(n)}$ is the Fejér – Korovkin kernel with $m(n)=n-1$, then it is known that (see [16])

$$\rho_{1,n-1}=\cos\frac{\pi}{n+1},\quad \rho_{2,n-1}=\frac{n}{n+1}\cos\frac{2\pi}{n+1}+\frac{1}{n+1}. \tag{17}$$

Using the relations (17) we get

$$G_{n-1}\left((e_1-x)^2;x)\right)\le\left|\frac{3}{2}-2\rho_{1,n-1}+\frac{1}{2}\rho_{2,n-1}\right|+\frac{1}{2}\left|1-\rho_{2,n-1}\right|\le$$

$$\le 3\left(\frac{\pi}{n+1}\right)^2+\left(\frac{\pi}{n+1}\right)^2=4\left(\frac{\pi}{n+1}\right)^2.$$

***Remark* 6.** If we consider in Theorem 2 the convolution-type operators with the Fejér – Korovkin kernel we have

$$|D(f;g)|=|G_{n-1}(fg;x)-G_{n-1}(f;x)G_{n-1}(g;x)|\le$$

$$\le\frac{1}{4}\tilde{\omega}\left(f;4\sqrt{2}\frac{\pi}{n+1}\right)\tilde{\omega}\left(g;4\sqrt{2}\frac{\pi}{n+1}\right)=\mathcal{O}\left(\tilde{\omega}\left(f;\frac{1}{n}\right)\tilde{\omega}\left(g;\frac{1}{n}\right)\right).$$

This is an improvement of what we obtained for the Bernstein and Hermite – Fejér operators.

6. Estimates via Cauchy's mean value theorem. Let $L\colon C[a,b]\to\mathbb{R}$ be a linear positive functional. We denote by

$$T(f,g)=L(fg)-L(f)L(g),\quad f,g\in C[a,b].$$

In this section our aim is to establish a Grüss inequality for the functional L using Cauchy's mean value theorem. Our work is motivated by B.G. Pachpatte's result obtained in [23] for the functional $L(f)=\dfrac{1}{\int_a^b w(x)dx}\displaystyle\int_a^b w(x)f(x)dx$, where $w\colon[a,b]\to[0,\infty)$ is an integrable function such that $\displaystyle\int_a^b w(x)dx>0$.

Theorem 4. *If $L\colon C[a,b]\to\mathbb{R}$ is a linear positive functional, with $L(1)=1$, then*

i) *there is $(\eta,\theta)\in[a,b]\times[a,b]$ such that $T(f,g)=\dfrac{f'(\eta)}{h'(\eta)}\dfrac{g'(\theta)}{h'(\theta)}T(h,h)$;*

ii) $|T(f,g)|\le\left\|\dfrac{f'}{h'}\right\|_\infty\left\|\dfrac{g'}{h'}\right\|_\infty|T(h,h)|$, *where $f,g,h\in C^1[a,b]$ and $h'(t)\ne 0$ for each $t\in[a,b]$.*

Proof. Let $x,y\in[a,b]$ with $y\ne x$. Applying Cauchy's mean value theorem, there exist points ξ_1 and ξ_2 between y and x such that

$$f(x) - f(y) = \frac{f'(\xi_1)}{h'(\xi_1)}(h(x) - h(y)), \tag{18}$$

$$g(x) - g(y) = \frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y)). \tag{19}$$

Multiplying the left-hand sides and right-hand sides of (18) and (19), we get

$$(f(x) - f(y))(g(x) - g(y)) = \frac{f'(\xi_1)}{h'(\xi_1)}\frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2.$$

If we apply the functional L with respect to x and y it follows

$$2T(f,g) = L_y L_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}\frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2\right). \tag{20}$$

If we denote by

$$m = \min_{(x,y)\in[a,b]\times[a,b]}\frac{f'(x)}{h'(x)}\frac{g'(y)}{h'(y)},$$

$$M = \max_{(x,y)\in[a,b]\times[a,b]}\frac{f'(x)}{h'(x)}\frac{g'(y)}{h'(y)},$$

then we can write $m \leq \dfrac{f'(\xi_1)}{h'(\xi_1)}\dfrac{g'(\xi_2)}{h'(\xi_2)} \leq M$, namely

$$m(h(x) - h(y))^2 \leq \frac{f'(\xi_1)}{h'(\xi_1)}\frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2 \leq M(h(x) - h(y))^2.$$

If apply the functional L with respect to x and y, we get

$$2mT(h,h) \leq L_y L_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}\frac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2\right) \leq 2MT(h,h).$$

Since

$$m \leq \frac{L_y L_x\left(\dfrac{f'(\xi_1)}{h'(\xi_1)}\dfrac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2\right)}{2T(h,h)} \leq M,$$

it follows that there is $(\eta,\theta) \in [a,b]\times[a,b]$ such that

$$\frac{L_y L_x\left(\dfrac{f'(\xi_1)}{h'(\xi_1)}\dfrac{g'(\xi_2)}{h'(\xi_2)}(h(x) - h(y))^2\right)}{2T(h,h)} = \frac{f'(\eta)}{h'(\eta)}\frac{g'(\theta)}{h'(\theta)}.$$

Using the above relation in (20), it follows

$$T(f,g) = \frac{f'(\eta)}{h'(\eta)}\frac{g'(\theta)}{h'(\theta)}T(h,h). \tag{21}$$

From (21) we have

$$|T(f,g)| \leq \left\|\frac{f'}{h'}\right\|_\infty \left\|\frac{g'}{h'}\right\|_\infty |T(h,h)|.$$

Theorem 4 is proved.

***Remark* 7.** If in Theorem 4 we take $h(x) = x$, $x \in [a,b]$, and $L(f) = \frac{1}{b-a}\int_a^b f(x)dx$, then

(i) there is $(\eta,\theta) \in [a,b]\times[a,b]$ such that

$$\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int\limits_a^b f(x)dx\int\limits_a^b g(x)dx = \frac{(b-a)^2}{12}f'(\eta)g'(\theta),$$

this identity was found by Ostrowski [21] in 1970;

(ii) $$\left|\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int\limits_a^b f(x)dx\int\limits_a^b g(x)dx\right| \le$$

$$\le \frac{(b-a)^2}{12}\sup_{x\in[a,b]}|f'(x)|\sup_{x\in[a,b]}|g'(x)|;$$

this inequality was proved by Čebyšev [4] in 1882, the constant $\frac{(b-a)^2}{12}$ is best possible, as can be seen for $[a,b]=[0,1]$, $f(x)=g(x)=x$.

Theorem 5. *If $L\colon C[a,b]\to\mathbb{R}$ is a linear positive functional, with $L(1)=1$, then the following inequality holds*:

$$|T(f,h)+T(g,h)| \le |T(h,h)|\left(\left\|\frac{f'}{h'}\right\|_\infty + \left\|\frac{g'}{h'}\right\|_\infty\right),$$

where $f,g,h\in C^1[a,b]$ and $h'(t)\neq 0$ for each $t\in[a,b]$.

Proof. Multiplying both sides of (18) and (19) by $h(x)-h(y)$ and adding the resulting identities we get

$$(f(x)-f(y))(h(x)-h(y)) + (g(x)-g(y))(h(x)-h(y)) =$$

$$= \frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))^2 + \frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2.$$

If we apply the functional L with respect to x and y, we get

$$2T(f,h)+2T(g,h) = L_yL_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))^2\right) +$$

$$+L_yL_x\left(\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2\right). \tag{22}$$

In a similar way with the proof of Theorem 4 it can be shown that there are $\eta,\theta\in[a,b]$ such that

$$L_yL_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))^2\right) = 2T(h,h)\frac{f'(\eta)}{h'(\eta)},$$

$$L_yL_x\left(\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))^2\right) = 2T(h,h)\frac{g'(\theta)}{h'(\theta)}.$$

Using the above identities in (22), we get

$$T(f,h)+T(g,h)=\left(\frac{f'(\eta)}{h'(\eta)}+\frac{g'(\theta)}{h'(\theta)}\right)T(h,h).$$

Therefore

$$|T(f,h)+T(g,h)|\le\left(\left\|\frac{f'}{h'}\right\|_\infty+\left\|\frac{g'}{h'}\right\|_\infty\right)|T(h,h)|.$$

Theorem 5 is proved.

In the paper [21] Ostrowski defined the concept of synchronous functions. The functions $f,g\colon [a,b]\to\mathbb{R}$ are called synchronous, if we have, for any couple of points x,y from $[a,b]$, $f(x)\ge f(y)$ if and only if $g(x)\ge g(y)$.

In the case that f,g are synchronous, we get $T(f,g)\ge 0$.

Theorem 6. *If $L\colon C[a,b]\to\mathbb{R}$ is a linear positive functional, with $L(1)=1$, then the following inequality is verified*:

$$|T(f,g)|\le\frac{1}{2}\left[\left\|\frac{f'}{h'}\right\|_\infty|T(g,h)|+\left\|\frac{g'}{h'}\right\|_\infty|T(f,h)|\right], \tag{23}$$

where $f,g,h\in C^1[a,b]$, $h'(t)\ne 0$ for each $t\in[a,b]$ and the functions f,g, respectively g,h are synchronous.

Proof. Multiplying both sides of (18) and (19) by $g(x)-g(y)$ and $f(x)-f(y)$, respectively, adding the resulting identities, and applying the functionals L with respect to x and y, we get

$$4T(f,g)=L_yL_x\left(\frac{f'(\xi_1)}{h'(\xi_1)}(h(x)-h(y))(g(x)-g(y))\right)+$$

$$+L_yL_x\left(\frac{g'(\xi_2)}{h'(\xi_2)}(h(x)-h(y))(f(x)-f(y))\right).$$

Using this identity and the reasoning from the proof of the above theorems inequality (23) follows.

Theorem 6 is proved.

7. Grüss inequality for more than two functions. In this section we will prove a Grüss inequality on a compact metric space for more than two functions.

Lemma 2. *Let $C(X)$ be a compact metric space and $f_k\in C(X)$, $1\le k\le n$, $n\ge 1$. Then the following inequality holds*:

$$\theta(f_1f_2\ldots f_n)\le\sum_{i=1}^{n}\theta(f_i)\prod_{k=1,k\ne i}^{n}\|f_k\|_\infty, \tag{24}$$

where $\theta(f):=\max\limits_X f-\min\limits_X f,\ f\in C(X)$.

Proof. Inequality (24) can be proved using induction.

Theorem 7. *Let $A\colon C(X)\to\mathbb{R}$ be a positive linear functional, $A(1)=1$, defined on the metric space $C(X)$. The inequality*

$$|A(f_1f_2\ldots f_n)-A(f_1)A(f_2)\ldots A(f_n)|\le\frac{1}{4}\sum_{i,j=1,i<j}^{n}\theta(f_i)\theta(f_j)\prod_{k=1,k\ne i,j}^{n}\|f_k\|_\infty \tag{25}$$

holds.

Proof. The inequality (25) can be proved using induction and relation (24).

***Remark* 8.** If $f_3, \ldots, f_n$ are constant, relation (25) reduces to

$$|A(f_1f_2) - A(f_1)A(f_2)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where $M_i = \max\limits_X f_i$, $m_i = \min\limits_X f_i$, $i \in \{1, 2\}$.

The following result is an extension of Ostrowski's inequality (1):

Theorem 8. *If $f_i \in C[a, b]$, $1 \leq i \leq n$, then the following inequality is true*:

$$|f_1(x) \ldots f_n(x) - \mu(f_1) \ldots \mu(f_n)| \leq \sum_{i=1}^{n} \tilde{\omega}(f_i; \varphi(x)) \prod_{k=1, k \neq i}^{n} \|f_k\|_\infty,$$

where $\mu(f) := \dfrac{1}{b-a} \displaystyle\int_a^b f(t)dt, \ \varphi(x) := \dfrac{(x-a)^2 + (b-x)^2}{2(b-a)}$.

Proof. The inequality can be proved using induction.

***Remark* 9.** Since $\varphi(x) \leq \dfrac{b-a}{2}$, $x \in [a, b]$, we get

$$|f_1(x) \ldots f_n(x) - \mu(f_1) \ldots \mu(f_n)| \leq \sum_{i=1}^{n} \tilde{\omega}\left(f_i; \frac{b-a}{2}\right) \prod_{k=1, k \neq i}^{n} \|f_k\|_\infty.$$

This relation yields the Grüss-type inequality

$$|\mu(f_1 \ldots f_n) - \mu(f_1) \ldots \mu(f_n)| \leq \sum_{i=1}^{n} \tilde{\omega}\left(f_i; \frac{b-a}{2}\right) \prod_{k=1, k \neq i}^{n} \|f_k\|_\infty.$$

Appendix. Here we give the proof of inequality (16) for the classical Hermite–Fejér interpolation operator based on the roots of Čebyšev polynomials of the first kind, defined in (13).

Lemma 3. *For the continuous function $|e_1 - x|$, $x \in [-1, 1]$ fixed, and $n \geq 1$ one has*

$$L_n(|e_1 - x|, x) \leq \frac{c}{n} |T_n(x)| \left\{(1 - x^2)^{1/2} \ln n + 1\right\}.$$

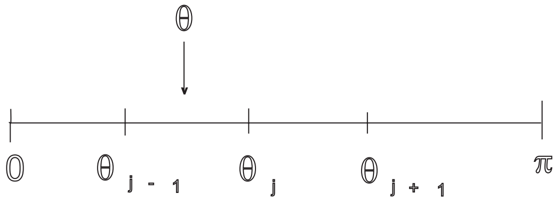
Here $e_1(t) = t$ for $|t| \leq 1$, and c is a constant independent of n and x and satisfying $1 \leq c \leq 4$.

Proof. The existence of a constant c independent of n and x in the above estimate can be derived from papers of R. N. Misra [18] or S. J. Goodenough and T. M. Mills [10]. In order to show that c is bounded from above by 4 we give a proof similar to that of Goodenough and Mills using the technique of O. Kiš [13].

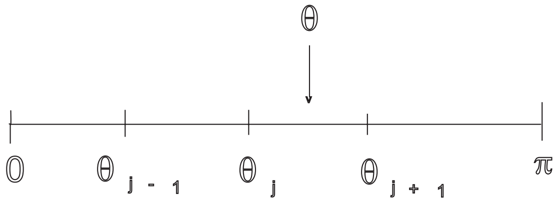
Let $x \in [-1, 1]$ be fixed. We may assume that $x \neq x_k$, $1 \leq k \leq n$, since otherwise the estimate is apparently true.

For $n = 1$ we have $x_1 = 0$ such that $L_1(|e_1 - x|; x) = |x| = |T_1(x)|$. Hence the estimate holds for $n = 1$. We assume in the sequel that $n \geq 2$. Because of $x = \cos\theta$, $0 \leq \theta \leq \pi$, and $x_k = \cos\theta_k$ with $\theta_k = (2k-1)(2n)^{-1}\pi$, we may proceed as follows. Let j be chosen in a way such that θ_j is closest to θ among all θ_k's (if θ has the same distance from θ_k and θ_{k+1}, say, we may choose either of them). Thus the following situations may occur:

"Left" case:



"Right" case:



Note that in the "left" case θ_{j-1} need not exist (θ close to 0); a corresponding remark applies in the "right" case to θ_{j+1} (θ close to π).

After fixing j as described above, we write

$$L_n(|e_1 - x|, x) = \sum_{k=1}^{n} |x_k - x| \frac{(1 - x x_k) T_n^2(x)}{n^2 (x - x_k)^2} =: \sum_{k=1}^{n} |x_k - x| h_k(x) =: \sum_{k=1}^{n} W_k(x) =$$

$$= \sum_{k=1}^{j-1} W_k(x) + W_j(x) + \sum_{k=j+1}^{n} W_k(x) =: I_1 + I_2 + I_3.$$

Clearly, if $j = 1$ or $j = n$, then one of the two sums will not be present. First observe that for $1 \leq k \leq n$ we have

$$h_k(x) = \frac{(1 - x x_k) T_n^2(x)}{n^2 (x - x_k)^2} = \frac{(1 - x^2) T_n^2(x)}{n^2 (x - x_k)^2} + \frac{x T_n^2(x)}{n^2 (x - x_k)},$$

which implies

$$W_k(x) = |x_k - x| h_k(x) \leq \frac{(1 - x^2) T_n^2(x)}{n^2 |x - x_k|} + \frac{|x| T_n^2(x)}{n^2}.$$

While the second term in this upper bound does not cause difficulties, the first one may be written in the following way:

$$\frac{(1 - x^2) T_n^2(x)}{n^2 |x - x_k|} = \frac{\sqrt{1 - x^2}\, T_n^2(x)}{n^2} \frac{\sin \theta}{|\cos \theta - \cos \theta_k|}.$$

Now only the second ratio requires further consideration. First observe that

$$\frac{\sin \theta}{|\cos \theta - \cos \theta_k|} \frac{1}{|\sin \frac{1}{2}(\theta - \theta_k)|} \leq \frac{\pi}{|\theta - \theta_k|};$$

here the first inequality may be obtained by using Lemma 2 (a) in Goodenough's and Mills' paper [10], while the second one is a consequence of $\alpha \leq \dfrac{1}{2}\pi \sin \alpha$, $0 \leq \alpha \leq \dfrac{1}{2}\pi$.

Consequently, it remains to investigate the quantities $1/|\theta - \theta_k|$. It is at this point that we check the "left" and the "right" case separately. In both cases we first estimate

$$I_1 + I_3 = \sum_{k=1}^{j-1} W_k(x) + \sum_{k=j+1}^{n} W_k(x) \text{ and add } W_j(x) \text{ afterwards;}$$

we have

$$I_1 + I_3 = \sum_{k=1}^{j-1} W_k(x) + \sum_{k=j+1}^{n} W_k(x) \leq$$

$$\leq \frac{\sqrt{1-x^2}T_n^2(x)}{n^2}\pi \left(\sum_{k=1}^{j-1} \frac{1}{|\theta - \theta_k|} + \sum_{k=j+1}^{n} \frac{1}{|\theta - \theta_k|} \right) + (n-1)\frac{|x|T_n^2(x)}{n^2}.$$

For the "left" case (i.e., $\theta_{j-1} < \theta < \theta_j$) we have for $1 < j < n$ and $k \leq j-1$

$$\theta - \theta_k \geq (\theta_{j-1} - \theta_k) + \frac{1}{2}(\theta_j - \theta_{j-1}) = (2i-1)(2n)^{-1}\pi, \text{ if } k = j - i,$$

and for $k \geq j+1$

$$\theta_k - \theta \geq \theta_k - \theta_j = in^{-1}\pi, \text{ if } k = j + i.$$

In this case

$$\sum_{k=1}^{j-1} \frac{1}{|\theta - \theta_k|} + \sum_{k=j+1}^{n} \frac{1}{|\theta - \theta_k|} \leq 2n\pi^{-1} \left(\sum_{k=1}^{j-1} \frac{1}{2k-1} + \sum_{k=1}^{n-j} \frac{1}{2k} \right) \leq$$

$$\leq 2n\pi^{-1} \left[1 + \frac{1}{2}\ln(2j-3) + \frac{1}{2}\left(1 + \ln(n-j)\right) \right] \leq 2n\pi^{-1} \left[\frac{3}{2} + \ln\left(\frac{1}{\sqrt{2}} n \right) \right].$$

Note that this estimate is also true if $j = 1$ or $j = n$. For the "right" case (i.e., $\theta_j < \theta <$ $< \theta_{j+1}$) our estimate for $1 < j < n$ and $k \leq j-1$ is

$$\theta - \theta_k \geq \theta_j - \theta_k = in^{-1}\pi, \text{ if } k = j - i$$

and for $k \geq j+1$

$$\theta_k - \theta \geq \theta_k - \theta_{j+1} + \frac{1}{2}(\theta_{j+1} - \theta_j) = (2i-1)(2n)^{-1}\pi, \text{ if } k = j + i.$$

Thus for the "right" case we arrive at the symmetric inequality

$$\sum_{k=1}^{j-1} \frac{1}{|\theta - \theta_k|} + \sum_{k=j+1}^{n} \frac{1}{|\theta - \theta_k}| \leq 2n\pi^{-1} \left(\sum_{k=1}^{j-1} \frac{1}{2k} + \sum_{k=1}^{n-j} \frac{1}{2k-1} \right) \leq$$

$$\leq 2n\pi^{-1} \left[\frac{1}{2} + \frac{1}{2}\ln(j-1) + 1 + \frac{1}{2}\ln(2(n-j)-1) \right] \leq 2n\pi^{-1} \left[\frac{3}{2} + \ln\left(\frac{1}{\sqrt{2}} n \right) \right].$$

Note that this inequality is also true for $j = 1$ or $j = n$.

The common estimate obtained for both the "left" and the "right" cases is thus

$$I_1 + I_3 \le \frac{\sqrt{1-x^2}T_n^2(x)}{n^2}\pi\left(\sum_{k=1}^{j-1}\frac{1}{|\theta-\theta_k|} + \sum_{k=j+1}^{n}\frac{1}{|\theta-\theta_k|}\right) + (n-1)\frac{|x|T_n^2(x)}{n^2} \le$$

$$\le \frac{\sqrt{1-x^2}T_n^2(x)}{n^2}\pi 2n\pi^{-1}\left(\frac{3}{2} + \ln\left(\frac{1}{\sqrt{2}}n\right)\right) + \frac{|x|T_n^2(x)}{n} \le$$

$$\le \frac{\sqrt{1-x^2}T_n^2(x)}{n}\left(3 + 2\ln\left(\frac{1}{\sqrt{2}}n\right)\right) + \frac{|x|T_n^2(x)}{n}.$$

Using Goodenough's and Mills' [10] Lemma 3 we also have that

$$I_2 = W_j(x) = |x_j - x|h_j(x) \le \pi(2n)^{-1}|\cos n\theta| = \pi(2n)^{-1}|T_n(x)|,$$

so that for $n \ge 2$ the following inequality holds:

$$I_1 + I_2 + I_3 \le \frac{\sqrt{1-x^2}T_n^2(x)}{n}\left(3 + 2\ln\left(\frac{1}{\sqrt{2}}n\right)\right) + \frac{|x|T_n^2(x)}{n} + \frac{\pi}{2n}|T_n(x)| =$$

$$= \frac{\sqrt{1-x^2}T_n^2(x)}{n}\left(2 + 2\ln\left(\frac{1}{\sqrt{2}}n\right)\right) + \frac{\sqrt{1-x^2}T_n^2(x)}{n} +$$

$$+\frac{|x|T_n^2(x)}{n} + \frac{\pi}{2n}|T_n(x)| \le$$

$$\le \frac{|T_n(x)|}{n}\left[\sqrt{1-x^2}\left(2+2\ln\left(\frac{1}{\sqrt{2}}n\right)\right) + \sqrt{1-x^2}|T_n(x)| + |x||T_n(x)| + \frac{\pi}{2}\right] \le$$

$$\le \frac{|T_n(x)|}{n}\left[\sqrt{1-x^2}\left(2 + 2\ln\left(\frac{1}{\sqrt{2}}n\right)\right) + 2 + \frac{\pi}{2}\right] \le$$

$$\le \frac{|T_n(x)|}{n}\left[\sqrt{1-x^2}4\ln n + 4\right] = 4\frac{|T_n(x)|}{n}\left[\sqrt{1-x^2}\ln n + 1\right].$$

In order to show that $c \ge 1$, it is only necessary to evaluate the left and the right-hand side of the inequality in the above claim at the point $x = 1$, say. We have

$$L_n(|e_1 - 1|, 1) = -L_n(e_1 - 1, 1) = \frac{1}{n}T_n(1)T_{n-1}(1) = \frac{1}{n}.$$

Using the same point on the right-hand side shows that

$$\frac{c}{n}|T_n(1)|\left\{(1-1)^{1/2}\ln n + 1\right\} = \frac{c}{n}$$

and thus $c \ge 1$.

Lemma 3 is proved.

***Remark* 10.** Numerical evidence suggests that the constant c in Lemma 3 equals 1. However, it does not seem to be possible to use the technique of O. Kiš to obtain such a good estimate.

1. *Acu A. M., Gonska H.* Ostrowski-type inequalities and moduli of smoothness // Results Math. – 2009. – **53**. – P. 217 – 228.

2. *Andrica D., Badea C.* Grüss' inequality for positive linear functionals // Period. math. hung. – 1988. – **19**. – P. 155 – 167.
3. *Anastassiou G. A.* Ostrowski type inequalities // Proc. Amer. Math. Soc. – 1995. – **123**. – P. 3775 – 3781.
4. *Čebyšev P. L.* Sur les expressions approximatives des intégrales définies par les autres prises entre les méme limites // Proc. Math. Soc. Kharkov. – 1882. – **2**. – P. 93 – 98 (in Russian) (Transl.: Oeuvres. – 1907. – **2**. – P. 716 – 719).
5. *Cheng X. L.* Improvement of some Ostrowski – Grüss type inequalities // Comput. Math. Appl. – 2001. – **42**. – P. 109 – 114.
6. *Dragomir S. S.* On the Ostrowski integral inequality for Lipschitzian mappings and applications // Comput. Math. Appl. – 1999. – **38**. – P. 33 – 37.
7. *Dragomir S. S., Wang S.* An inequality of Ostrowski – Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules // Comput. Math. Appl. – 1997. – **33**, № 11. – P. 15 – 20.
8. *Gavrea B., Gavrea I.* Ostrowski type inequalities from a linear functional point of view // J. Inequal. Pure and Appl. Math. – 2000. – **1**, Article 11.
9. *Gonska H.* Quantitative Approximation in $C(X)$ // Habilitationsschrift. – Univ. Duisburg, 1986.
10. *Goodenough S. J., Mills T. M.* A new estimate for the approximation of functions by Hermite – Fejér interpolation polynomials // J. Approxim. Theory. – 1981. – **31**. – P. 253 – 260.
11. *Grüss G.* Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$ // Math. Z. – 1935. – **39**. – S. 215 – 226.
12. *Karamata J.* Inégalités relatives aux quotients et à la différence de $\int fg$ et $\int f \int g$ // Bull. Acad. Serbe. Sci. Math. Natur. A. – 1948. – P. 131 – 145.
13. *Kiš O.* Remarks on the rapidity of convergence of Lagrange interpolation (in Russian) // Ann. Univ. sci. Budapest. Sect. math. – 1968. – **11**. – P. 27 – 40.
14. *Landau E.* Über einige Ungleichungen von Herrn G. Grüss // Math. Z. – 1935. – **39**. – S. 742 – 744.
15. *Lehnhoff H. G.* A simple proof of A.F. Timan's theorem // J. Approxim. Theory. – 1983. – **38**. – P. 172 – 176.
16. *Matsuoka Y.* On the degree of approximation of functions by some positive linear operators // Sci. Rep. Kagoshima Univ. – 1960. – **9**. – P. 11 – 16.
17. *Mercer A. Mc. D., Mercer P. R.* New proofs of the Grüss inequality // Austral. J. Math. Anal. and Appl. – 2004. – **1**, Issue 2. – P. 1 – 6.
18. *Misra R. N.* On the rate of convergence of Hermite – Fejér interpolation polynomials // Period. math. hung. – 1982. – **13**. – P. 15 – 20.
19. *Mitrinović D. S., Pečarić J. E., Fink A. M.* Classical and new inequalities in analysis. – Dordrecht: Kluwer, 1993.
20. *Ostrowski A.* Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert // Comment. math. helv. – 1938. – **10**. – P. 226 – 227.
21. *Ostrowski A.* On an integral inequality // Aequat. math. – 1970. – **4**. – P. 358 – 373.
22. *Pachpatte B. G.* A note on Ostrowski like inequalities // J. Inequal. Pure and Appl. Math. – 2005. – **6**, Article 114.
23. *Pachpatte B. G.* A note on Grüss type inequalities via Cauchy's mean value theorem // Math. Inequal. Appl. – 2007. – **11**, № 1. – P. 75 – 80.
24. *Semenov E. M., Mitjagin B. S.* Lack of interpolation of linear operators in spaces of smooth functions // Mat. USSR. Izv. – 1977. – **11**. – S. 1229 – 1266.

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